Lecture

AXIOMS OF PROBABILITY

- **Additivity:** If $A_1, \ldots, A_n$ are n disjoint events ($A_i \cap A_j = \emptyset$ for $i \neq j$), then the probability of their union satisfies
$$\mathbf{Pr}[\bigcup_{i=1}^{n} A_i] = \sum_{i=1}^{n} \mathbf{Pr}[A_i]$$
- **Normalization:** The probability of the entire sample space Ω is equal to one, i.e $\mathbf{Pr}[\Omega] = 1$
- **Nonnegativity:** $\mathbf{Pr}[A] \geq 0$ for every event A.

Exercises

1. Give a mathematical derivation of the formula
$$\mathbf{Pr}[(A \cap \overline{B}) \cup (\overline{A} \cap B)] = \mathbf{Pr}[A] + \mathbf{Pr}[B] - 2\mathbf{Pr}[A \cap B].$$
Your derivation should be a sequence of steps, with each step justified by appealing to one of the probability axioms. When you use the additivity axiom, make sure to explain why two sets are disjoint.

2. Consider the following scenario: out of the students in a class, 60% are geniuses, 70% love chocolate, and 40% fall into both categories. To aid in your solution, prove two general statements first, and then use them to answer the question.
   - For events A and B in any sample space Ω, prove that $1 = \mathbf{Pr}[A \cup B] + \mathbf{Pr}[\ \overline{(A \cup B)}\ ]$.
   - Prove that $\mathbf{Pr}[A \cup B] = \mathbf{Pr}[A] + \mathbf{Pr}[B] - \mathbf{Pr}[A \cap B]$.
   - Find the probability that a randomly selected student is neither a genius nor a chocolate lover. To do so, define events A and B so that you can use the things you've proved.

3. Waiting in line are three irritated customers and three content customers. Due to their moods, if two irritated customers are standing next to each other, then they will get in a fight. Also, if a content customer is at the front of the line, then the irritated customers will get angry. If the six people line up at random, and all lines are equally likely, what is the probability that no one gets angry or gets in a fight? Solve this problem in three different ways:
   (a) Assume that all six people are distinguishable. How many total lines are there? How many lines lead to no anger and no violence?
   (b) Now, assume that the people are indistinguishable, except for their mood. How many total lines are there? How many lines lead to no anger and no violence?
   (c) Finally, assume that the six customers walk in one at a time until the line is full, and again calculate the probability that no one gets angry or gets in a fight.
   (d) You should have gotten the same answer for each calculation. Why?

4. For these next problems, recall the definition of conditional probability, for events X and Y with $\mathbf{Pr}[Y] > 0$,

$$\mathbf{Pr}[X \mid Y] = \frac{\mathbf{Pr}[X \cap Y]}{\mathbf{Pr}[Y]}.$$

An important point to remember – for this rule to work, all of the probabilities and events have to use the **same** outcomes and **same** probability space.

(a) (easy check your understanding) Prove that $\mathbf{Pr}[X \mid Y] \cdot \mathbf{Pr}[Y] = \mathbf{Pr}[Y \mid X] \cdot \mathbf{Pr}[X]$.

(b) We roll two fair 6-sided dice. Each one of the 36 possible outcomes is assumed to be equally likely.

i. Find the probability that doubles are rolled.
ii. Given that the roll results in a sum of 4 or less, find the conditional probability that doubles are rolled.
iii. Find the probability that at least one die roll is a 6.
iv. Given that the two dice land on different numbers, find the conditional probability that at least one die roll is a 6.

Challenge: The (famous) Derangement Problem

Let's revisit the dinner party scenario from the homework. At a dinner party, all of the n people present are to be seated at a circular table. Suppose there is a name tag at each place at the table. Now, suppose that the n people sit down uniformly at random. What is the probability that nobody sits down in their correct name tag? (this is called a *derangement*).

Since this problem is a little tricky, you should start by first working it out explicitly for the case of three and four people.

For those of you who are interested, here is a nice exposition of two of many possible solutions, which you will learn a lot from if you take the time to understand the whole thing!
http://math.ucr.edu/home/baez/qg-winter2004/derangement.pdf

Solution to problem 1.

Question: Give a mathematical derivation of the formula

$$\mathbf{Pr}[(A \cap \overline{B}) \cup (\overline{A} \cap B)] = \mathbf{Pr}[A] + \mathbf{Pr}[B] - 2\mathbf{Pr}[A \cap B].$$

Solution We will work from right to left, starting with $\mathbf{Pr}[A] + \mathbf{Pr}[B] - 2\mathbf{Pr}[A \cap B]$. First, let's rewrite the sets A and B is a more accessible way. The steps involved are pretty straightforward, but make sure you understand them!

$$\begin{aligned} A &= A \cap \Omega & B &= B \cap \Omega \\ &= A \cap (B \cup \overline{B}) & &= B \cap (A \cup \overline{A}) \\ &= (A \cap B) \cup (A \cap \overline{B}) & &= (B \cap A) \cup (B \cap \overline{A}). \end{aligned}$$

Then, we can write, by additivity and disjointness

$$\begin{aligned} \mathbf{Pr}[A] &= \mathbf{Pr}[(B \cap A) \cup (B \cap \overline{A})] \\ &= \mathbf{Pr}[(B \cap A)] + \mathbf{Pr}[(B \cap \overline{A})], \end{aligned}$$

and similarly,

$$\begin{aligned} \mathbf{Pr}[B] &= \mathbf{Pr}[(A \cap B) \cup (A \cap \overline{B})] \\ &= \mathbf{Pr}[(A \cap B)] + \mathbf{Pr}[(A \cap \overline{B})]. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{Pr}[A] + \mathbf{Pr}[B] - 2\mathbf{Pr}[A \cap B] &= \mathbf{Pr}[(A \cap \overline{B})] + \mathbf{Pr}[(B \cap \overline{A})] + 2\mathbf{Pr}[(B \cap A)] - 2\mathbf{Pr}[A \cap B] \\ &= \mathbf{Pr}[(A \cap \overline{B})] + \mathbf{Pr}[(B \cap \overline{A})]. \end{aligned}$$