CSE 312
Foundations II:
2. Counting

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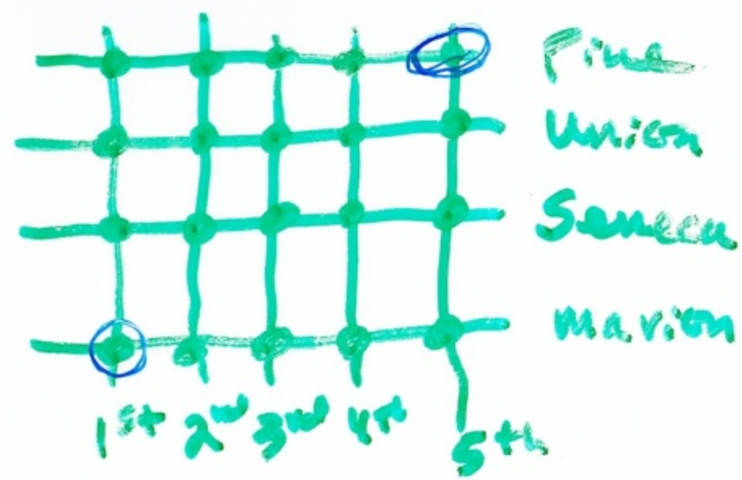
W.L. Ruzzo

Counting - As easy as 1, 2, 3, ... ?

How many ways are there to do X ?

E.g. X = "chose an int 1,...,10"

E.g. X = "Get from 1st & Marion to 5th & Pine going N, E"



The Point:

Counting gets hard when numbers are big and/or constraints are complex.

Systematic approaches help.

"The basic principle of Counting"

$\boxed{\text{n outcomes of A followed by m outcomes of B} \rightarrow n \cdot m \text{ outcomes of A+B}}$

aka "The Product Rule"

Easily generalized to more events

Example

Q. How many n-bit numbers are there

A. $\overbrace{2 \cdot 2 \cdot \cdots 2}^{n} = 2^n$

Example

Q. How many 4-character passwords are there, if each char is a...z, 0..9

A: $36 \cdot 36 \cdot 36 \cdot 36 \cong 1.7$ million

Q. Ditto, but no char may be repeated

A: $36 \cdot 35 \cdot 34 \cdot 33 \cong 1.4$ million

Permutations

How many arrangements of 1, 2, 3 are possible (each used once, no repeats)

123   231

132   312

213   321

More generally

n choices for 1st

$(n-1)$ " " 2nd

$(n-2)$ " " 3rd

⋮

1 choice for last

$$n\cdot(n-1)\cdot(n-2)\cdots 2\cdot 1 = n! \text{ (factorial)}$$

Example

Q. How many permutation of DOGIE are there?

A. $5! = 120$

Q. How many permutations of DOGGY are there?

A. $5!/2 = 60$

$DOG_1G_2Y \doteq DOG_2G_1Y$

$ODG_1YG_2 \doteq ODG_2YG_1$

$\vdots$

Q. ... GODOGGY

A: $\dfrac{7!}{2!\cdot 3!\cdot 1!\cdot 1!}$

Combinations

Your elf-lord avatar can carry

3 objects chosen from

1. Sword
2. Knife
3. magic belt
4. water jug
5. iPad w/ magic WiFi

How many ways can you equip him/her?

A: $\dfrac{5 \cdot 4 \cdot 3}{3!} = \dfrac{5!}{3! \cdot 2!}$

Combinations: r things chosen from n

$$\binom{n}{r} \quad \text{"n choose r"} \quad = \frac{n!}{r!(n-r!)}$$

aka binomial coefficients

Important special case:

how many (unordered) pairs of n objects

$$\binom{n}{2} = \frac{n(n-1)}{2} = \Theta(n^2)$$

Many Identities

$$\binom{n}{r} = \binom{n}{n-r}$$ ← symmetry of defn

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$ ← "obj #1" is in or out

The Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

proof 1: induction

proof 2: $(x+y)\cdot(x+y)\cdot \ldots \cdot(x+y)$

"pick either x or y from 1st binomial"
" . . . . 2nd . . . "
" . . . . . nth . . . "

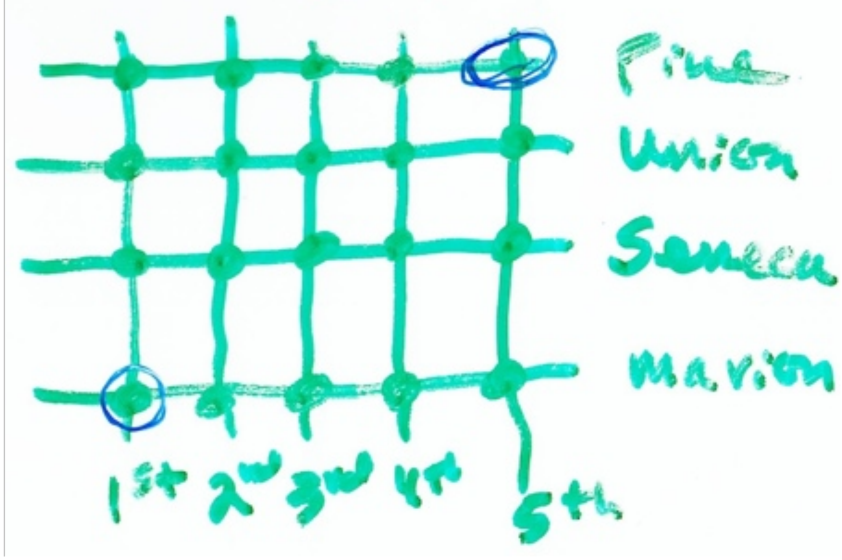
in all possible ways. How many ways to get exactly k x's? : $\binom{n}{k}$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof: $= (1+1)^n$

How many ways are there to

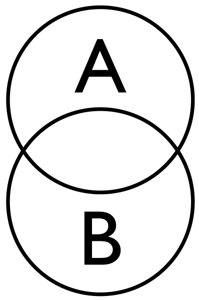
"Get from 1st & Marion to 5th & Pine going N, E"

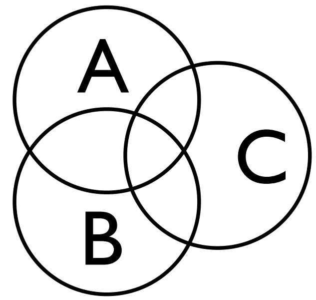


A: 7 choose 3 = 35:

you walk 7 blocks; at each intersection choose N or E; must choose N exactly 3 times.

Inclusion-Exclusion





$|A \cup B| =$

$|A|+|B|-|A \cap B|$

$|A \cup B \cup C| =$

$|A| + |B| + |C|$

$-|A \cap B|-|A \cap C|-|B \cap C|$

$+ |A \cap B \cap C|$

General: +singles - pairs + triples - quads + ...

pigeonhole principle

pigeonhole principle

If there are *n* pigeons in *k* holes and *n* > *k*, then some hole contains more than one pigeon.
More precisely, some hole contains at least $\lceil n/k \rceil$ pigeons.

There are two people in London who have the same number of hairs on their head.
- Typical head ~ 150,000 hairs
- Let's say max-harry-head ~ 1,000,000 hairs
- Since there are more than 1,000,000 people in London…