3. Discrete Probability

CSE 312
Autumn 2012
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Change to Syllabus

Introducing: The Daily Puzzler

One short problem

Due at start of lecture, MWF

Graded: ☺ or ☹

Electronic turnin only

Due 10/5: What is the probability of getting "2 pairs" when dealt 5 cards in poker?

(will be on web later today)

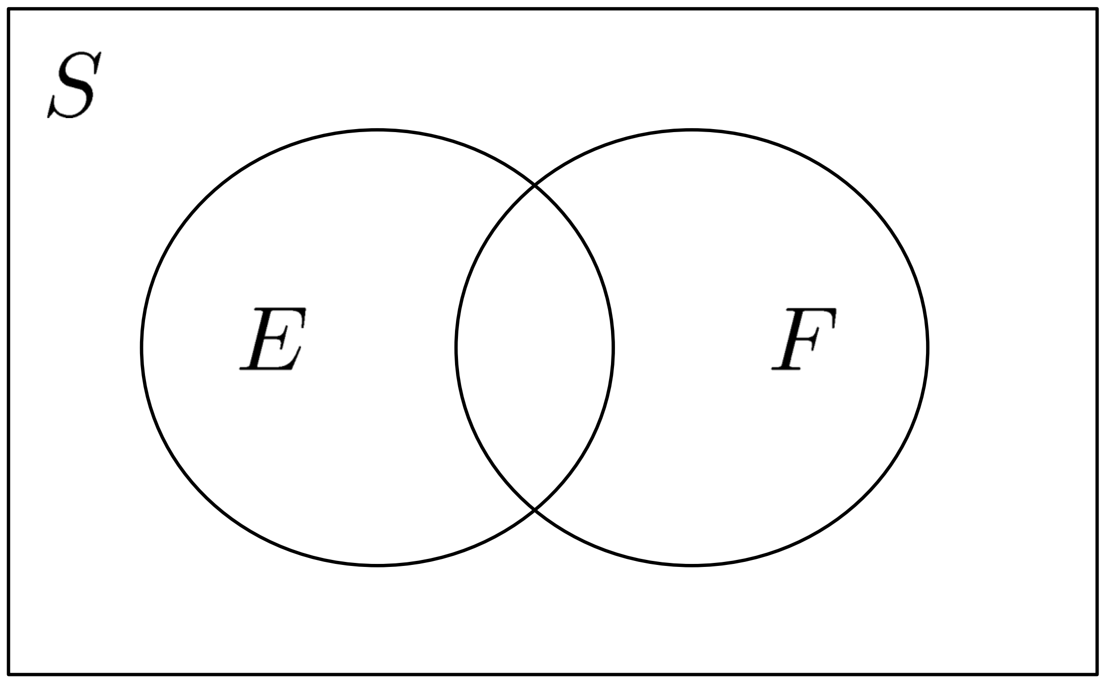
Sample space: *S* is the set of all possible outcomes of an experiment (often Ω in text books–Greek uppercase omega)

Coin flip:	S = {Heads, Tails}
Flipping two coins:	S = {(H,H), (H,T), (T,H), (T,T)}
Roll of one 6-sided die:	S = {1, 2, 3, 4, 5, 6}
# emails in a day:	$S = \{ x : x \in Z,\ x \geq 0 \}$
YouTube hrs. in a day:	$S = \{ x : x \in R, 0 \leq x \leq 24 \}$

Events: **E** $\subseteq$ **S** is an arbitrary subset of the sample space

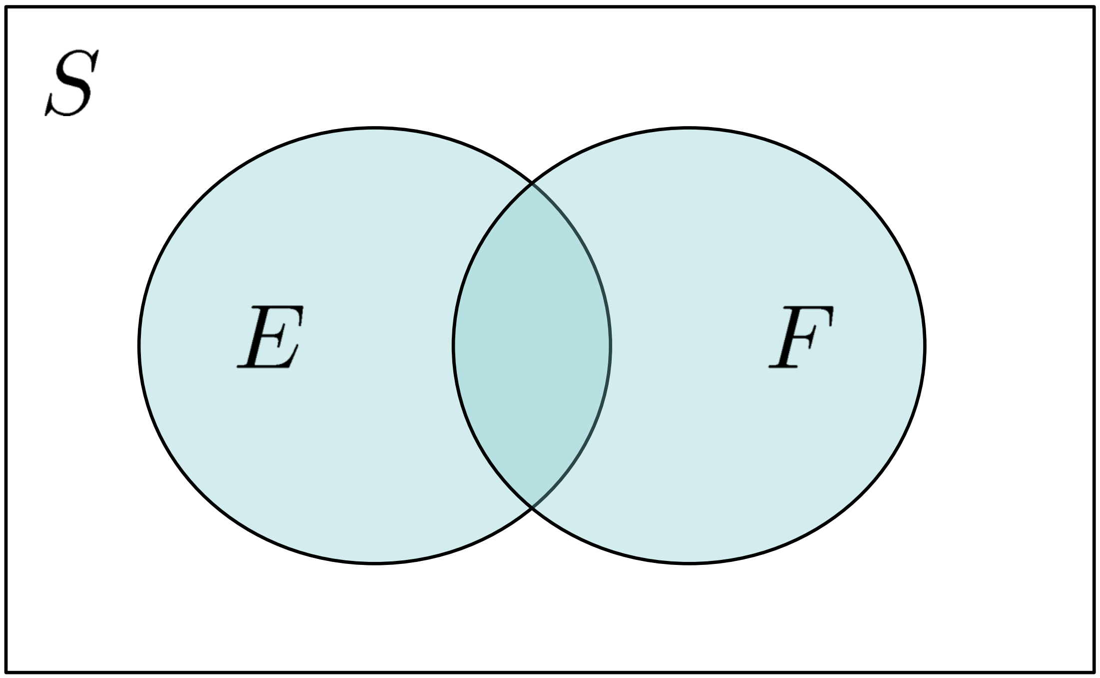
Coin flip is heads:	$E = \{Head\}$
At least one head in 2 flips:	$E = \{(H,H), (H,T), (T,H)\}$
Roll of die is odd:	$E = \{1, 3, 5\}$
# emails in a day < 20:	$E = \{ x : x \in Z,\ 0 \leq x < 20 \}$
# emails in a day is prime:	$E = \{ 2, 3, 5, 7, 11, 13, \ldots \}$
Wasted day (>5 YT hrs):	$E = \{ x : x \in R,\ x > 5 \}$

E and F are events in the sample space S



E and F are events in the sample space S

Event "E OR F", written $E \cup F$

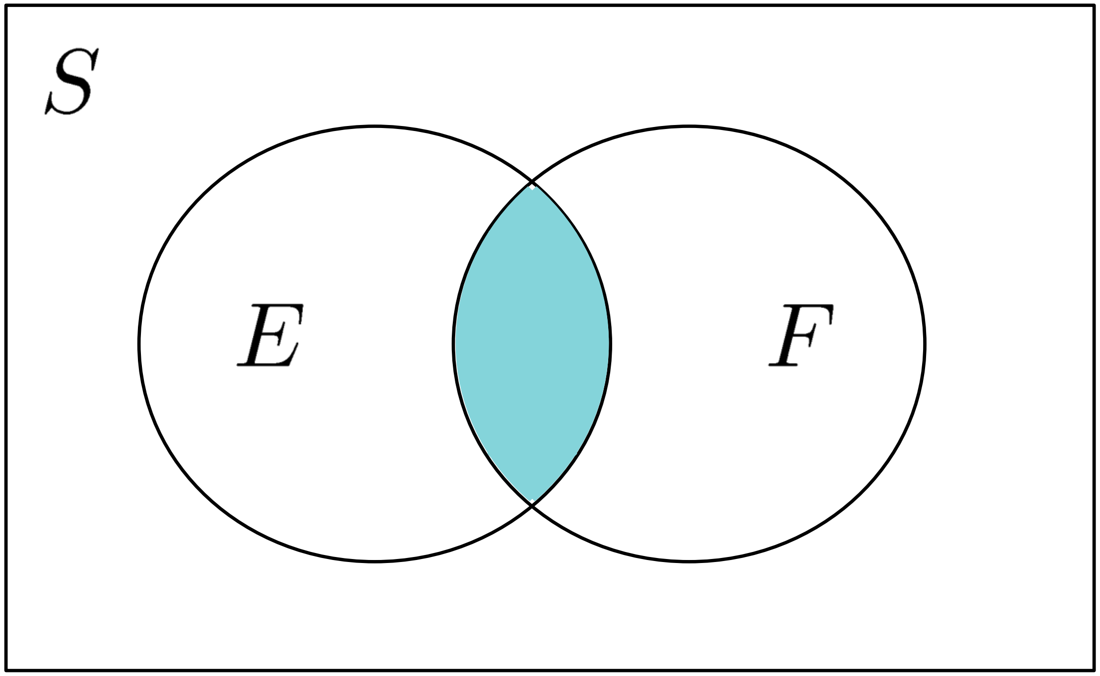


S = {1,2,3,4,5,6}
outcome of one die roll

E = {1,2}, F = {2,3}
$E \cup F$ = {1, 2, 3}

E and F are events in the sample space S

Event "E AND F", written $E \cap F$ or EF

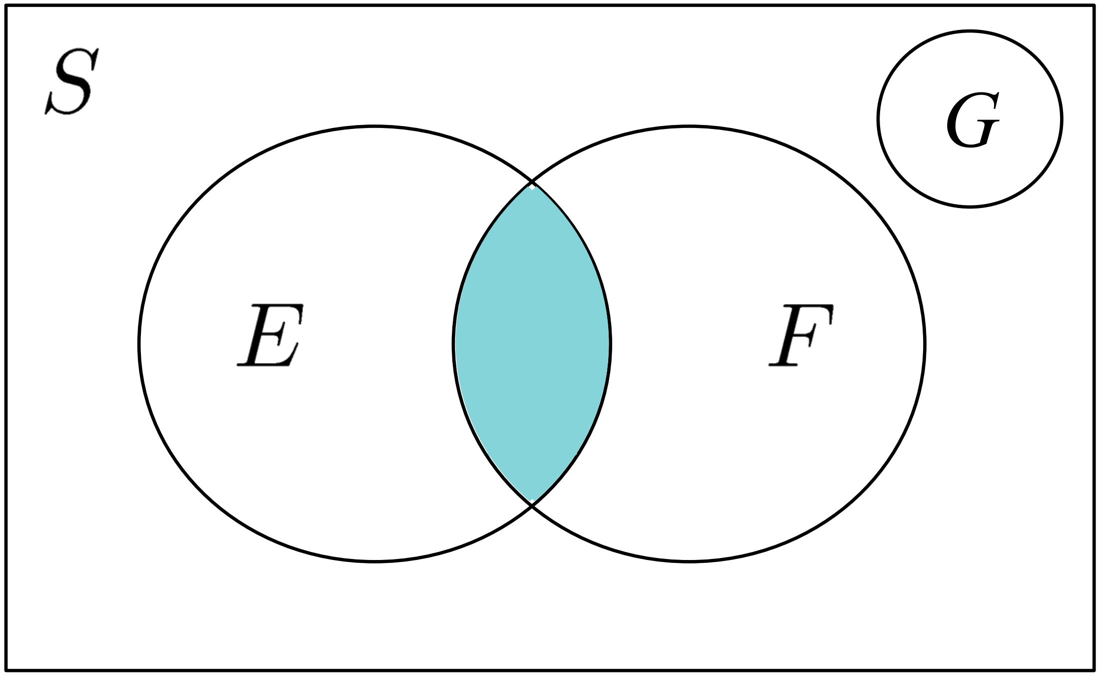


S = {1,2,3,4,5,6}
outcome of one die roll

E = {1,2}, F = {2,3}
$E \cap F$ = {2}

E and F are events in the sample space S

EF = ∅ ⇔ E,F are "mutually exclusive"

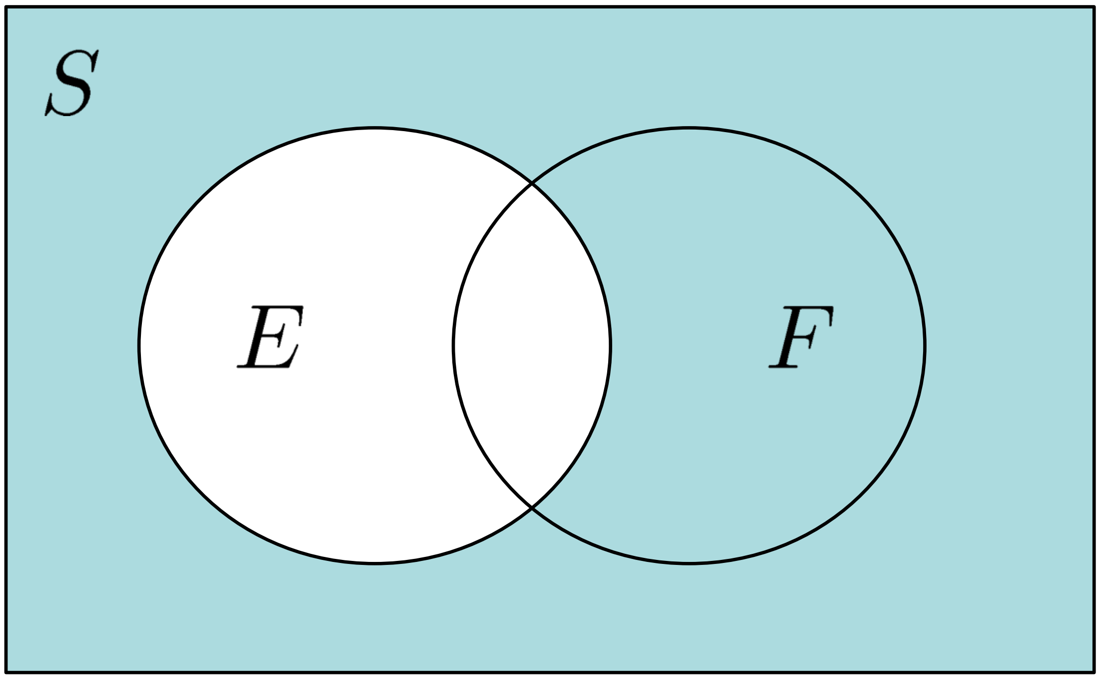


S = {1,2,3,4,5,6}
outcome of one die roll

E = {1,2}, F = {2,3}, G={5,6}
EF = {2}, *not* mutually exclusive, but E,G and F,G are

E and F are events in the sample space S

Event "not E," written $\overline{E}$ or $\neg E$

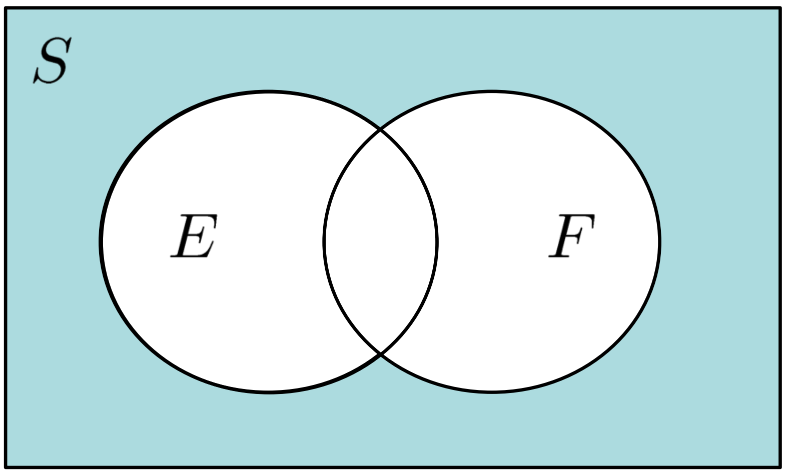


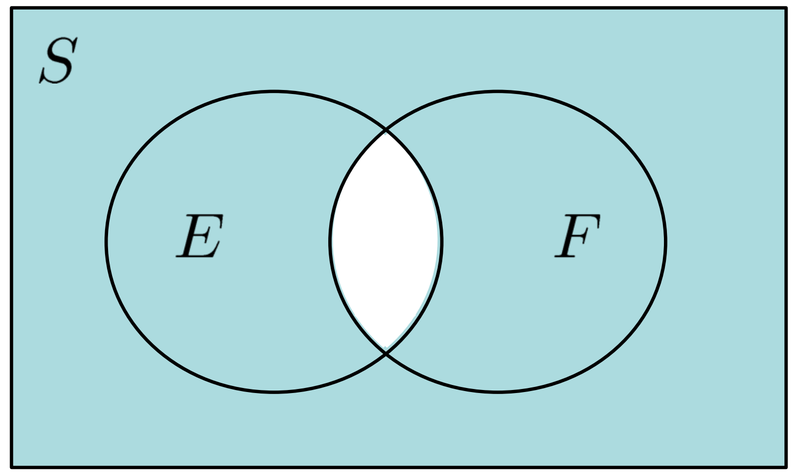
S = {1,2,3,4,5,6}
outcome of one die roll

$E = \{1, 2\}$ $\neg E = \{3, 4, 5, 6\}$

DeMorgan's Laws

$$\overline{E \cup F} = \bar{E} \cap \bar{F} \qquad \overline{E \cap F} = \bar{E} \cup \bar{F}$$





axioms of probability

Intuition: Probability as the relative frequency of an event

$\Pr(E) = \lim_{n \to \infty}$ (# of occurrences of E in n trials)/n

Axiom 1: $0 \le \Pr(E) \le 1$

Axiom 2: $\Pr(S) = 1$

Axiom 3: If E and F are mutually exclusive ($EF = \emptyset$), then

$$\Pr(E \cup F) = \Pr(E) + \Pr(F)$$

For any sequence $E_1, E_2, \ldots, E_n$ of mutually exclusive events,

$$\Pr\left(\bigcup_{i=1}^{n} E_i\right) = \Pr(E_1) + \cdots + \Pr(E_n)$$

- $\Pr(\bar{E}) = 1 - \Pr(E)$

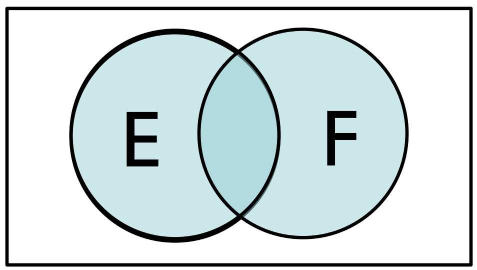
$$\Pr(\bar{E}) = \Pr(S) - \Pr(E) \text{ because } S = E \cup \bar{E}$$

- If $E \subseteq F$, then $\Pr(E) \leq \Pr(F)$

$$\Pr(F) = \Pr(E) + \Pr(F - E) \geq \Pr(E)$$

- $\Pr(E \cup F) = \Pr(E) + \Pr(F) - \Pr(EF)$

inclusion-exclusion formula



- And many others

equally likely outcomes

Simplest case: sample spaces with equally likely outcomes.

Coin flips: S = {Heads, Tails}

Flipping two coins: S = {(H,H),(H,T),(T,H),(T,T)}

Roll of 6-sided die: S = {1, 2, 3, 4, 5, 6}

Pr(each outcome) = $\frac{1}{|S|}$

In that case,

$$\Pr(E) = \frac{\text{number of outcomes in } E}{\text{number of outcomes in } S} = \frac{|E|}{|S|}$$

Why? Axiom 3 plus fact that E = union of singletons in E

Roll two 6-sided dice. What is Pr(sum of dice = 7) ?

S = { (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
(2,1), (2,2), (2,3), (2,4), (2,5), (2,6),
(3,1), (3,2), (3,3), (3,4), (3,5), (3,6),
(4,1), (4,2), (4,3), (4,4), (4,5), (4,6),
(5,1), (5,2), (5,3), (5,4), (5,5), (5,6),
(6,1), (6,2), (6,3), (6,4), (6,5), (6,6) }

Side point: *S* is small; can write out explicitly, but how would you visualize the analogous problem with 10^3-sided dice?

E = { (6,1), (5,2), (4,3), (3,4), (2,5), (1,6)}

Pr(sum = 7) = |E|/|S| = 6/36 = 1/6.

twinkies and ding dongs

4 Twinkies and 3 DingDongs in a bag. 3 drawn.
What is Pr(one Twinkie and two DingDongs drawn) ?

Ordered: (S ordered triple with 3 of 7 distinguishable objects)

- Pick 3, one after another: |S| = 7 • 6 • 5 = 210
- Pick Twinkie as either 1st, 2nd, or 3rd item:
  |E| = (4•3•2) + (3•4•2) + (3•2•4) = 72
- Pr(1 Twinkie and 2 DingDongs) = 72/210 = 12/35.

Unordered: (S *un*ordered triple with 3 of 7 distinguishable objects)

- Grab 3 at once: |S| $= \binom{7}{3} = 35$
- |E| $= \binom{4}{1}\binom{3}{2} = 12$
- Pr(1 Twinkie and 2 DingDongs) = 12/35.

Exercise: a 3rd way – S is ordered list of 7, E is "1st 3 OK"; same answer?

birthdays

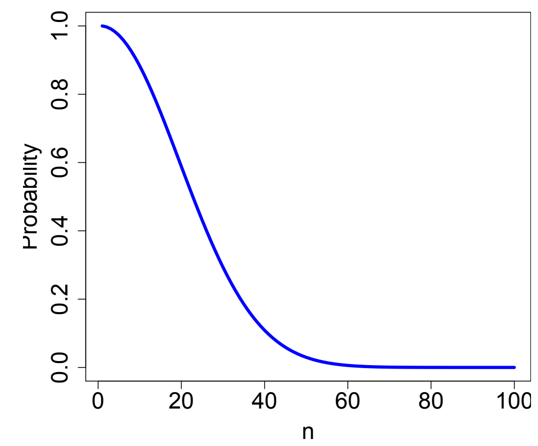
What is the probability that, of n people, none share the same birthday?

$|S| = (365)^n$

$|E| = (365)(364)(363)\cdots(365-n+1)$

Pr(no matching birthdays) = |E|/|S|

$= (365)(364)\ldots(365-n+1)/(365)^n$



Some values of n…

n = 23: Pr(no matching birthdays) < 0.5

n = 77: Pr(no matching birthdays) < 1/5000

n = 100: Pr(no matching birthdays) < 1/3,000,000

n = 150: Pr(…) < 1/3,000,000,000,000,000

n = 366?

Pr = 0

Above formula gives this, since

$(365)(364)\ldots(365-n+1)/(365)^n == 0$

when n = 366 (or greater).

Even easier to see via pigeon hole principle.

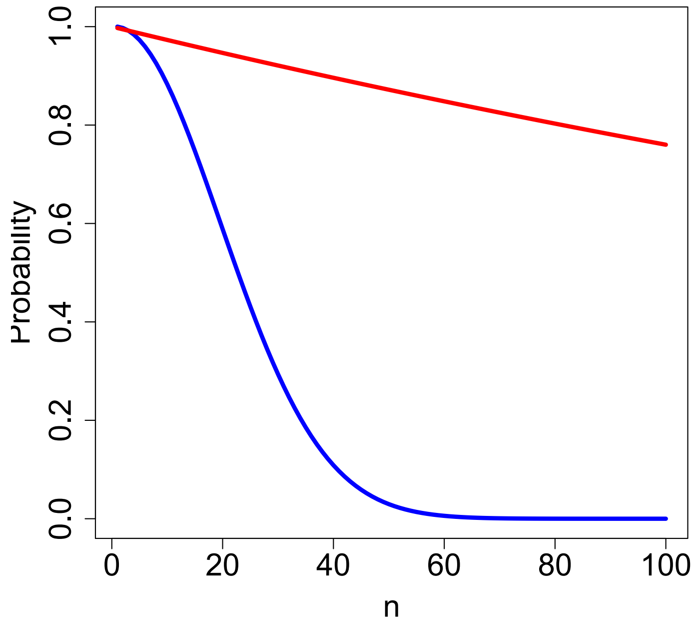
What is the probability that, of n people, none share the same birthday as you?

$|S| = (365)^n$

$|E| = (364)^n$

Pr(no birthdays = yours)

$= |E|/|S| = (364)^n/(365)^n$



Some values of n…

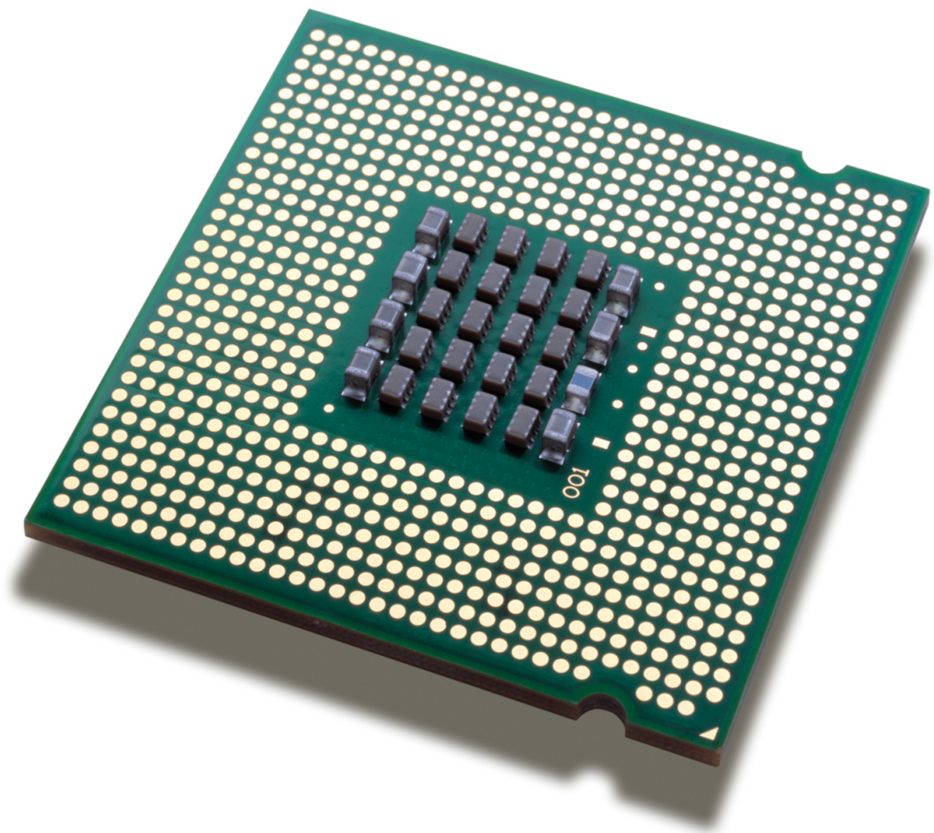
n = 23: Pr(no matching birthdays) ≈ 0.9388

n = 77: Pr(no matching birthdays) ≈ 0.8096

n = 253: Pr(no matching birthdays) ≈ 0.4995

Exercise: p^n is not linear, but red line looks straight. Why?

chip defect detection



n chips manufactured, one of which is defective
k chips randomly selected from n for testing

What is Pr(defective chip is in k selected chips) ?

|S| = $\binom{n}{k}$ |E| = $\binom{1}{1}\binom{n-1}{k-1}$

Pr(defective chip is among k selected chips)

$$= \frac{\binom{1}{1}\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{\frac{(n-1)!}{(k-1)!(n-k)!}}{\frac{n!}{k!(n-k)!}} = \frac{k}{n}$$

n chips manufactured, one of which is defective
k chips randomly selected from n for testing

What is Pr(defective chip is in k selected chips) ?

Different analysis:

- Select k chips at random by permuting all n chips and then choosing the first k.
- Let E_i = event that i^{th} chip is defective.
- Events $E_1, E_2, \ldots, E_k$ are mutually exclusive
- $Pr(E_i) = 1/n$ for $i=1,2,\ldots,k$
- Thus Pr(defective chip is selected)

$$= Pr(E_1) + \cdots + Pr(E_k) = k/n.$$

n chips manufactured, *two* of which are defective

k chips randomly selected from n for testing

What is Pr(a defective chip is in k selected chips) ?

|S| = $\binom{n}{k}$ |E| = (1 chip defective) + (2 chips defective)

$$= \binom{2}{1}\binom{n-2}{k-1} + \binom{2}{2}\binom{n-2}{k-2}$$

Pr(a defective chip is in k selected chips)

$$= \frac{\binom{2}{1}\binom{n-2}{k-1} + \binom{2}{2}\binom{n-2}{k-2}}{\binom{n}{k}}$$

n chips manufactured, *two* of which are defective

k chips randomly selected from n for testing

What is Pr(a defective chip is in k selected chips) ?

Another approach:

Pr(a defective chip is in k selected chips) = 1-Pr(none)

Pr(none):

$$|S| = \binom{n}{k}, |E| = \binom{n-2}{k}, Pr(\text{none}) = \frac{\binom{n-2}{k}}{\binom{n}{k}}$$

Pr(a defective chip is in k selected chips) = $1 - \frac{\binom{n-2}{k}}{\binom{n}{k}}$

(Same as above? Check it!)

poker hands

5 card poker hands (ordinary 52 card deck, no jokers etc.)

flush, 1 pair, 3 of a kind, 2 pairs, full house, …

Sample Space?

Imagine sorted tableau of cards, pick 5:

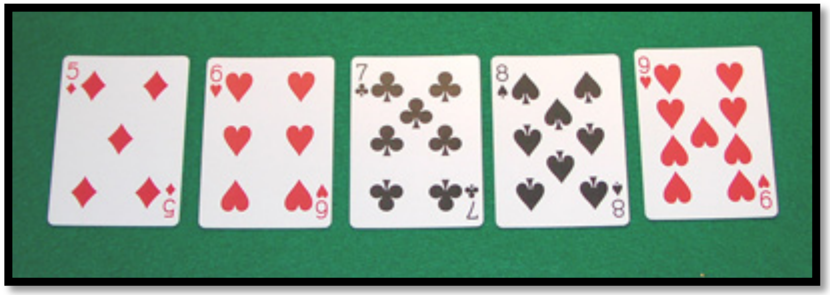
A♥	2♥	3♥	…	10♥	J♥	Q♥	K♥
A♣	2♣	3♣	…	10♣	J♣	Q♣	K♣
A♦	2♦	3♦	…	10♦	J♦	Q♦	K♦
A♠	2♠	3♠	…	10♠	J♠	Q♠	K♠

$|S| = \binom{52}{5}$

any straight in poker

Consider 5 card poker hands.

A "straight" is 5 consecutive rank cards ignoring suit (Ace low or high, but not both. E.g., A,2,3,4,5 or 10,J,Q,K,A)



What is Pr(straight) ?

S as on previous slide, $|S| = \binom{52}{5}$

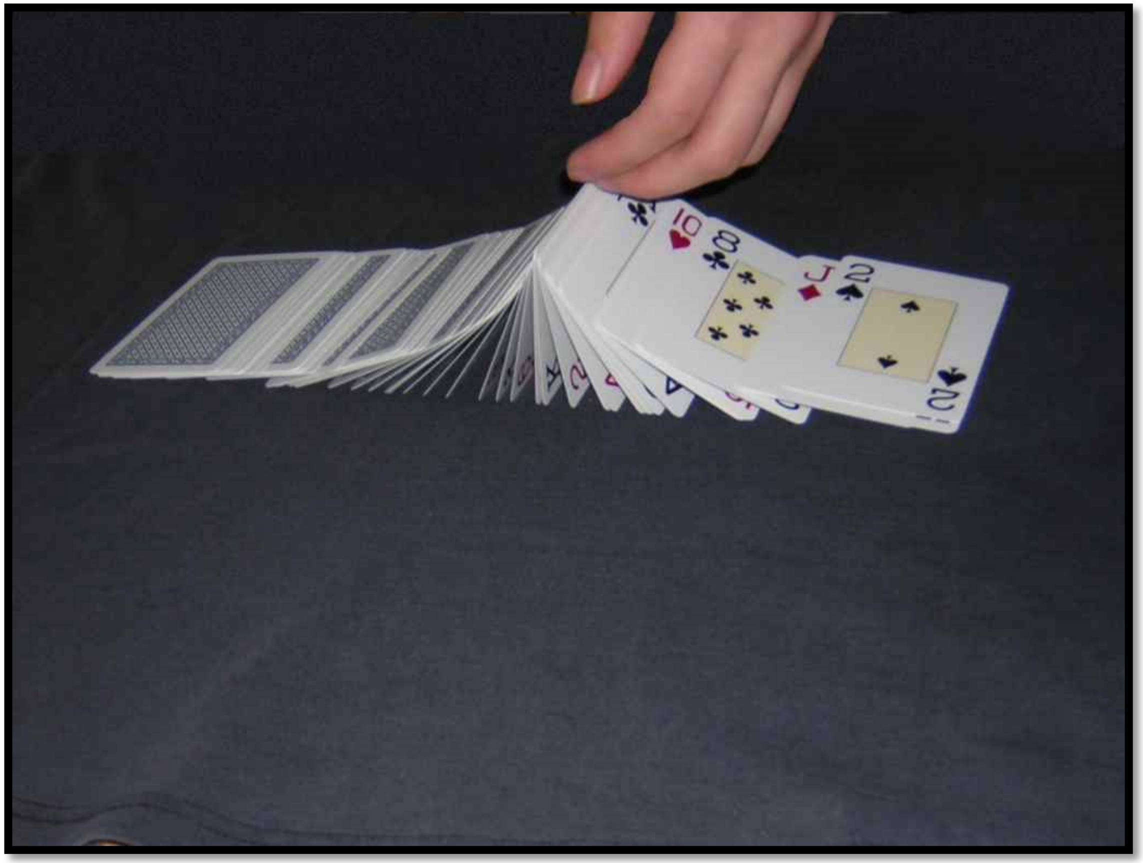
What's E?

E = Pick col A, 2, … 10, then 1 of 4 in next 5 cols (wrapping)

$|E| = 10 \cdot \binom{4}{1}^5$

$$\Pr(\text{straight}) = \frac{10\binom{4}{1}^5}{\binom{52}{5}} \approx 0.00394$$

card flipping



52 card deck. Cards flipped one at a time.

After first ace (of any suit) appears, consider next card

Pr(next card = ace of spades) < Pr(next card = 2 of clubs) ?

Maybe, Maybe Not …

S = all permutations of 52 cards, |S| = 52!

Event 1: Next = Ace of Spades.

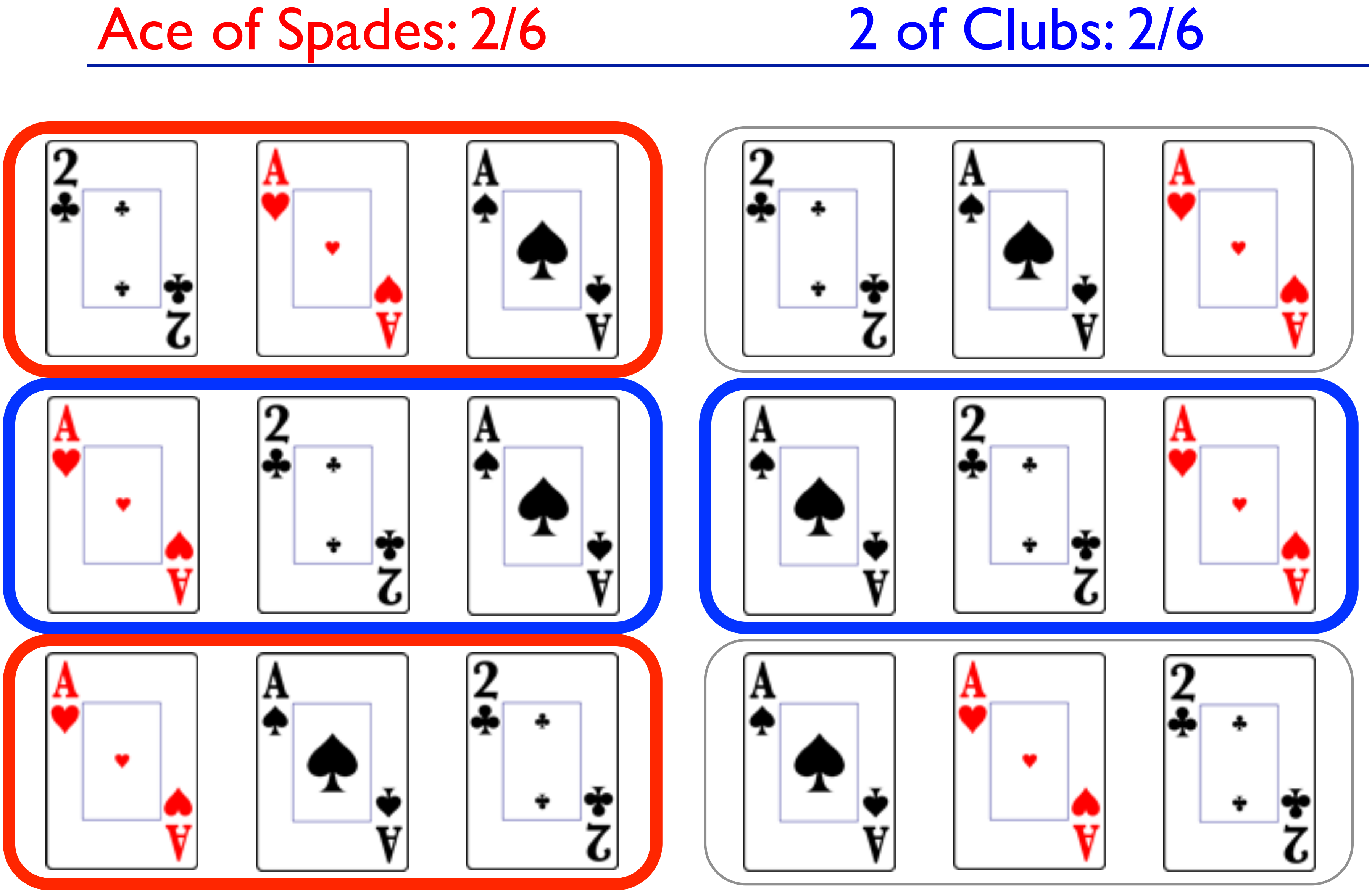
Remove A♠, shuffle remaining 51 cards, add A♠ after first Ace

$|E_1| = 51!$ (only 1 place A♠ can be added)

Event 2: Next = 2 of Clubs

Do the same thing with 2♣; E_1 and E_2 have same size

So, $Pr(E_1) = Pr(E_2) = 51!/52! = 1/52$



Theory is the same for a 3-card deck; Pr = 2!/3! = 1/3

Card images from http://www.eludication.org/playingcards.html

hats

n persons at a party throw hats in middle, select at random. What is Pr(no one gets own hat)?

Pr(no one gets own hat) =
    1 – Pr(someone gets own hat)

Pr(someone gets own hat) = $\Pr(\cup_{i=1}^{n} E_i)$, where
E_i = event that person i gets own hat

$\Pr(\cup_{i=1}^{n} E_i) = \Sigma_i P(E_i) - \Sigma_{i<j} \Pr(E_i\, E_j) + \Sigma_{i<j<k} \Pr(E_i\, E_j\, E_k) \ldots$

Visualizing the sample space *S*:

People:
Hats:

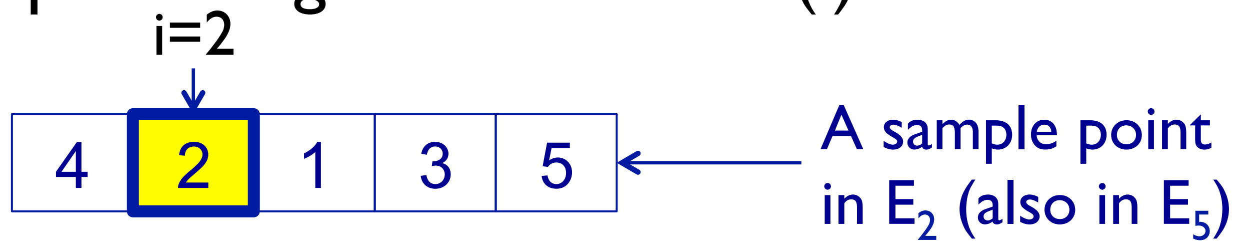
P_1	P_2	P_3	P_4	P_5
H_4	H_2	H_5	H_1	H_3

I.e., a sample point is a *permutation* π of 1, …, n

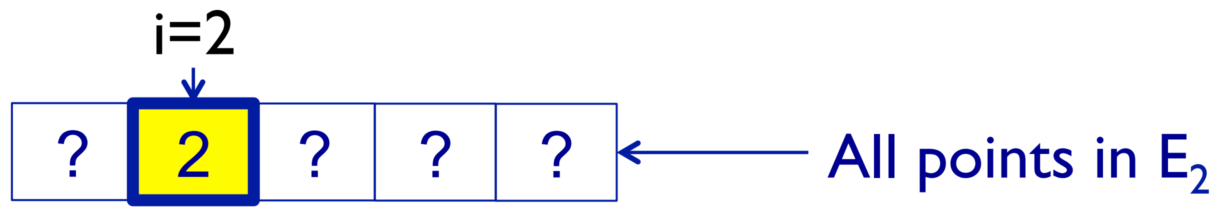
4	2	5	1	3

$|S| = n!$

E_i = event that person i gets own hat: $\pi(i) = i$



Counting single events:

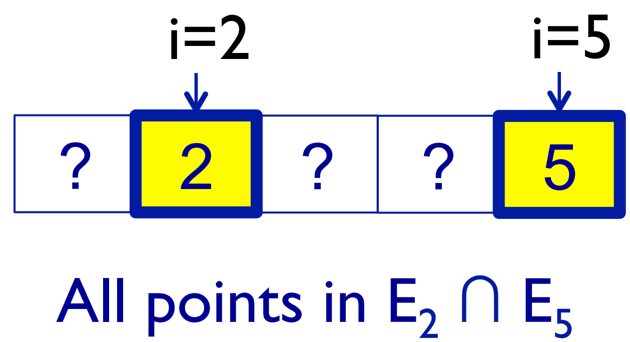


$|E_i| = (n-1)!$ for all i

Counting pairs:

E_iE_j : $\pi(i) = i$ & $\pi(j) = j$

$|E_iE_j| = (n-2)!$ for all i, j



n persons at a party throw hats in middle, select at random. What is Pr(no one gets own hat)?

E_i = event that person i gets own hat

$Pr(\cup_{i=1}^{n} E_i) = \Sigma_i P(E_i) - \Sigma_{i<j} Pr(E_i E_j) + \Sigma_{i<j<k} Pr(E_i E_j E_k) \ldots$

Pr(k fixed people get own back) = (n-k)!/n!

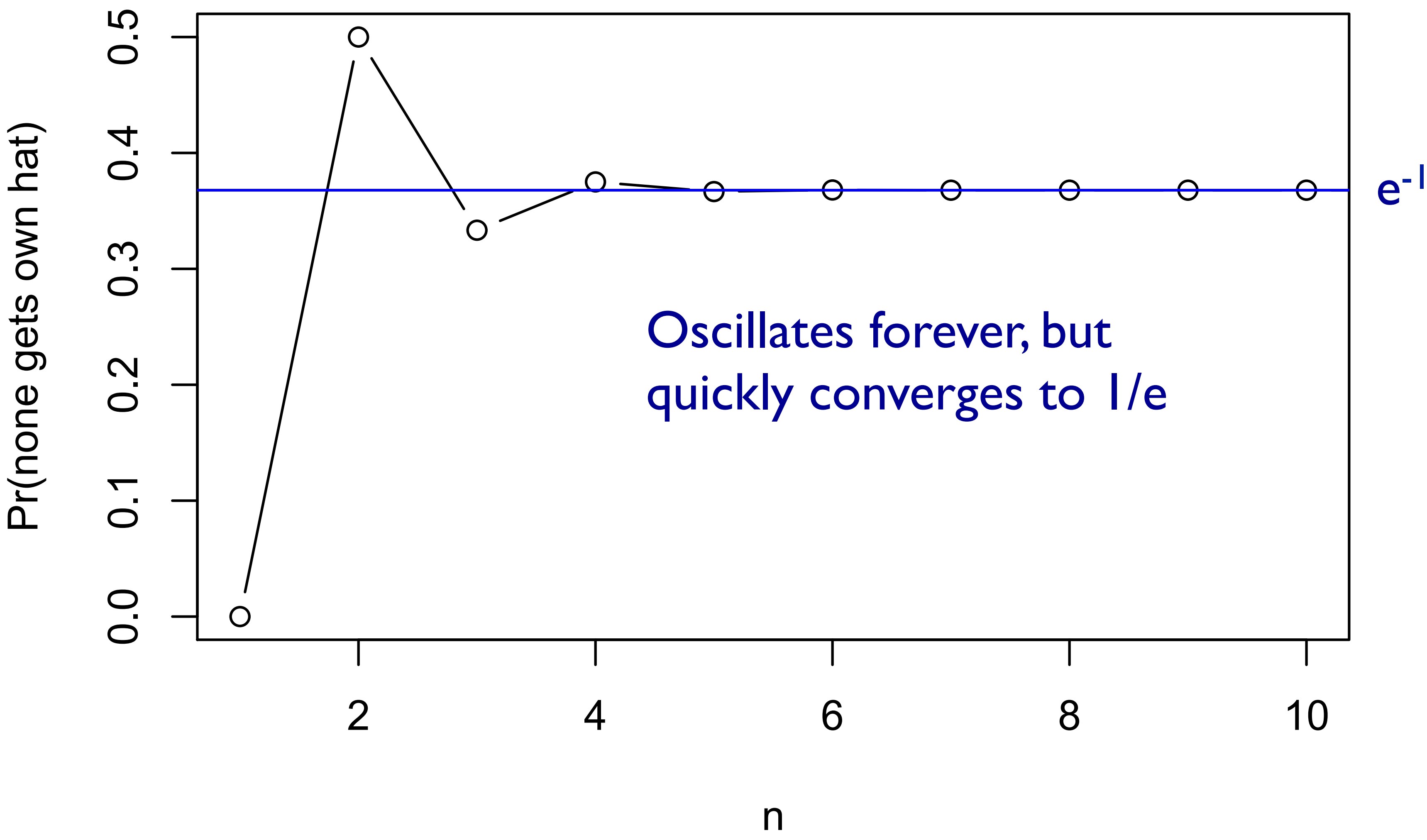
$\binom{n}{k}$ times that $= \dfrac{n!}{k!(n-k)!} \dfrac{(n-k)!}{n!} = 1/k!$

Pr(none get own) = 1-Pr(some do) =

$1 - 1/1! + 1/2! - 1/3! + 1/4! \ldots + (-1)^n/n! \approx 1/e \approx .37$

Pr(none get own) = 1 - Pr(some do) =

$1 - 1 + 1/2! - 1/3! + 1/4! \ldots + (-1)^n/n! \approx e^{-1} \approx .37$



Sample spaces
Events } Visualize!

Set theory

Axioms

Simple identities

Equally likely outcomes (counting)

Examples

- All good for building your skills
- Birthdays is particularly important for applications
- Hats is important as example of inclusion/exclusion