5. independence

Two events E and F are *independent* if

$P(EF) = P(E)\,P(F)$

equivalently: $P(E|F) = P(E)$

otherwise, they are called *dependent*

Roll two dice, yielding values D_1 and D_2

$E = \{ D_1 = 1 \}$

$F = \{ D_2 = 1 \}$

$P(E) = 1/6$, $P(F) = 1/6$, $P(EF) = 1/36$

$P(EF) = P(E) \cdot P(F) \Rightarrow$ E and F *independent*

$G = \{D_1 + D_2 = 5\} = \{(1,4),(2,3),(3,2),(4,1)\}$

$P(E) = 1/6$, $P(G) = 4/36 = 1/9$, $P(EG) = 1/36$

not independent!

E, G *dependent* events

Two events E and F are *independent* if

$P(EF) = P(E)\,P(F)$

equivalently: $P(E|F) = P(E)$

otherwise, they are called *dependent*

Three events E, F, G are independent if

$P(EF) = P(E)P(F)$, $P(EG) = P(E)P(G)$, $P(FG) = P(F)P(G)$

and $P(EFG) = P(E)\,P(F)\,P(G)$

Example: Let X, Y be each $\{-1, 1\}$ with equal prob

$E = \{X = 1\}$, $F = \{Y = 1\}$, $G = \{ XY = 1\}$

$P(EF) = P(E)P(F)$, $P(EG) = P(E)P(G)$, $P(FG) = P(F)P(G)$

but *P(EFG) = 1/4* !!! (because $P(G|EF) = 1$)

In general, events $E_1, E_2, \ldots, E_n$ are independent if for every subset S of {1,2,…, n}, we have

$$P\left(\bigcap_{i\in S} E_i\right) = \prod_{i\in S} P(E_i)$$

(Sometimes this property holds only for small subsets S. E.g., E,F,G on the previous slide are *pairwise* independent, but not fully independent.)

Theorem: E, F independent $\Rightarrow$ E, F^c independent

Proof:
$$
\begin{aligned}
P(EF^c) &= P(E) - P(EF) \\
&= P(E) - P(E)\,P(F) \\
&= P(E)\,(1 - P(F)) \\
&= P(E)\,P(F^c)
\end{aligned}
$$

Theorem:

E, F independent $\Leftrightarrow$ $P(E|F)=P(E)$ $\Leftrightarrow$ $P(F|E) = P(F)$

Proof: Note $P(EF) = P(E|F)\,P(F)$, regardless of in/dep.
Assume independent. Then

$$P(E)P(F) = P(EF) = P(E|F)\,P(F) \Rightarrow P(E|F)=P(E) \quad (\div \text{ by } P(F))$$

Conversely, $P(E|F)=P(E) \Rightarrow P(E)P(F) = P(EF)$ $(\times$ by $P(F))$

Biased coin comes up heads with probability p.

P(heads on n flips)

$= p^n$

P(tails on n flips)

$= (1-p)^n$

P(exactly k heads in n flips)

$$= \binom{n}{k} p^k (1-p)^{n-k}$$

Aside: note that the probability of *some* number of heads = $\sum_k \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1$ as it should, by the binomial theorem.

m strings hashed (uniformly) into a table with n buckets

Each string hashed is an *independent* trial

E = at least one string hashed to first bucket

What is P(E) ?

Solution:

F_i = string i *not* hashed into first bucket (i=1,2,…,m)

$P(F_i) = 1 - 1/n = (n-1)/n$ for all $i=1,2,\ldots,m$

Event $(F_1\ F_2\ \ldots\ F_m)$ = no strings hashed to first bucket

$$
\begin{aligned}
P(E) &= 1 - P(F_1\ F_2 \cdots F_m) \\
&= 1 - P(F_1)\ P(F_2) \cdots P(F_m) \qquad \text{indp} \\
&= 1 - ((n-1)/n)^m
\end{aligned}
$$

m strings hashed (non-uniformly) to table w/ n buckets

Each string hashed is an *independent* trial, with probability p_i of getting hashed to bucket i

E = At least 1 of buckets 1 to k gets $\geq$ 1 string

What is P(E) ?

Solution:

F_i = at least one string hashed into i-th bucket

$$P(E) = P(F_1 \cup \cdots \cup F_k) = 1 - P((F_1 \cup \cdots \cup F_k)^c)$$

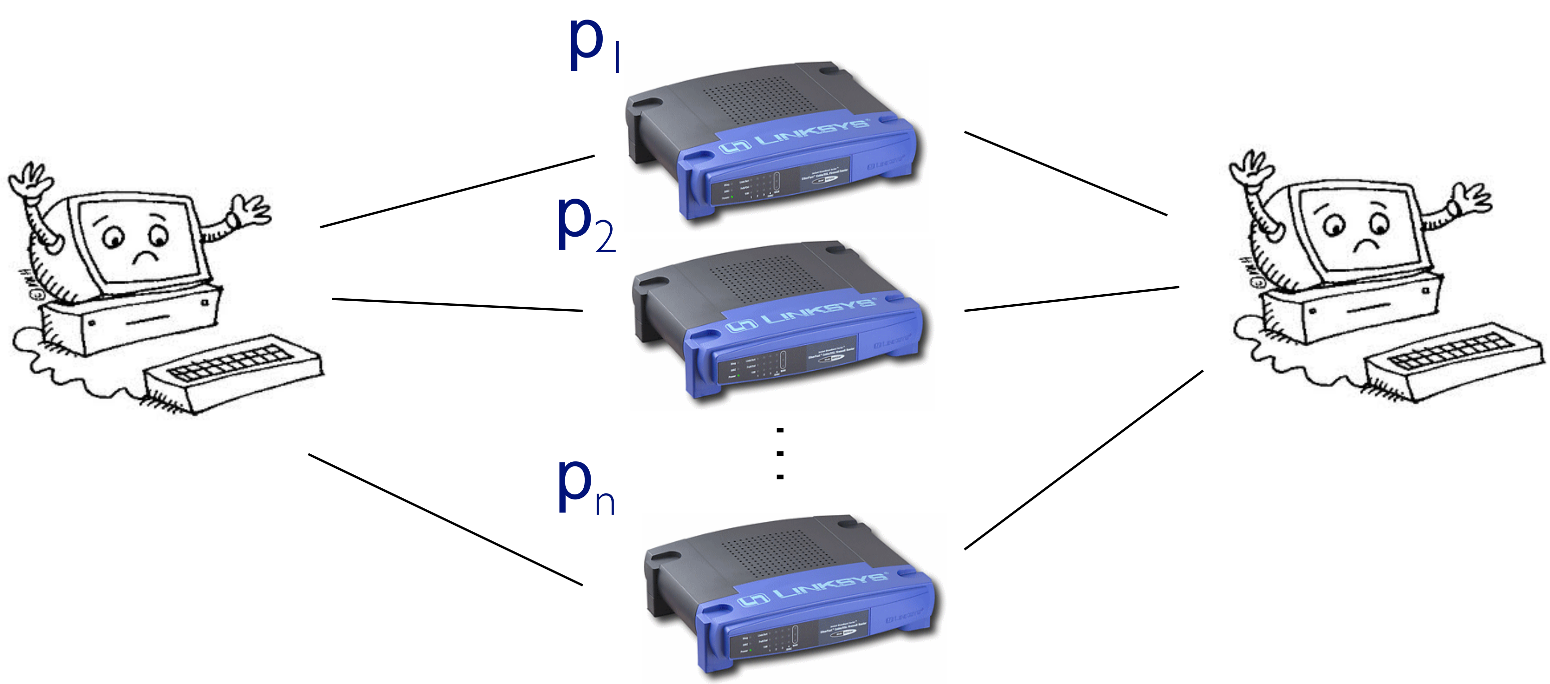
$$= 1 - P(F_1^c F_2^c \ldots F_k^c)$$

$$= 1 - P(\text{no strings hashed to buckets 1 to k})$$

$$= 1 - (1 - p_1 - p_2 - \cdots - p_k)^m$$

network failure

Consider the following parallel network

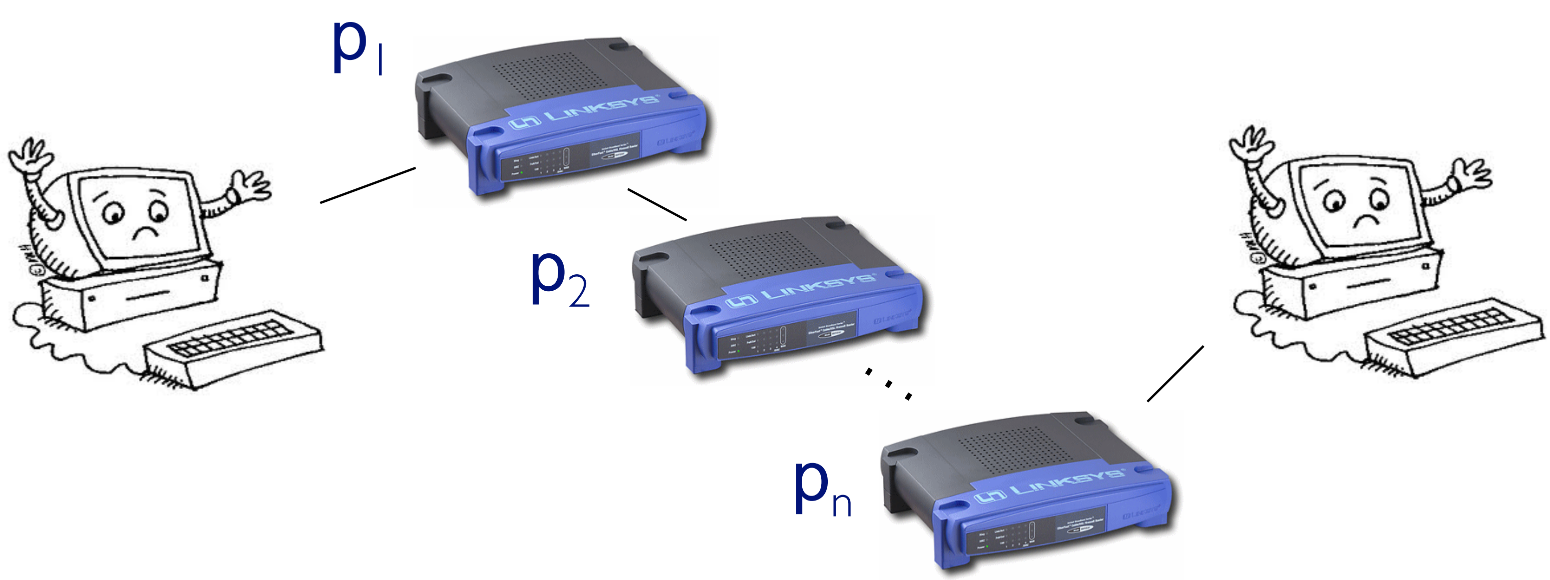


n routers, i^{th} has probability p_i of failing, independently

P(there is functional path) = 1 – P(all routers fail)

$$= 1 - p_1 p_2 \cdots p_n$$

Contrast: a *series* network



n routers, i^{th} has probability p_i of failing, independently

P(there is functional path) =

P(*no* routers fail) = $(1 - p_1)(1 - p_2) \cdots (1 - p_n)$

Recall: Two events E and F are independent if

$P(EF) = P(E)\,P(F)$

If E & F are independent, does that tell us anything about

$P(EF|G), P(E|G), P(F|G),$

when G is an arbitrary event? In particular, is

$P(EF|G) = P(E|G)\,P(F|G)$?

In general, *no*.

Roll two 6-sided dice, yielding values D_1 and D_2

$E = \{ D_1 = 1 \}$

$F = \{ D_2 = 6 \}$

$G = \{ D_1 + D_2 = 7 \}$

E and F are independent

$P(E|G) = 1/6$

$P(F|G) = 1/6$, but

$P(EF|G) = 1/6$, *not* $1/36$

so E|G and F|G are not independent!

Two events E and F are called *conditionally independent given G*, if

$$P(EF|G) = P(E|G)\,P(F|G)$$

Or, equivalently,

$$P(E|FG) = P(E|G)$$

do CSE majors get fewer A's?

Say you are in a dorm with 100 students

10 are CS majors: $P(CS) = 0.1$

30 get straight A's: $P(A) = 0.3$

3 are CS majors who get straight A's

$P(CS,A) = 0.03$

$P(CS,A) = P(CS)\,P(A)$, so CS and A *independent*

At faculty night, only CS majors and A students show up

So 37 students arrive

Of 37 students, 10 are CS ⇒

$P(CS \mid CS \text{ or } A) = 10/37 = 0.27 < .3 = P(A)$

Seems CS major *lowers* your chance of straight A's ☹

Weren't they supposed to be independent?

In fact, CS and A are *conditionally dependent* at fac night

Say you have a lawn
- It gets watered by rain or sprinklers
- These two events are independent
- You come outside and the grass is wet.
  - You know that the sprinklers were on
  - Does that lower the probability that it rained?

This is a phenomenon is called "explaining away" –
- One cause of an observation makes another cause less likely

Only CS majors and A students come to faculty night
- Knowing you came because you're a CS major makes it less likely you came because you get straight A's

conditioning can also break DEPENDENCE

Randomly choose a day of the week

A = { It is not a Monday }

B = { It is a Saturday }

C = { It is the weekend }

A and B are dependent events

$P(A) = 6/7$, $P(B) = 1/7$, $P(AB) = 1/7$.

Now condition both A and B on C:

$P(A|C) = 1$, $P(B|C) = ½$, $P(AB|C) = ½$

$P(AB|C) = P(A|C)\,P(B|C) \Rightarrow A|C$ and $B|C$ independent

Dependent events can become independent by conditioning on additional information!

gamblers ruin

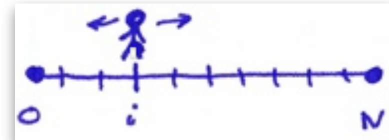
Ross 3.4, ex 41

2 Gamblers: Alice & Bob.

A has i dollars; B has (N-i)

Flip a coin. Heads – A wins \$1; Tails – B wins \$1

Repeat until A or B has all N dollars



aka "Drunkard's Walk"

$\boxed{\text{What is P(A wins)?}}$

Let E_i = event that A wins starting with \$i

Approach: Condition on outcome of 1st flip; H = heads

nice example of the utility of conditioning: future decomposed into two crisp cases instead of being a blurred superposition thereof

$$p_i = P(E_i) = P(E_i \mid H)\,P(H) + P(E_i \mid \bar{H})\,P(\bar{H})$$

$$p_i = \tfrac{1}{2}(p_{i+1} + p_{i-1})$$

$$2p_i = p_{i+1} + p_{i-1}$$

$$p_{i+1} - p_i = p_i - p_{i-1}$$

$$p_2 - p_1 = p_1 - p_0 = p_1 \quad \text{since } p_0 = 0$$

$$\text{so } p_2 = 2p_1$$

$$\vdots$$

$$p_i = i\,p_1$$

$$p_N = N p_1 = 1$$

$$\text{So } p_i = i/N$$

See book for more

Ross 3.5

Theorem

Let S be any sample Space and F be any event in S with $P(F) \neq 0$. Then "$P(- \mid F)$", conditional Probabilities given F, satisfy the axioms of probability

(a) $0 \leq P(E \mid F) \leq 1$

(b) $P(S \mid F) = 1$

(c) If E_i are mutually exclusive, then

$$P\left(\bigcup_i E_i \mid F\right) = \sum_i P(E_i \mid F)$$

Proof: See book (some algebra + some set theory) but the idea is very simple: Every event of interest is "$- \cap F$", so just as if S shrinks to F.

Child is born with (A,a) gene pair (event $B_{A,a}$)

Mother has (A,A) gene pair

Two possible fathers: M_1 = (a,a), M_2 = (a,A)

$P(M_1) = p$, $P(M_2) = 1-p$

What is $P(M_1 \mid B_{A,a})$?

All terms implicitly conditioned on the observed genotypes AA, Aa, ...

Solution:

$$P(M_1 \mid B_{Aa})$$

$$= \frac{P(B_{Aa} \mid M_1)P(M_1)}{P(B_{Aa} \mid M_1)P(M_1) + P(B_{Aa} \mid M_2)P(M_2)}$$

$$= \frac{1 \cdot p}{1 \cdot p + 0.5(1-p)} = \frac{2p}{1+p} > \frac{2p}{1+1} = p$$

independence: summary

Events E & F are *independent* if

P(EF) = P(E) P(F), or, equivalently P(E|F) = P(E)

More than 2 events are indp if, for *all subsets*, joint probability = product of separate event probabilities

Independence can greatly simplify calculations

For fixed G, conditioning on G gives a probability measure, P(E|G)

But "conditioning" and "independence" are orthogonal:

- Events E & F that are (unconditionally) independent may become dependent when conditioned on G
- Events that are (unconditionally) dependent may become independent when conditioned on G