6. random variables

T T T T H T H H

let X = index of

A *random variable* is some (usually numeric) *function* of the outcome, not in the outcome itself.

Ex.

Let H be the number of Heads when 20 coins are tossed

Let T be the total of 2 dice rolls

Let X be the number of coin tosses needed to see 1^{st} head

Note; even if the underlying experiment has "equally likely outcomes," the associated random variable may not

Outcome	*H*	*P(H)*
TT	0	P(H=0) = 1/4
TH	1	} P(H=1) = 1/2
HT	1	
HH	2	P(H=2) = 1/4

20 balls numbered 1, 2, ..., 20

Draw 3 without replacement

Let X = the maximum of the numbers on those 3 balls

What is P(X ≥ 17)

$$
\begin{aligned}
P(X = 20) &= \binom{19}{2} \Big/ \binom{20}{3} = \frac{3}{20} = 0.150 \\
P(X = 19) &= \binom{18}{2} \Big/ \binom{20}{3} = \frac{18 \cdot 17/2!}{20 \cdot 19 \cdot 18/3!} \approx 0.134 \\
&\vdots
\end{aligned}
$$

$$\sum_{i=17}^{20} P(X = i) \approx 0.508$$

Alternatively:

$$P(X \geq 17) = 1 - P(X < 17) = 1 - \binom{16}{3} \Big/ \binom{20}{3} \approx 0.508$$

Flip a (biased) coin repeatedly until 1st head observed

How many flips? Let X be that number.

P(X=1) = P(H) = p

P(X=2) = P(TH) = (1-p)p

P(X=3) = P(TTH) = $(1-p)^2p$

...

Check that it is a valid probability distribution:

$$P\left(\bigcup_{i\geq 1}\{X=i\}\right)=\sum_{i\geq 1}(1-p)^{i-1}p=p\sum_{i\geq 0}(1-p)^i=p\frac{1}{1-(1-p)}=1$$

A *discrete* random variable is one taking on a countable number of possible values.

Ex:

X = sum of 3 dice, $3 \le X \le 18$, $X \in N$

Y = number of 1st head in seq of coin flips, $1 \le Y$, $Y \in N$

Z = largest prime factor of (1+Y), $Z \in \{2, 3, 5, 7, 11, \ldots\}$

If X is a discrete random variable taking on values from a countable set $T \subseteq R$, then

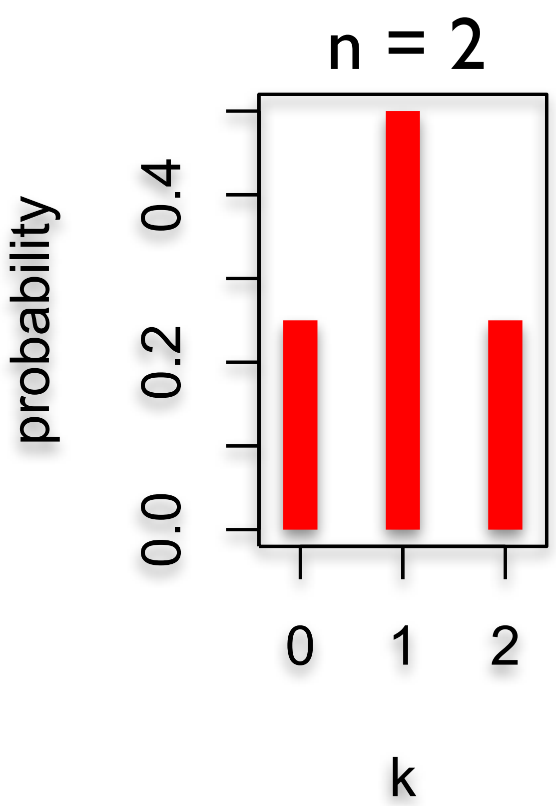
$$p(a) = \begin{cases} P(X = a) & \text{for } a \in T \\ 0 & \text{otherwise} \end{cases}$$

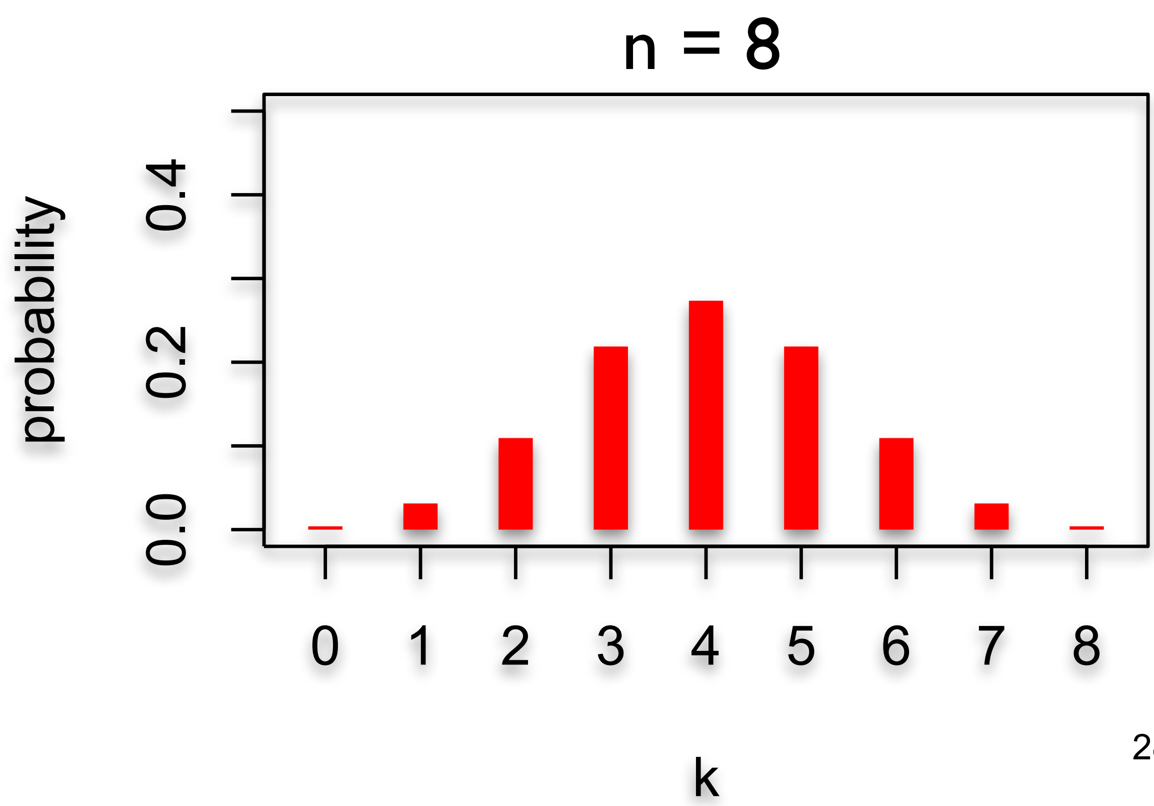
is called the *probability mass function*. Note: $\sum_{a \in T} p(a) = 1$

Let X be the number of heads observed in n coin flips

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \text{ where } p = P(H)$$

Probability mass function:





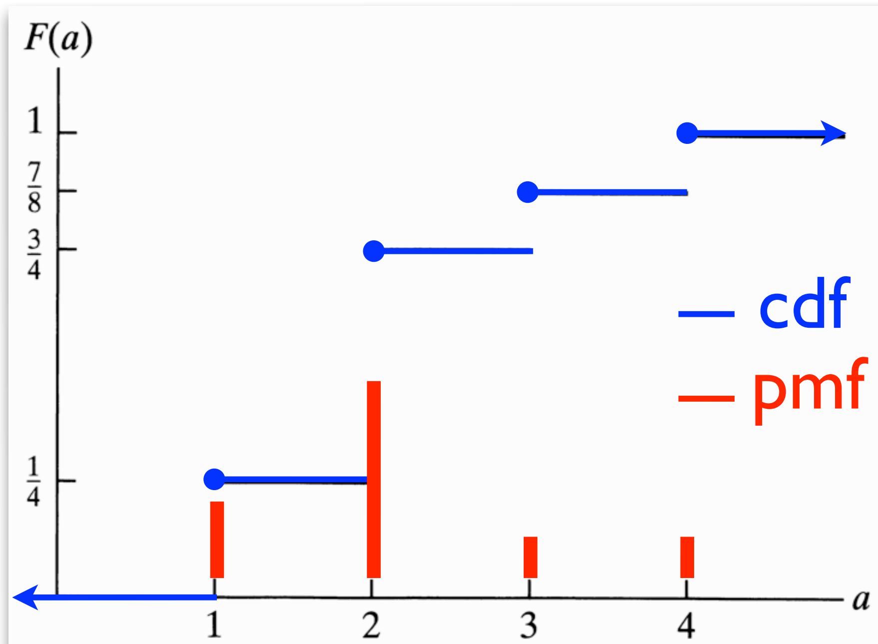
The *cumulative distribution function* for a random variable X is the function $F: \mathbb{R} \to [0,1]$ defined by

$$F(a) = P[X \le a]$$

Ex: if X has probability mass function given by:

$$p(1) = \frac{1}{4} \qquad p(2) = \frac{1}{2} \qquad p(3) = \frac{1}{8} \qquad p(4) = \frac{1}{8}$$

$$F(a) = \begin{cases} 0 & a < 1 \\ \frac{1}{4} & 1 \le a < 2 \\ \frac{3}{4} & 2 \le a < 3 \\ \frac{7}{8} & 3 \le a < 4 \\ 1 & 4 \le a \end{cases}$$



NB: for discrete random variables, be careful about "≤" vs "<"

For a discrete r.v. X with p.m.f. p(•), the *expectation of X*, aka *expected value* or *mean*, is

$$\boxed{E[X] = \Sigma_x \, xp(x)}$$

average of random values, weighted by their respective probabilities

For the equally-likely outcomes case, this is just the average of the possible random values of X

For *un*equally-likely outcomes, it is again the average of the possible random values of X, weighted by their respective probabilities

Ex 1: Let X = value seen rolling a fair die p(1), p(2), ..., p(6) = 1/6

$$E[X] = \sum_{i=1}^{6} ip(i) = \frac{1}{6}(1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5$$

Ex 2: Coin flip; X = +1 if H (win \$1), -1 if T (lose \$1)

E[X] = (+1)•p(+1) + (-1)•p(-1) = 1•(1/2) +(-1)•(1/2) = 0

For a discrete r.v. X with p.m.f. p(•), the *expectation of X*, aka *expected value* or *mean*, is

$$E[X] = \Sigma_x \, xp(x)$$

average of random values, weighted by their respective probabilities

Another view: A 2-person gambling game. If X is how much you win playing the game once, how much would you expect to win, on average, per game when repeatedly playing?

Ex 1: Let X = value seen rolling a fair die p(1), p(2), ..., p(6) = 1/6

If you win X dollars for that roll, how much do you expect to win?

$$E[X] = \sum_{i=1}^{6} ip(i) = \frac{1}{6}(1 + 2 + \cdots + 6) = \frac{21}{6} = 3.5$$

Ex 2: Coin flip; X = +1 if H (win \$1), -1 if T (lose \$1)

E[X] = (+1)•p(+1) + (-1)•p(-1) = 1•(1/2) +(-1)•(1/2) = 0

“a fair game”: in repeated play you expect to win as much as you lose. Long term net gain/loss = 0.

Let X be the number of flips up to & including 1st head observed in repeated flips of a biased coin. If I pay you $1 per flip, how much money would you expect to make?

$$\begin{aligned} P(H) &= p; \quad P(T) = 1-p = q \\ p(i) &= pq^{i-1} \\ E(x) &= \textstyle\sum_{i\geq 1} ip(i) = \sum_{i\geq 1} ipq^{i-1} = p\sum_{i\geq 1} iq^{i-1} \qquad (*) \end{aligned}$$

A calculus trick:

$$\sum_{i\geq 1} iy^{i-1} = \sum_{i\geq 1} \frac{d}{dy} y^i = \sum_{i\geq 0} \frac{d}{dy} y^i = \frac{d}{dy} \sum_{i\geq 0} y^i = \frac{d}{dy} \frac{1}{1-y} = \frac{1}{(1-y)^2}$$

$dy^0/dy = 0$

So $(*)$ becomes:

$$E[X] = p\sum_{i\geq i} iq^{i-1} = \frac{p}{(1-q)^2} = \frac{p}{p^2} = \frac{1}{p}$$

How much would you pay to play?

E.g.:

p=1/2; on average head every 2nd flip

p=1/10; on average, head every 10th flip.

expectation of a *function* of a random variable

Calculating E[g(X)]:

Y=g(X) is a new r.v. Calc P[Y=j], then apply defn:

X = sum of 2 dice rolls

i	p(i) = P[X=i]	i•p(i)
2	1/36	2/36
3	2/36	6/36
4	3/36	12/36
5	4/36	20/36
6	5/36	30/36
7	6/36	42/36
8	5/36	40/36
9	4/36	36/36
10	3/36	30/36
11	2/36	22/36
12	1/36	12/36

$E[X] = \Sigma_i\, ip(i) = \boxed{252/36} = 7$

Y = g(X) = X mod 5

j	q(j) = P[Y = j]	j•q(j)
0	4/36+3/36 =7/36	0/36
1	5/36+2/36 =7/36	7/36
2	1/36+6/36+1/36 =8/36	16/36
3	2/36+5/36 =7/36	21/36
4	3/36+4/36 =7/36	28/36

$E[Y] = \Sigma_j\, jq(j) = \boxed{72/36} = 2$

expectation of a *function* of a random variable

Calculating E[g(X)]: Another way – add in a different order, using P[X=...] instead of calculating P[Y=...]

X = sum of 2 dice rolls

i	p(i) = P[X=i]	g(i)•p(i)
2	1/36	2/36
3	2/36	6/36
4	3/36	12/36
5	4/36	0/36
6	5/36	5/36
7	6/36	12/36
8	5/36	15/36
9	4/36	16/36
10	3/36	0/36
11	2/36	2/36
12	1/36	2/36

$E[g(X)] = \Sigma_i\, g(i)p(i) = \boxed{72/36} = 2$

Y = g(X) = X mod 5

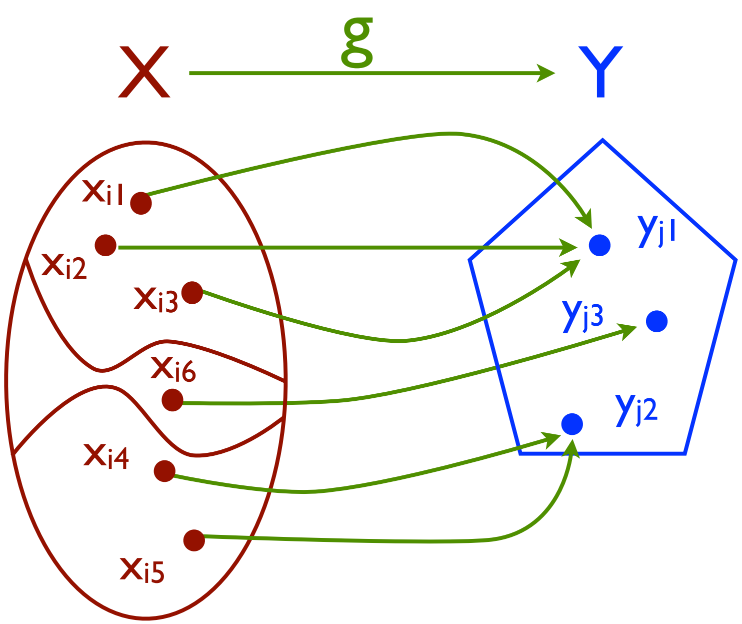
j	q(j) = P[Y = j]	j•q(j)
0	4/36+3/36 =7/36	0/36
1	5/36+2/36 =7/36	7/36
2	1/36+6/36+1/36 =8/36	16/36
3	2/36+5/36 =7/36	21/36
4	3/36+4/36 =7/36	28/36

$E[Y] = \Sigma_j\, jq(j) = \boxed{72/36} = 2$

Above example is not a fluke.

Theorem: if $Y = g(X)$, then $E[Y] = \Sigma_i\, g(x_i)p(x_i)$, where x_i, $i = 1, 2, \ldots$ are all possible values of X.

Proof: Let y_j, $j = 1, 2, \ldots$ be all possible values of Y.



Note that $S_j = \{ x_i \mid g(x_i)=y_j \}$ is a partition of the domain of g.

$$\begin{aligned}
\sum_i g(x_i)p(x_i) &= \sum_j \sum_{i:g(x_i)=y_j} g(x_i)p(x_i) \\
&= \sum_j \sum_{i:g(x_i)=y_j} y_j p(x_i) \\
&= \sum_j y_j \sum_{i:g(x_i)=y_j} p(x_i) \\
&= \sum_j y_j P\{g(X) = y_j\} \\
&= E[g(X)]
\end{aligned}$$

A & B each bet \$1, then flip 2 coins:

HH	A wins \$2
HT	Each takes back \$1
TH	
TT	B wins \$2

Let X be A's net gain: +1, 0, -1, resp.:

$P(X = +1) = 1/4$
$P(X = 0) = 1/2$
$P(X = -1) = 1/4$

What is E[X]?

$$E[X] = 1 \cdot 1/4 + 0 \cdot 1/2 + (-1) \cdot 1/4 = 0$$

What is $E[X^2]$?

$$E[X^2] = 1^2 \cdot 1/4 + 0^2 \cdot 1/2 + (-1)^2 \cdot 1/4 = 1/2$$

Note:
$E[X^2] \neq E[X]^2$

Linearity of expectation, I

For any constants a, b: $\boxed{E[aX + b] = aE[X] + b}$

Proof:

$$\begin{aligned} E[aX+b] &= \sum_x (ax+b)\cdot p(x) \\ &= a\sum_x xp(x) + b\sum_x p(x) \\ &= aE[X] + b \end{aligned}$$

Example:

Q: In the 2-person coin game above, what is E[2X+1]?

A: E[2X+1] = 2E[X]+1 = 2•0 + 1 = 1

Linearity, II

Let X and Y be two random variables derived from outcomes of a single experiment. Then

$$E[X+Y] = E[X] + E[Y]$$

True even if X,Y *dependent*

Proof: Assume the sample space S is countable. (The result is true without this assumption, but I won't prove it.) Let X(s), Y(s) be the values of these r.v.'s for outcome s∈S.

Claim: $E[X] = \sum_{s \in S} X(s) \cdot p(s)$

Proof: similar to that for "expectation of a function of an r.v.," i.e., the events "X=x" partition S, so sum above can be rearranged to match the definition of $E[X] = \sum_x x \cdot P(X = x)$

Then:

$$E[X+Y] = \Sigma_{s \in S}(X[s] + Y[s])\ p(s)$$
$$= \Sigma_{s \in S}X[s]\ p(s) + \Sigma_{s \in S}Y[s]\ p(s) = E[X] + E[Y]$$

Example

X = # of heads in *one* coin flip, where P(X=1) = p.

What is $E(X)$?

$E[X] = 1 \cdot p + 0 \cdot (1-p) = p$

Let X_i, $1 \leq i \leq n$, be # of H in flip of coin with $P(X_i=1) = p_i$

What is the expected number of heads when all are flipped?

$E[\Sigma_i X_i] = \Sigma_i E[X_i] = \Sigma_i p_i$

Special case: $p_1 = p_2 = ... = p$:

$E[\#\text{ of heads in } n \text{ flips}] = pn$

Note:

Linearity is special!

It is *not* true in general that

$$E[X \cdot Y] = E[X] \cdot E[Y]$$

$$E[X^2] = E[X]^2 \quad \leftarrow \text{counterexample above}$$

$$E[X/Y] = E[X] / E[Y]$$

$$E[\operatorname{asinh}(X)] = \operatorname{asinh}(E[X])$$

$$\vdots$$

Alice & Bob are gambling (again). X = Alice's gain per flip:

$$X = \begin{cases} +1 & \text{if Heads} \\ -1 & \text{if Tails} \end{cases}$$

E[X] = 0

. . . Time passes . . .

Alice (yawning) says "let's raise the stakes"

$$Y = \begin{cases} +1000 & \text{if Heads} \\ -1000 & \text{if Tails} \end{cases}$$

E[Y] = 0, as before.

Are you (Bob) equally happy to play the new game?

E[X] measures the "average" or "central tendency" of X.
What about its *variability*?

If $E[X] = \mu$, then $E[|x-\mu|]$ seems like a natural quantity to look at: how much do we expect X to deviate from its average. Unfortunately, it's a bit inconvenient mathematically; following is easier/more common.

Definition

The *variance* of a random variable X with mean $E[X] = \mu$ is $\mathrm{Var}[X] = E[(X-\mu)^2]$, often denoted σ^2.

The *standard deviation* of X is $\sigma = \sqrt{\mathrm{Var}[X]}$

The *variance* of a random variable X with mean $E[X] = \mu$ is $Var[X] = E[(X-\mu)^2]$, often denoted σ^2.

I:

Square always ≥ 0, and exaggerated as X moves away from μ, so Var[X] emphasizes *deviation* from the mean.

II:

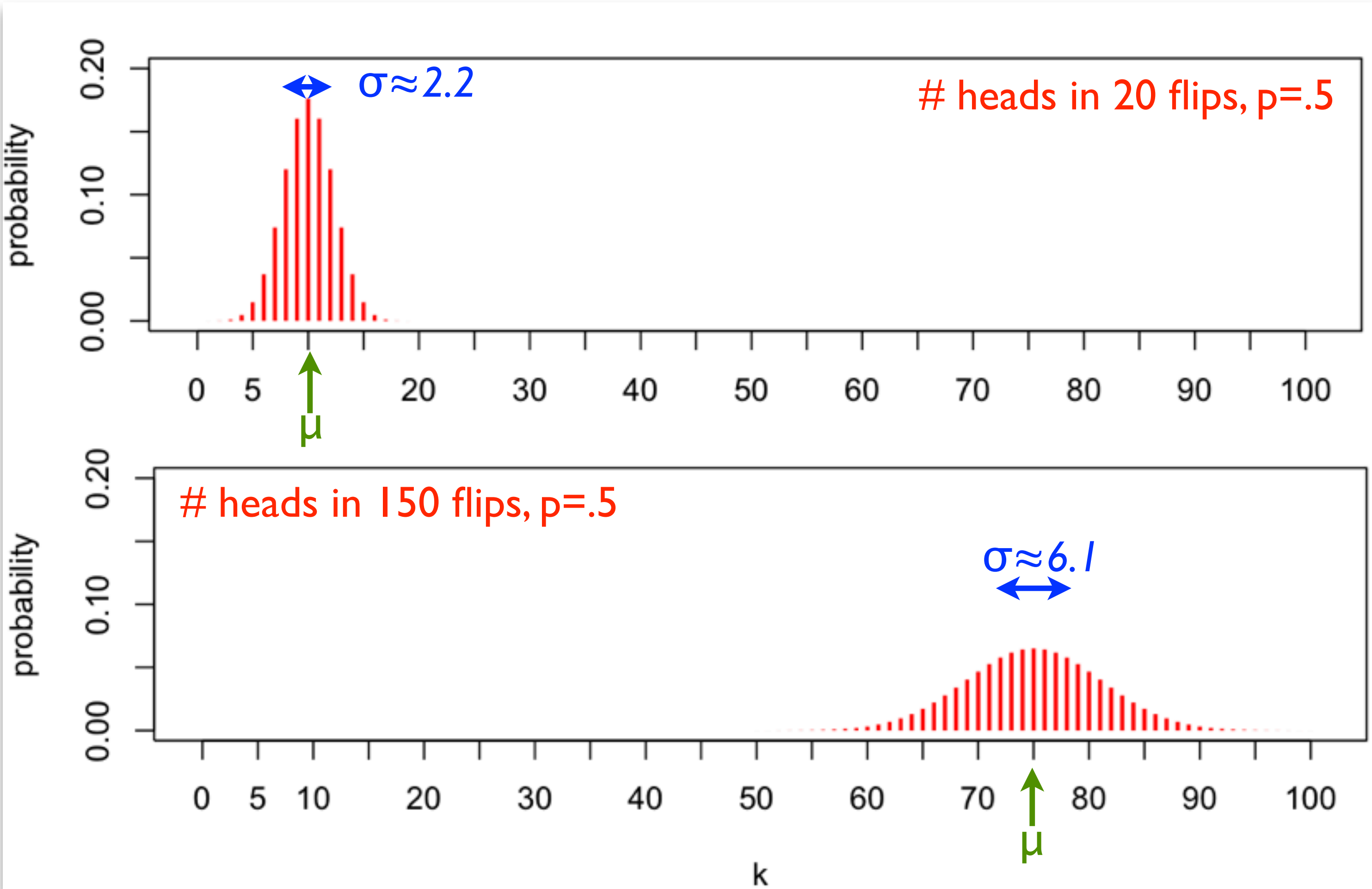
Numbers vary a lot depending on exact distribution of X, but typically X is

within $\mu \pm \sigma$ ~66% of the time, and

within $\mu \pm 2\sigma$ ~95% of the time.

(We'll see the reasons for this soon.)

$\mu = E[X]$ is about *location;* $\sigma = \sqrt{Var(X)}$ is about *spread*



Alice & Bob are gambling (again). X = Alice's gain per flip:

$$X = \begin{cases} +1 & \text{if Heads} \\ -1 & \text{if Tails} \end{cases}$$

E[X] = 0

Var[X] = 1

… Time passes …

Alice (yawning) says "let's raise the stakes"

$$Y = \begin{cases} +1000 & \text{if Heads} \\ -1000 & \text{if Tails} \end{cases}$$

E[Y] = 0, as before.

Var[Y] = 1,000,000

Are you (Bob) equally happy to play the new game?

Two games:

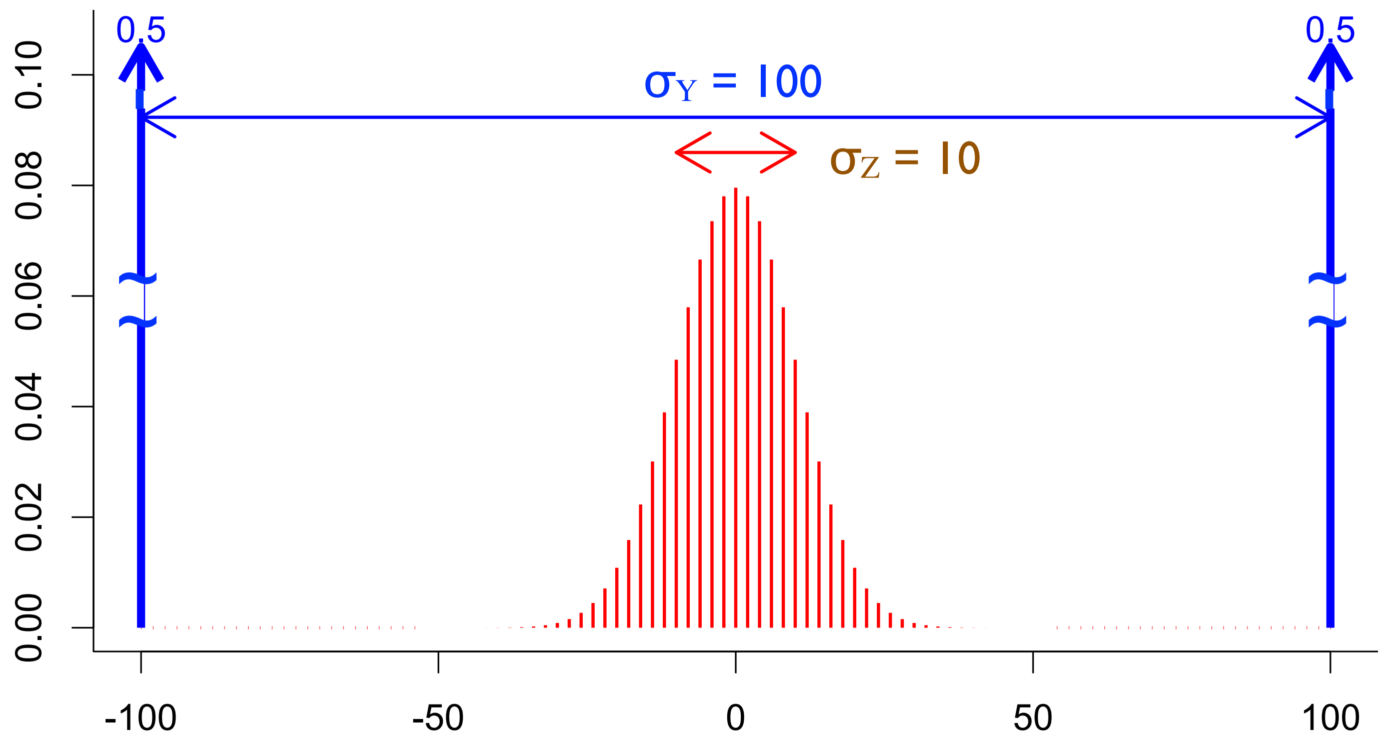
a) flip 1 coin, win Y = \$100 if heads, \$-100 if tails

b) flip 100 coins, win Z = (#(heads) - #(tails)) dollars

Same expectation in both: E[Y] = E[Z] = 0

Same extremes in both: max gain = \$100; max loss = \$100

But *variability* is very different:



$$\boxed{\mathrm{Var}(X) = E[X^2] - (E[X])^2}$$

$$\begin{aligned}
\mathrm{Var}(X) &= E[(X - \mu)^2] \\
&= \sum_x (x - \mu)^2 p(x) \\
&= \sum_x (x^2 - 2\mu x + \mu^2) p(x) \\
&= \sum_x x^2 p(x) - 2\mu \sum_x x p(x) + \mu^2 \sum_x p(x) \\
&= E[X^2] - 2\mu^2 + \mu^2 \\
&= E[X^2] - \mu^2
\end{aligned}$$

Example:

What is Var[X] when X is outcome of one fair die?

$$E[X^2] = 1^2\left(\frac{1}{6}\right) + 2^2\left(\frac{1}{6}\right) + 3^2\left(\frac{1}{6}\right) + 4^2\left(\frac{1}{6}\right) + 5^2\left(\frac{1}{6}\right) + 6^2\left(\frac{1}{6}\right)$$
$$= \left(\frac{1}{6}\right)(91)$$

E[X] = 7/2, so

$$\mathrm{Var}(X) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$$

$$\boxed{\mathrm{Var}[aX+b] = a^2\,\mathrm{Var}[X]}$$

$$\begin{aligned}\mathrm{Var}(aX + b) &= E[(aX + b - a\mu - b)^2] \\ &= E[a^2(X - \mu)^2] \\ &= a^2E[(X - \mu)^2] \\ &= a^2\mathrm{Var}(X)\end{aligned}$$

Ex:

$$X = \begin{cases} +1 & \text{if Heads} \\ -1 & \text{if Tails} \end{cases}$$

$E[X] = 0$

$\mathrm{Var}[X] = 1$

$$Y = \begin{cases} +1000 & \text{if Heads} \\ -1000 & \text{if Tails} \end{cases}$$

$Y = 1000\,X$

$E[Y] = E[1000\,X] = 1000\,E[x] = 0$

$\mathrm{Var}[Y] = \mathrm{Var}[1000\,X]$

$= 10^6\mathrm{Var}[X] = 10^6$

In general:

$$\mathrm{Var}[X+Y] \neq \mathrm{Var}[X] + \mathrm{Var}[Y]$$

Ex 1:

Let $X = \pm 1$ based on 1 coin flip

As shown above, $E[X] = 0, \mathrm{Var}[X] = 1$

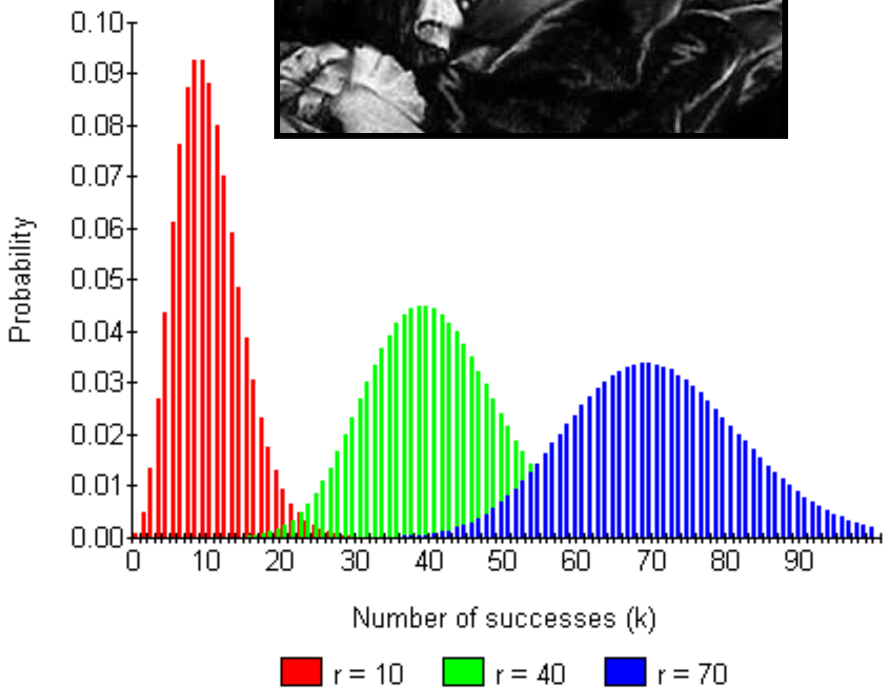
Let $Y = -X$; then $\mathrm{Var}[Y] = (-1)^2\mathrm{Var}[X] = 1$

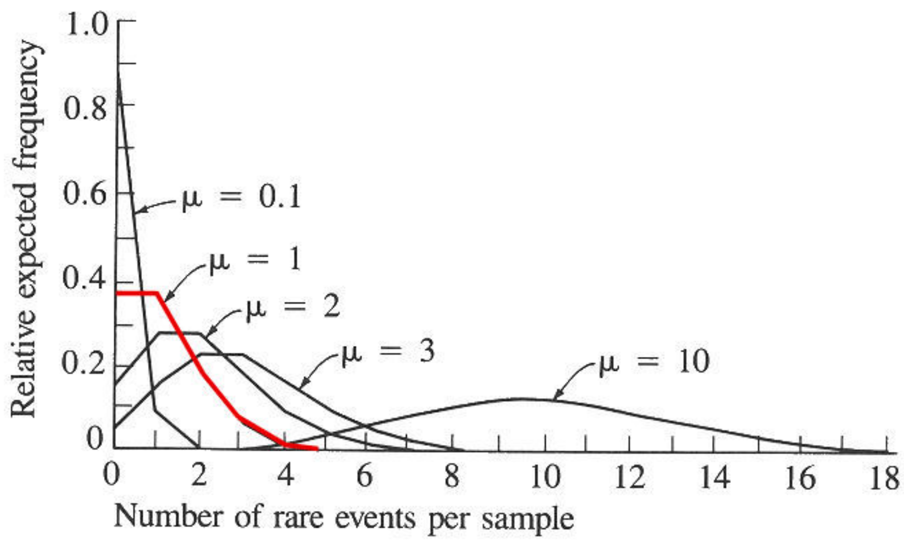
But $X+Y = 0$, always, so $\mathrm{Var}[X+Y] = 0$

Ex 2:

As another example, is $\mathrm{Var}[X+X] = 2\mathrm{Var}[X]$?

a zoo of (discrete) random variables





An experiment results in "Success" or "Failure"

X is a random *indicator variable* (1=success, 0=failure)

$P(X=1) = p$ and $P(X=0) = 1-p$

X is called a *Bernoulli* random variable: $X \sim Ber(p)$

$E[X] = E[X^2] = p$

$Var(X) = E[X^2] - (E[X])^2 = p - p^2 = p(1-p)$

Examples:

coin flip

random binary digit

whether a disk drive crashed

Jacob (aka James, Jacques) Bernoulli, 1654 – 1705

Consider n independent random variables $Y_i \sim \text{Ber}(p)$

- $X = \Sigma_i Y_i$ is the number of successes in n trials
- X is a *Binomial* random variable: $X \sim \text{Bin}(n,p)$

$$P(X = i) = \binom{n}{i} p^i (1-p)^{n-i} \quad i = 0, 1, \ldots, n$$

By Binomial theorem, $\sum_{i=0}^{n} P(X = i) = 1$

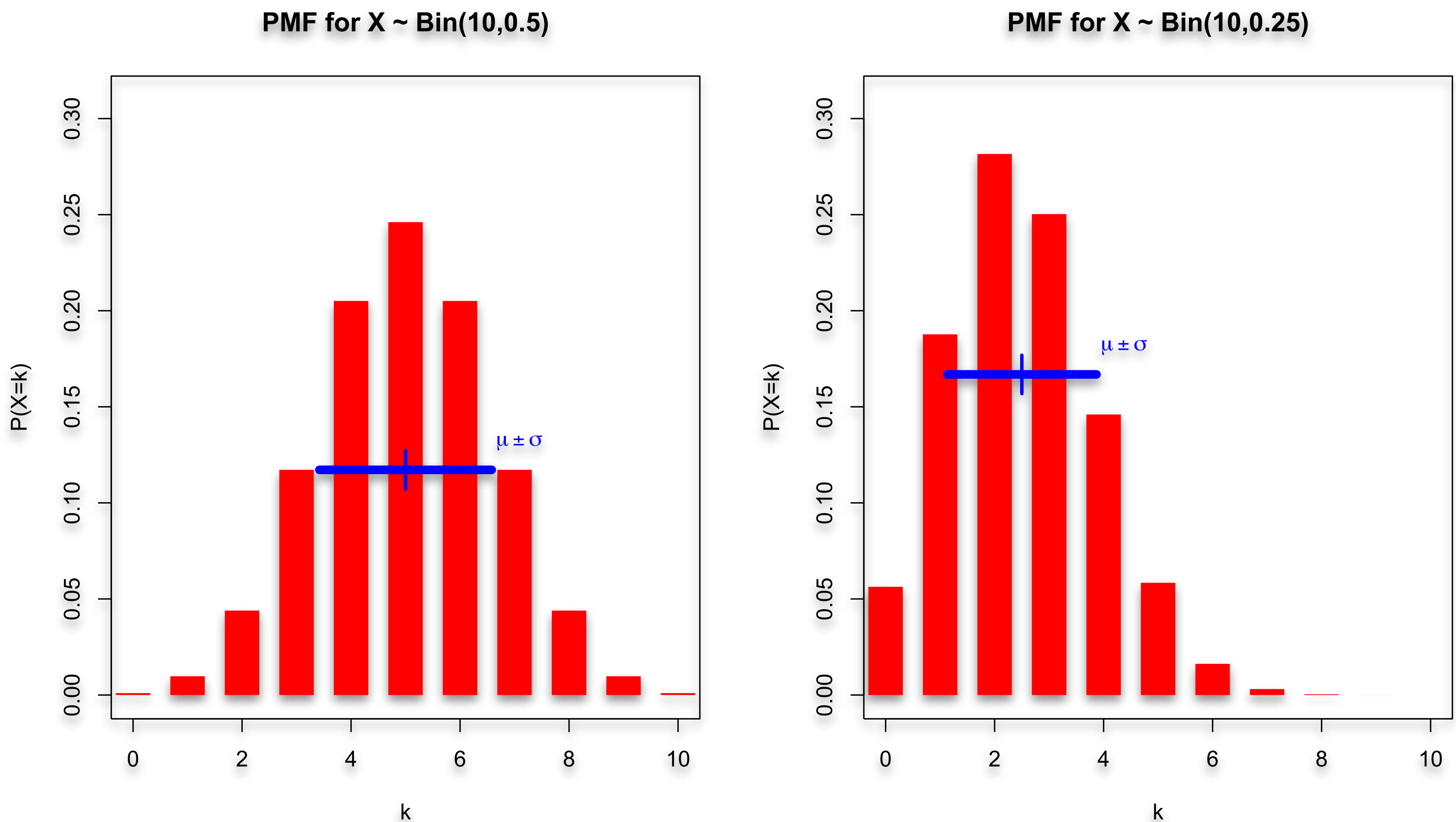
Examples

- # of heads in n coin flips
- # of 1's in a randomly generated length n bit string
- # of disk drive crashes in a 1000 computer cluster

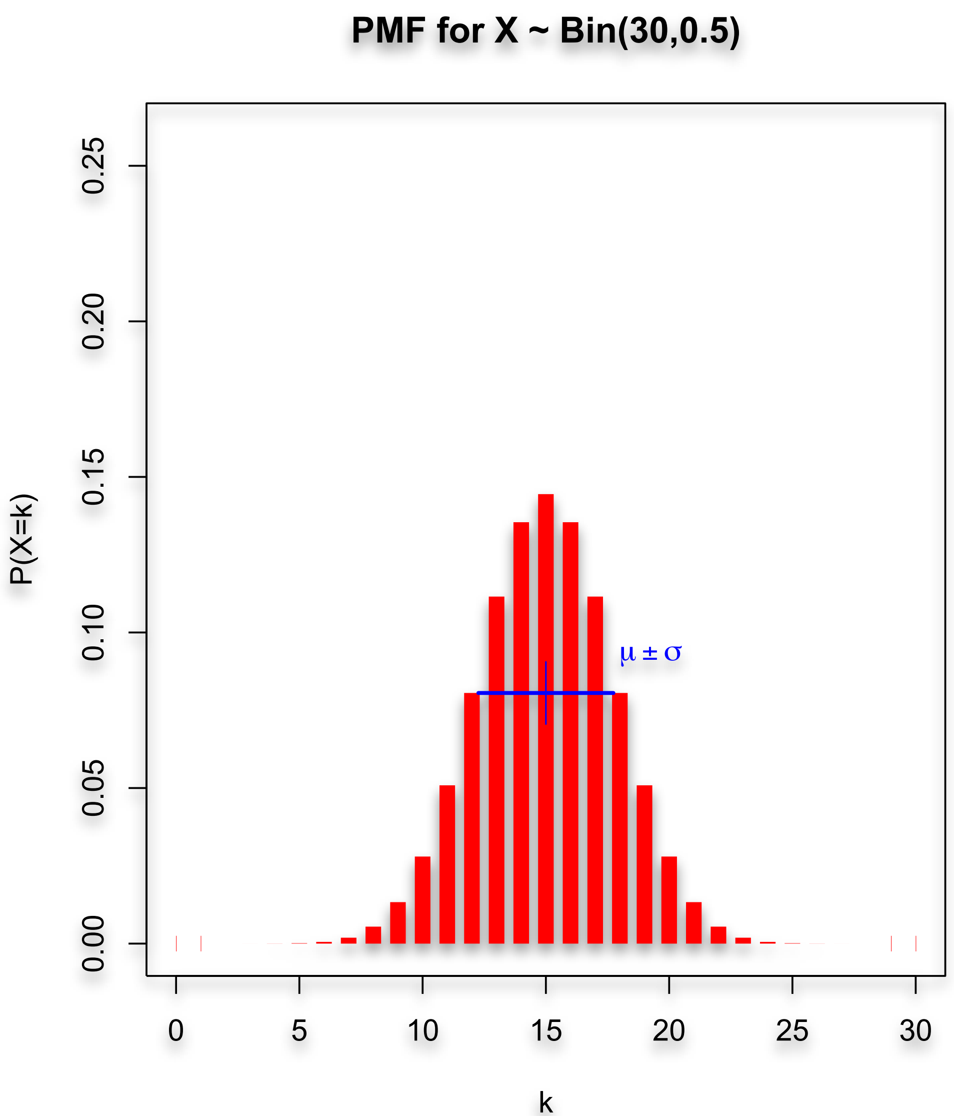
$E[X] = pn$

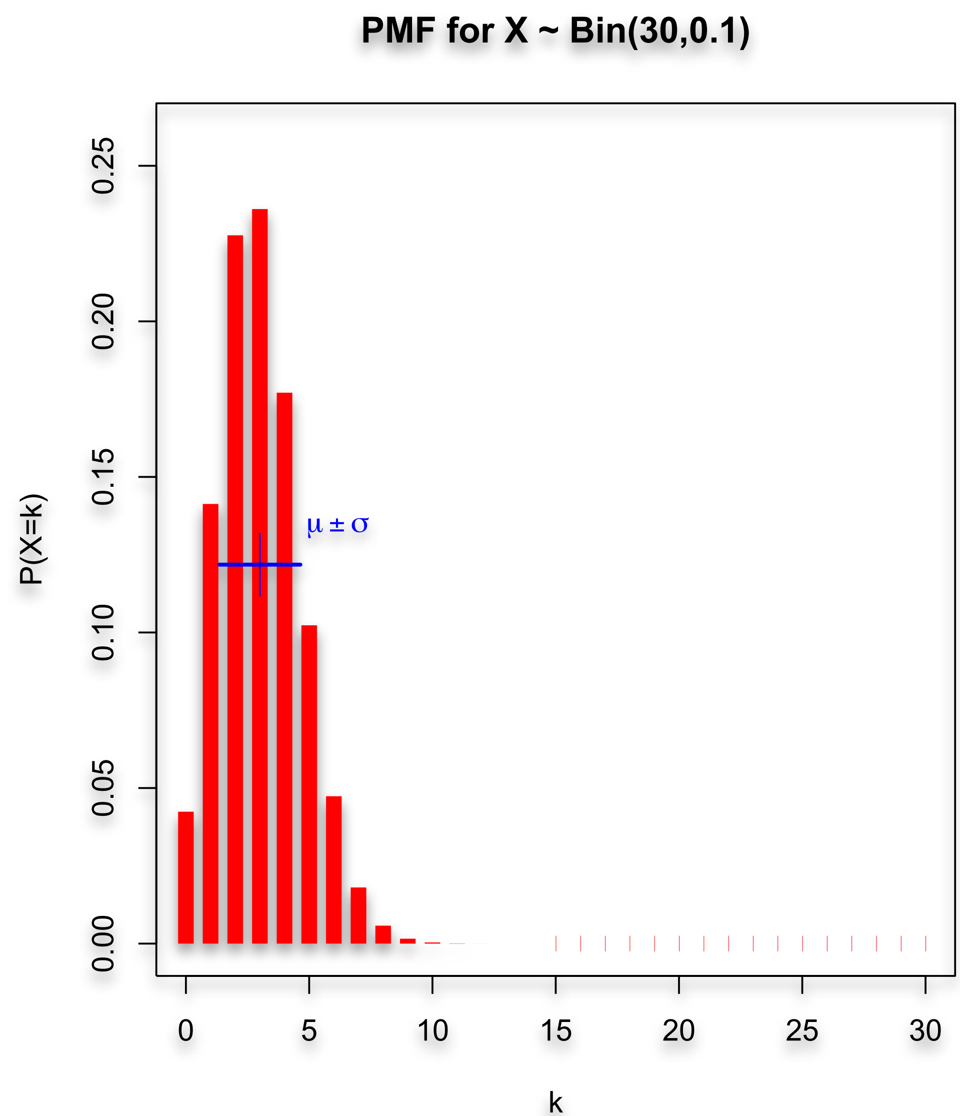
$\text{Var}(X) = p(1\text{-}p)n$ ←(proof below, twice)

binomial pmfs



binomial pmfs





$$E[X^k] = \sum_{i=0}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} i^k \binom{n}{i} p^i (1-p)^{n-i}$$

using

$$i\binom{n}{i} = n\binom{n-1}{i-1}$$

$$E[X^k] = np \sum_{i=1}^{n} i^{k-1} \binom{n-1}{i-1} p^{i-1} (1-p)^{n-i}$$

letting
j = i-1

$$= np \sum_{j=0}^{n-1} (j+1)^{k-1} \binom{n-1}{j} p^j (1-p)^{n-1-j}$$

$$= npE[(Y+1)^{k-1}]$$

where Y is a binomial random variable with parameters $n-1, p$.

k=1 gives: $E[X] = np$; k=2 gives $E[X^2]=np[(n-1)p+1]$

hence: $$\text{Var}(X) = E[X^2] - (E[X])^2$$
$$= np[(n-1)p+1] - (np)^2$$
$$= np(1-p)$$

Theorem: If X & Y are *independent*, then E[X•Y] = E[X]•E[Y]

Proof:

Let $x_i, y_i, i = 1, 2, \ldots$ be the possible values of X, Y.

$$\begin{aligned} E[X \cdot Y] &= \sum_i \sum_j x_i \cdot y_j \cdot P(X = x_i \wedge Y = y_j) \\ &= \sum_i \sum_j x_i \cdot y_j \cdot P(X = x_i) \cdot P(Y = y_j) \\ &= \sum_i x_i \cdot P(X = x_i) \cdot \left(\sum_j y_j \cdot P(Y = y_j) \right) \\ &= E[X] \cdot E[Y] \end{aligned}$$

independence

Note: *NOT* true in general; see earlier example $E[X^2] \neq E[X]^2$

variance of *independent* r.v.s is additive

(Bienaymé, 1853)

Theorem: If X & Y are *independent*, then

Var[X+Y] = Var[X]+Var[Y]

Proof: Let

$$\begin{aligned} \widehat{X} &= X - E[X] & \widehat{Y} &= Y - E[Y] \\ E[\widehat{X}] &= 0 & E[\widehat{Y}] &= 0 \\ Var[\widehat{X}] &= Var[X] & Var[\widehat{Y}] &= Var[Y] \end{aligned}$$

Var(aX+b) = a²Var(X)

$$\begin{aligned} Var[X+Y] &= Var[\widehat{X}+\widehat{Y}] \\ &= E[(\widehat{X}+\widehat{Y})^2] - (E[\widehat{X}+\widehat{Y}])^2 \\ &= E[\widehat{X}^2 + 2\widehat{X}\widehat{Y} + \widehat{Y}^2] - 0 \\ &= E[\widehat{X}^2] + 2E[\widehat{X}\widehat{Y}] + E[\widehat{Y}^2] \\ &= Var[\widehat{X}] + 0 + Var[\widehat{Y}] \\ &= Var[X] + Var[Y] \end{aligned}$$

If $Y_1, Y_2, \ldots, Y_n \sim \text{Ber}(p)$ and independent,

then $X = \sum_{i=1}^{n} Y_i \sim \text{Bin}(n, p)$.

$E[X] = E[\sum_{i=1}^{n} Y_i] = nE[Y_1] = np$

$\text{Var}[X] = \text{Var}[\sum_{i=1}^{n} Y_i] = n\text{Var}[Y_1] = np(1-p)$

A RAID-like disk array consists of *n* drives, each of which will fail independently with probability *p*. Suppose it can operate effectively if at least one-half of its components function, e.g., by "majority vote."
For what values of *p* is a 5-component system more likely to operate effectively than a 3-component system?

X_5 = # failed in 5-component system ~ Bin(5, p)
X_3 = # failed in 3-component system ~ Bin(3, p)

X_5 = # failed in 5-component system ~ Bin(5, p)

X_3 = # failed in 3-component system ~ Bin(3, p)

P(5 component system effective) = P(X_5 < 5/2)

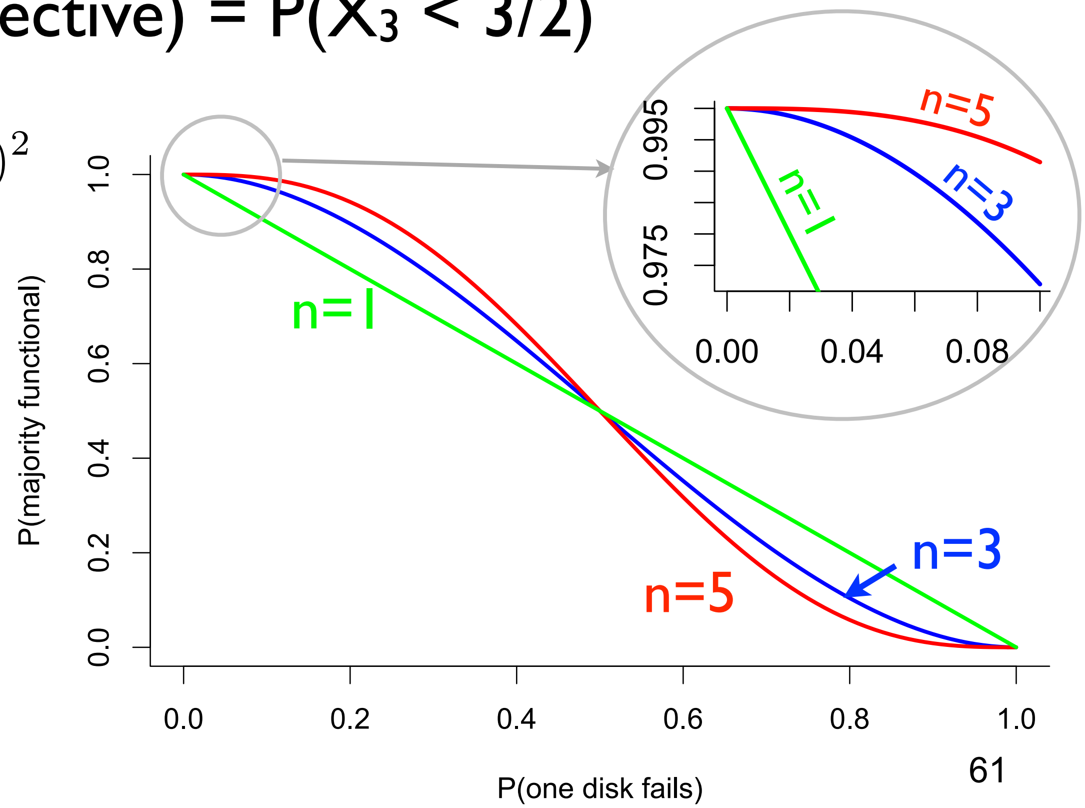
$$\binom{5}{0}p^0(1-p)^5 + \binom{5}{1}p^1(1-p)^4 + \binom{5}{2}p^2(1-p)^3$$

P(3 component system effective) = P(X_3 < 3/2)

$$\binom{3}{0}p^0(1-p)^3 + \binom{3}{1}p^1(1-p)^2$$

Calculation:

5-component system is better iff p < 1/2



Goal: send a 4-bit message over a noisy communication channel.

Say, 1 bit in 10 is flipped in transit, independently.

What is the probability that the message arrives correctly?

Let X = # of errors; X ~ Bin(4, 0.1)

P(correct message received) = P(X=0)

$$P(X = 0) = \binom{4}{0}(0.1)^0(0.9)^4 = 0.6561$$

Can we do better? Yes: error correction via redundancy.

E.g., send every bit in triplicate; use majority vote.

Let Y = # of errors in one trio; Y ~ Bin(3, 0.1); P(a trio is OK) =

$$P(Y \le 1) = \binom{3}{0}(0.1)^0(0.9)^3 + \binom{3}{1}(0.1)^1(0.9)^2 = 0.972$$

If X' = # errors in triplicate msg, X' ~ Bin(4, 0.028), and

$$P(X' = 0) = \binom{4}{0}(0.028)^0(0.972)^4 = 0.8926168$$

The Hamming(7,4) code:

Have a 4-bit string to send over the network (or to disk)

Add 3 "parity" bits, and send 7 bits total

If bits are $b_1 b_2 b_3 b_4$ then the three parity bits are

$parity(b_1 b_2 b_3)$, $parity(b_1 b_3 b_4)$, $parity(b_2 b_3 b_4)$

Each bit is independently corrupted (flipped) in transit with probability 0.1

Z = number of bits corrupted ~ Bin(7, 0.1)

The Hamming code allow us to *correct* all 1 bit errors.

(E.g., if b_1 flipped, 1st 2 parity bits, but not 3rd, will look wrong; the only single bit error causing this symptom is b_1. Similarly for any other single bit being flipped. Some, but not all, multi-bit errors can be detected, but not corrected.)

P(correctable message received) = $P(Z \leq 1)$

Using Hamming error-correcting codes: Z ~ Bin(7, 0.1)

$$P(Z \leq 1) = \binom{7}{0}(0.1)^0(0.9)^7 + \binom{7}{1}(0.1)^1(0.9)^6 \approx 0.8503$$

Recall, uncorrected success rate is

$$P(X = 0) = \binom{4}{0}(0.1)^0(0.9)^4 = 0.6561$$

And triplicate code error rate is:

$$P(X' = 0) = \binom{4}{0}(0.028)^0(0.972)^4 = 0.8926168$$

Hamming code is nearly as reliable as the triplicate code, with 5/12 ≈ 42% fewer bits. (& better with longer codes.)

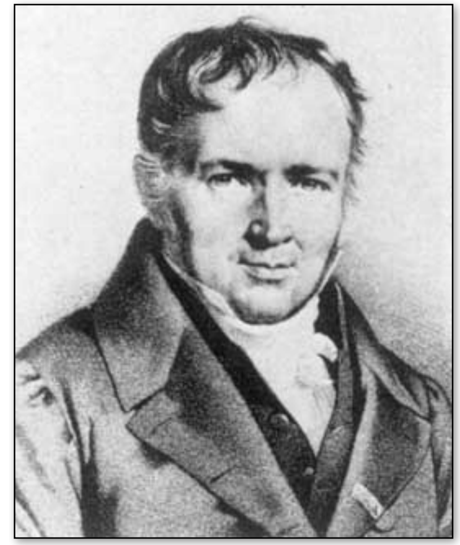
Sending a bit string over the network

- $n = 4$ bits sent, each corrupted with probability 0.1
- X = # of corrupted bits, $X \sim \text{Bin}(4, 0.1)$
- In real networks, large bit strings (length $n \approx 10^4$)
- Corruption probability is very small: $p \approx 10^{-6}$
- $X \sim \text{Bin}(10^4, 10^{-6})$ is unwieldy to compute

Extreme n and p values arise in many cases

- # bit errors in file written to disk
- # of typos in a book
- # of elements in particular bucket of large hash table
- # of server crashes per day in giant data center
- # facebook login requests sent to a particular server

Suppose "events" happen, independently, at an *average* rate of λ per unit time. Let X be the *actual* number of events happening in a given time unit. Then X is a *Poisson* r.v. *with parameter* λ (denoted X ~ Poi(λ)) and has distribution (PMF):


Siméon Poisson, 1781-1840

$$P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

Examples:

# of alpha particles emitted by a lump of radium in 1 sec.

# of traffic accidents in Seattle in one year

# of babies born in a day at UW Med center

# of visitors to my web page today

See B&T Section 6.2 for more on theoretical basis for Poisson.

X is a Poisson r.v. with parameter λ if it has PMF:

$$P(X=i) = e^{-\lambda}\frac{\lambda^i}{i!}$$

Is it a valid distribution? Recall Taylor series:

$$e^{\lambda} = \frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} + \cdots = \sum_{0 \le i} \frac{\lambda^i}{i!}$$

So

$$\sum_{0 \le i} P(X=i) = \sum_{0 \le i} e^{-\lambda}\frac{\lambda^i}{i!} = e^{-\lambda}\sum_{0 \le i}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1$$

$$\begin{aligned}
E[X] &= \sum_{0 \le i} i \cdot e^{-\lambda} \frac{\lambda^i}{i!} \\
&= \sum_{1 \le i} i \cdot e^{-\lambda} \frac{\lambda^i}{i!} \\
&= \lambda e^{-\lambda} \sum_{1 \le i} \frac{\lambda^{i-1}}{(i-1)!} \\
&= \lambda e^{-\lambda} \sum_{0 \le j} \frac{\lambda^j}{j!} \\
&= \lambda e^{-\lambda} e^{\lambda} \\
&= \lambda
\end{aligned}$$

i = 0 term is zero

j = i-1

As expected, given definition in terms of "average rate λ"

(Var[X] = λ, too; proof similar, see B&T example 6.20)

binomial random variable is Poisson in the limit

Poisson approximates binomial when n is large, p is small, and $\lambda = np$ is "moderate"

Different interpretations of "moderate"

$n > 20$ and $p < 0.05$

$n > 100$ and $p < 0.1$

Formally, Binomial is Poisson in the limit as $n \to \infty$ (equivalently, $p \to 0$) while holding $np = \lambda$

X ~ Binomial(n,p)

$$
\begin{aligned}
P(X=i) &= \binom{n}{i} p^i (1-p)^{n-i} \\
&= \frac{n!}{i!(n-i)!} \left(\frac{\lambda}{n}\right)^i \left(1-\frac{\lambda}{n}\right)^{n-i}, \text{ where } \lambda = pn \\
&= \frac{n(n-1)\cdots(n-i+1)}{n^i} \; \frac{\lambda^i}{i!} \; \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i} \\
&= \underbrace{\frac{n(n-1)\cdots(n-i+1)}{(n-\lambda)^i}} \; \frac{\lambda^i}{i!} \; \underbrace{(1-\lambda/n)^n} \\
&\approx \quad 1 \quad \cdot \frac{\lambda^i}{i!} \cdot \quad e^{-\lambda}
\end{aligned}
$$

I.e., Binomial ≈ Poisson for large n, small p, moderate i, λ.

Recall example of sending bit string over a network

Send bit string of length $n = 10^4$

Probability of (independent) bit corruption is $p = 10^{-6}$

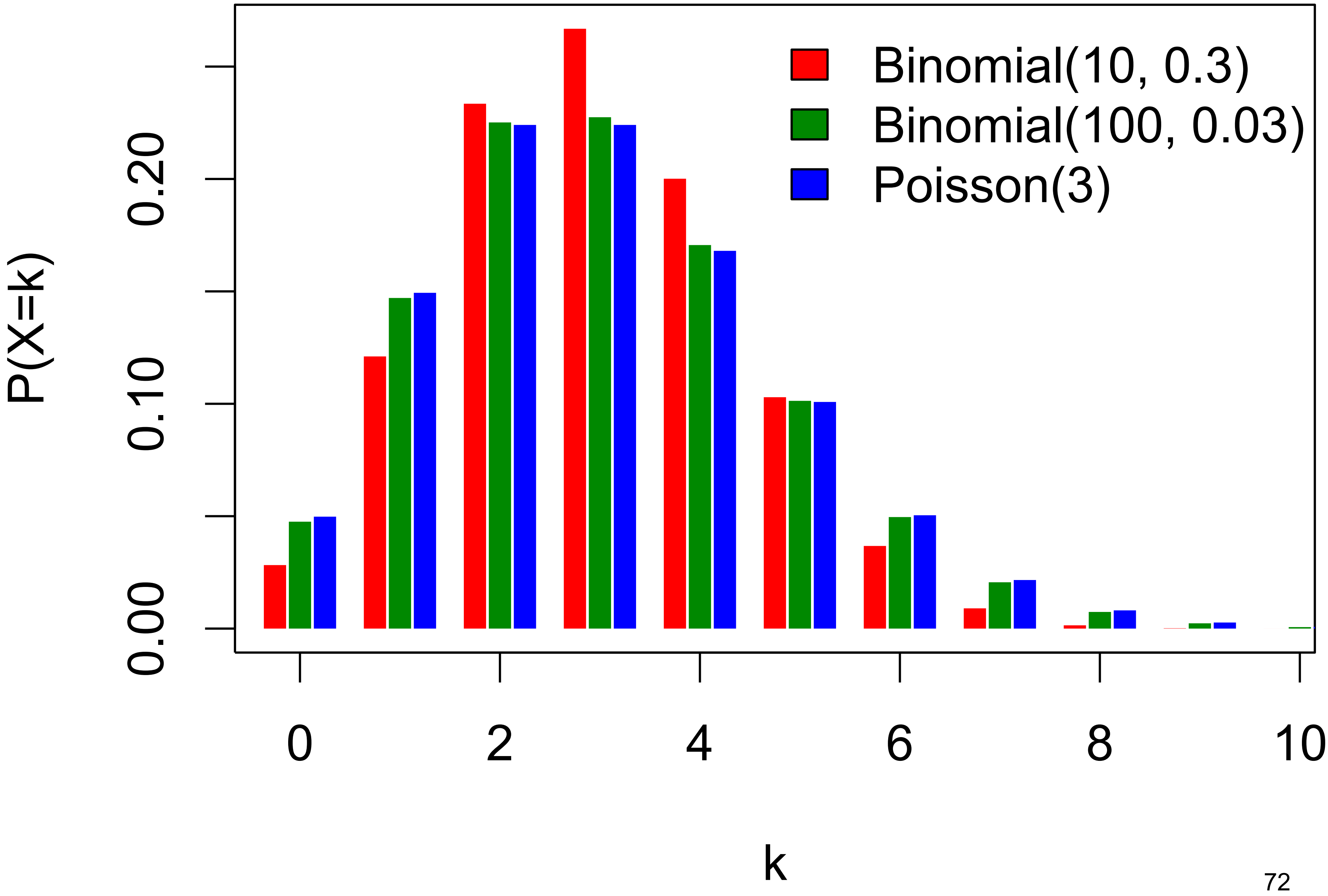
$X \sim \text{Poi}(\lambda = 10^4 \cdot 10^{-6} = 0.01)$

What is probability that message arrives uncorrupted?

$$P(X=0) = e^{-\lambda}\frac{\lambda^0}{0!} = e^{-0.01}\frac{0.01^0}{0!} \approx 0.990049\underline{834}$$

Using $Y \sim \text{Bin}(10^4, 10^{-6})$:

$P(Y=0) \approx 0.9900498\underline{29}$


Binomial(10, 0.3)
Binomial(100, 0.03)
Poisson(3)
P(X=k)
0.00
0.10
0.20
0
2
4
6
8
10
k

Recall: if $Y \sim \mathrm{Bin}(n,p)$, then:

$E[Y] = pn$

$\mathrm{Var}[Y] = np(1\text{-}p)$

And if $X \sim \mathrm{Poi}(\lambda)$ where $\lambda = np$ ($n \to \infty$, $p \to 0$) then

$E[X] = \lambda = np = E[Y]$

$\mathrm{Var}[X] = \lambda \approx \lambda(1\text{-}\lambda/n) = np(1\text{-}p) = \mathrm{Var}[Y]$

Expectation and variance of Poisson are the same (λ)

Expectation is the same as corresponding binomial

Variance almost the same as corresponding binomial

Note: when two different distributions share the same mean & variance, it suggests (but doesn't prove) that one may be a good approximation for the other.

Suppose a server can process 2 requests per second

Requests arrive at random at an average rate of 1/sec

Unprocessed requests are held in a *buffer*

Q. How big a buffer do we need to avoid ever dropping a request?

A. Infinite

Q. How big a buffer do we need to avoid dropping a request more often than once a day?

A. (approximate) If X is the number of arrivals in a second, then X is Poisson (λ=1). We want b s.t.
$P(X > b) < 1/(24*60*60) \approx 1.2 \times 10^{-5}$

$P(X = b) = e^{-1}/b!$ $\quad\quad \Sigma_{i\geq 8} P(X=i) \approx P(X=8) \approx 10^{-5}$

Also look at probability of 10 arrivals in 2 seconds, 12 in 3 seconds, etc.

In a series $X_1, X_2, \ldots$ of Bernoulli trials with success probability p, let Y be the index of the first success, i.e.,

$$X_1 = X_2 = \ldots = X_{Y-1} = 0 \;\&\; X_Y = 1$$

Then Y is a *geometric* random variable with parameter p.

Examples:

Number of coin flips until first head

Number of blind guesses on LSAT until I get one right

Number of darts thrown until you hit a bullseye

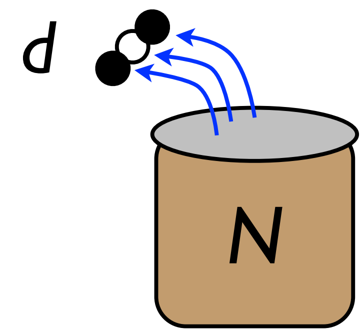
Number of random probes into hash table until empty slot

Number of wild guesses at a password until you hit it

$P(Y=k) = (1-p)^{k-1}p$; Mean $1/p$; Variance $(1-p)/p^2$

B&T, exercise 1.61

Draw *d* balls (without replacement) from an urn containing *N*, of which *w* are white, the rest black.



Let X = number of white balls drawn

$$P(X = i) = \frac{\binom{w}{i}\binom{N-w}{d-i}}{\binom{N}{d}}, \quad i = 0, 1, \ldots, d$$

(note: n choose k = 0 if k < 0 or k > n)

E[X] = dp, where p = w/N (the fraction of white balls)

proof: Let X_j be 0/1 indicator for j-th ball is white, $X = \Sigma X_j$

The X_j are *dependent*, but $E[X] = E[\Sigma X_j] = \Sigma E[X_j] = dp$

Var[X] = dp(1-p)(1-(d-1)/(N-1))

$N \approx 22500$ human genes, many of unknown function

Suppose in some experiment, d =1588 of them were observed (say, they were all switched on in response to some drug)

A big question: What are they doing?

One idea: The Gene Ontology Consortium (www.geneontology.org) has grouped genes with known functions into categories such as "muscle development" or "immune system." Suppose 26 of your d genes fall in the "muscle development" category.

Just chance?

Or call Coach & see if he wants to dope some athletes?

Hypergeometric: GO has 116 genes in the muscle development category. If those are the white balls among 22500 in an urn, what is the probability that you would see 26 of them in 1588 draws?

Table 2. Gene Ontology Analysis on Differentially Bound Peaks in Myoblasts versus Myotubes

GO Categories Enriched in Genes Associated with Myotube-Increased Peaks

GOID	Term	P Value	OR[a]	Count[b]	Size[c]	Ont[d]
GO:0005856	cytoskeleton	2.05E-11	2.40	94	490	CC
GO:0043292	contractile fiber	6.98E-09	5.85	22	58	CC
GO:0030016	myofibril	1.96E-08	5.74	21	56	CC
GO:0044449	contractile fiber part	2.58E-08	5.97	20	52	CC
GO:0030017	sarcomere	4.95E-08	6.04	19	49	CC
GO:0008092	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	MF
GO:0007519	skeletal muscle development	2.50E-06	4.13	20	65	BP
GO:0015629	actin cytoskeleton	4.73E-06	3.08	27	111	CC
GO:0003779	actin binding	[illegible]	[illegible]	[illegible]	159	MF
GO:0006936	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	BP
GO:0044430	cytoskeletal part	[illegible]	[illegible]	[illegible]	294	CC
GO:0031674	I band	2.27E-05	5.67	12	32	CC
GO:0003012	muscle system process	2.54E-05	4.11	16	52	BP
GO:0030029	actin filament-based process	2.89E-05	2.73	27	119	BP
GO:0007517	muscle development	5.06E-05	2.69	26	116	BP

probability of seeing this many genes from a set of this size by chance according to the hypergeometric distribution.

E.g., if you draw 1588 balls from an urn containing 490 white balls and ≈22000 black balls, P(94 white) $\approx 2.05\times10^{-11}$

A differentially bound peak was associated to the closest gene (unique Entrez ID) measured by distance to TSS within CTCF flanking domains. OR: ratio of predicted to observed number of genes within a given GO category. Count: number of genes with differentially bound peaks. Size: total number of genes for a given functional group. Ont: the Geneontology. BP = biological process, MF = molecular function, CC = cellular component.

Often care about 2 (or more) random variables *simultaneously*

measured X = height and Y = weight

X = cholesterol and Y = blood pressure

X_1, X_2, X_3 = work loads on servers A, B, C

Joint probability mass function:

$f_{XY}(x, y) = P(X = x \ \& \ Y = y)$

Joint cumulative distribution function:

$F_{XY}(x, y) = P(X \leq x \ \& \ Y \leq y)$

Two joint PMFs

W \ Z	1	2	3
1	2/24	2/24	2/24
2	2/24	2/24	2/24
3	2/24	2/24	2/24
4	2/24	2/24	2/24

X \ Y	1	2	3
1	4/24	1/24	1/24
2	0	3/24	3/24
3	0	4/24	2/24
4	4/24	0	2/24

$P(W = Z) = 3 * 2/24 = 6/24$

$P(X = Y) = (4 + 3 + 2)/24 = 9/24$

Can look at arbitrary relationships between variables this way

Two joint PMFs

W \ Z	1	2	3	$f_W(w)$
1	2/24	2/24	2/24	6/24
2	2/24	2/24	2/24	6/24
3	2/24	2/24	2/24	6/24
4	2/24	2/24	2/24	6/24
$f_Z(z)$	8/24	8/24	8/24	

X \ Y	1	2	3	$f_X(x)$
1	4/24	1/24	1/24	6/24
2	0	3/24	3/24	6/24
3	0	4/24	2/24	6/24
4	4/24	0	2/24	6/24
$f_Y(y)$	8/24	8/24	8/24	

Marginal distribution of one r.v.: sum over the other:

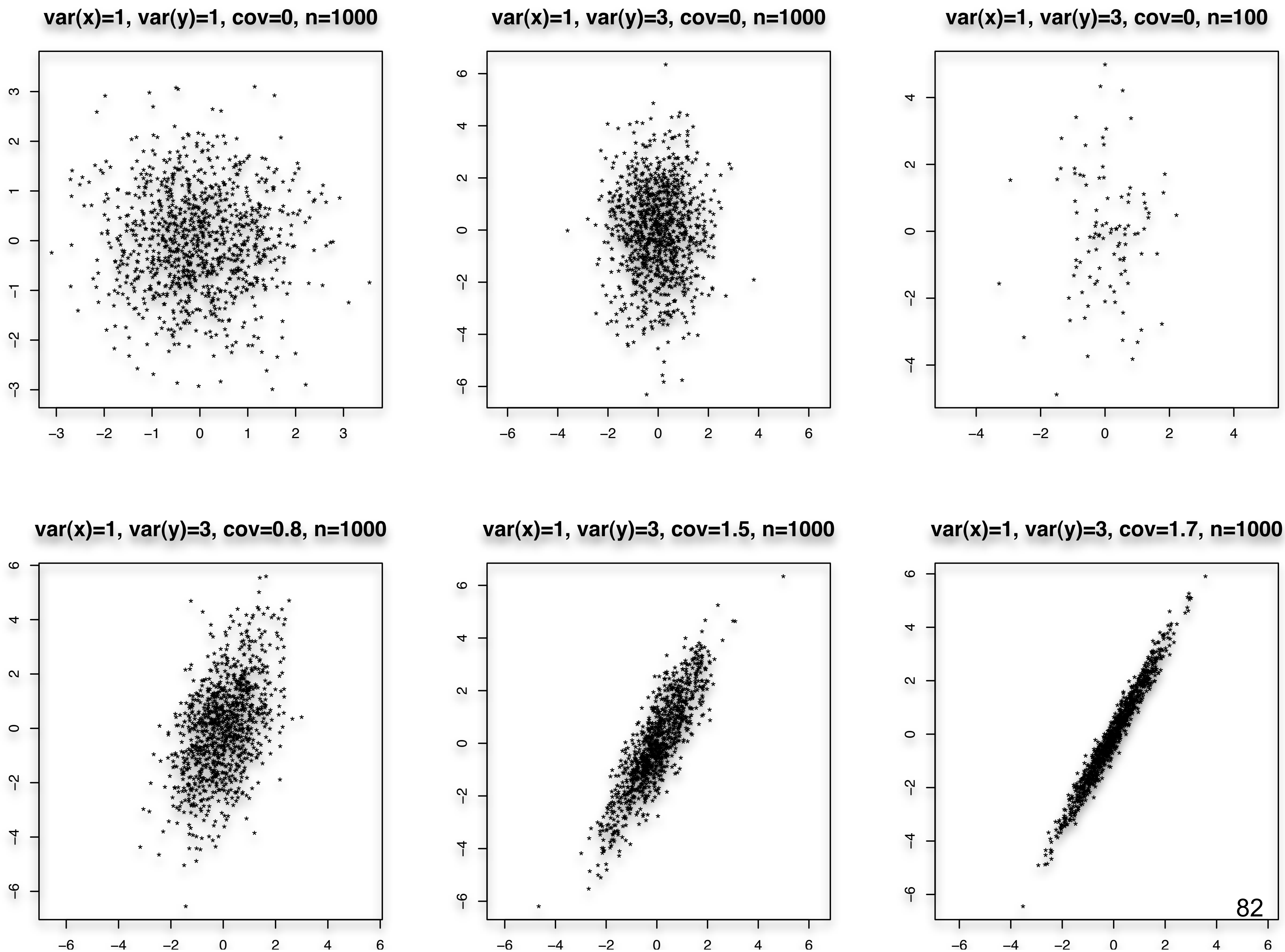
$f_Y(y) = \Sigma_x f_{XY}(x,y)$ $f_X(x) = \Sigma_y f_{XY}(x,y)$

Question: Are W & Z independent? Are X & Y independent?

sampling from a (continuous) joint distribution

Top row; independent variables

bottom row: dependent variables



A function g(X,Y) defines a new random variable.

Its expectation is:

$$E[g(X, Y)] = \Sigma_x \Sigma_y\, g(x, y) f_{XY}(x,y)$$

Expectation is linear. I.e., if g is linear:

$$E[g(X, Y)] = E[a X + b Y + c] = a E[X] + b E[Y] + c$$

Example:

g(X,Y) = 2X-Y

E[g(X,Y)] = 72/24 = 3

E[g(X,Y)] = 2•2.5 - 2 = 3

X \ Y	1	2	3
1	1 • 4/24	0 • 1/24	-1 • 1/24
2	3 • 0/24	2 • 3/24	1 • 3/24
3	5 • 0/24	4 • 4/24	3 • 2/24
4	7 • 4/24	6 • 0/24	5 • 2/24

RV: a numeric function of the outcome of an experiment

Probability Mass Function p(x): prob that RV = x; $\Sigma p(x)=1$

Cumulative Distribution Function F(x): *probability that RV ≤ x*

Concepts generalize to *joint* distributions

Expectation:

- of a random variable: $E[X] = \Sigma_x\, xp(x)$
- of a function: if $Y = g(X)$, then $E[Y] = \Sigma_x\, g(x)p(x)$
- linearity:
  - $E[aX + b] = aE[X] + b$
  - $E[X+Y] = E[X] + E[Y]$; even if *dependent*
  - *this interchange of "order of operations" is quite special to linear combinations. E.g.* $E[XY] \neq E[X]*E[Y]$*, in general (but see below)*

Variance:

$\mathrm{Var}[X] = E[(X-E[X])^2] = E[X^2] - (E[X])^2]$

Standard deviation: $\sigma = \sqrt{\mathrm{Var}[X]}$

$\mathrm{Var}[aX+b] = a^2\,\mathrm{Var}[X]$

If X & Y are *independent*, then

$E[X \cdot Y] = E[X] \cdot E[Y]$;

$\mathrm{Var}[X+Y] = \mathrm{Var}[X]+\mathrm{Var}[Y]$

(These two equalities hold for *indp* rv's; but not in general.)

Important Examples:

Bernoulli: P(X=1) = p and P(X=0) = 1-p $\mu = p$, $\sigma^2 = p(1-p)$

Binomial: $P(X=i) = \binom{n}{i} p^i (1-p)^{n-i}$ $\mu = np$, $\sigma^2 = np(1-p)$

Poisson: $P(X=i) = e^{-\lambda} \frac{\lambda^i}{i!}$ $\mu = \lambda$, $\sigma^2 = \lambda$

Bin(n,p) ≈ Poi(λ) where λ = np fixed, n → ∞ (and so p=λ/n → 0)

Geometric $P(X=k) = (1-p)^{k-1}p$ $\mu = 1/p$, $\sigma^2 = (1-p)/p^2$

Many others, e.g., hypergeometric