## lectuuurrreeeee.

In the *longest increasing subsequence* problem, the input is a sequence of numbers $a_1, \ldots, a_n$. A *subsequence* is any subset of these numbers taken in order, of the form $a_{i_1}, a_{i_2}, \ldots, a_{i_k}$ where $1 \le i_1 < i_2 < \cdots < i_k \le n$, and an *increasing* subsequence is one in which the numbers are getting strictly larger. The task is to find the increasing subsequence of greatest length. For instance, the longest increasing subsequence of $5, 2, 8, 6, 3, 6, 9, 7$ is $2, 3, 6, 9$:

$$5 \quad 2 \quad 8 \quad 6 \quad 3 \quad 6 \quad 9 \quad 7$$

We will devise an $O(n^2)$-time algorithm for this task.

## exercises, on the easier side and kinda related to class.

1. **You are going on a long trip.** You start on the road at mile post 0. Along the way there are $n$ hotels, at mile posts $a_1 < a_2 < \cdots < a_n$, where each $a_i$ is measured from the starting point. The only places you are allowed to stop are at these hotels, but you can choose which of the hotels you stop at. You must stop at the final hotel (at distance $a_n$), which is your destination.

   You'd like to travel 200 miles a day, but this may not be possible, depending on the spacing of the hotels. If you travel $x$ miles during a day, the penalty for that day is $(200 - x)^2$. You want to plan your trip so as to minimize the sum, over all travel days, of the daily penalties.

   Give an efficient algorithm that determines the optimal sequence of hotels at which to stop.

2. **Counting heads.** Given integers $n$ and $k$, along with probabilities $p_1, \ldots, p_n \in [0, 1]$, you want to determine the probability of obtaining exactly $k$ heads when $n$ biased coins are tossed independently at random, where $p_i$ is the probability that the $i$th coin comes up heads.
   *Assume you can multiply and add two floating point numbers in the range $[0, 1]$ in $O(1)$ time.*

   (a) Give an $O(n^2)$ algorithm for this task.

   (b) (challenge, b/c advanced; not, too hard) Give an $O(n(\log n)^2)$ algorithm for this task.

## more exercises, about real-life greedy algorithms

In the US, when we have access to lots of coins, most of us use the following greedy allocation scheme for giving change: first, we hand out as many quarters as possible; then, as many dimes as possible; then as many nickels; then, finish the rest with pennies.

$ **Prove that for any amount**, 1 cent, 2 cents, . . . , 99 cents, greedy change-making in the US also results in giving out the fewest number of coins.

$ **Come up with a set of coin denominations** where there is a *non-greedy* strategy that gives out fewer coins when making change for some amount between 1 and 99 cents.

$ **What happens** with coin denominations $1, 2, 4, 8, 16, 32, 64$ cents?

$$ **Prove or disprove:** greedy allocation results in using the fewest number of coins for coin denominations $1, p^1, p^2, \ldots, p^k$ cents, for values $p = 2, 3, 4, 5, 6, 7, 8, 9$, where k is such that $p^k < 100 < p^{k+1}$.

$$$ (mathematically advanced) **Can you derive a general rule** for coin denominations $c_1, c_2, \ldots, c_k$ where greedy-allocation results in the fewest number of coins?

## a probabilistic algorithmic puzzle

A disgruntled TA plays a cruel game with his class of 100 students. He takes a stack of 100 quarters and goes around the room to each student one at a time. First, he flips one of the quarters. Then, he puts superglue on whichever side lands face up, and glues the quarter to the student's forehead, so that student cannot see the quarter but everyone else can. After he's done gluing all 100 quarters, each student must write down on a piece of paper, without talking to anyone else, an educated guess of which side of the quarter is glued to his or her forehead. Then, the TA collects the papers, and anyone who guessed wrong automatically gets on $\mathcal{F}$ on the final.

Not totally unmerciful, the TA tells the students about the game beforehand and gives them time to decide upon a strategy. What strategy should they come up with so that no more than half the students get $\mathcal{F}$'s?