5. independence

Defn: Two events E and F are *independent* if

$P(EF) = P(E) P(F)$

If $P(F)>0$, this is equivalent to: $P(E|F) = P(E)$ (proof below)

Otherwise, they are called *dependent*

Roll two dice, yielding values D_1 and D_2

1) $E = \{ D_1 = 1 \}$

   $F = \{ D_2 = 1 \}$

   $P(E) = 1/6,\; P(F) = 1/6,\; P(EF) = 1/36$

   $P(EF) = P(E) \cdot P(F) \Rightarrow$ E and F *independent*

   Intuitive; the two dice are not physically coupled

2) $G = \{D_1 + D_2 = 5\} = \{(1,4),(2,3),(3,2),(4,1)\}$

   $P(E) = 1/6, P(G) = 4/36 = 1/9, P(EG) = 1/36$

   not independent!

   E, G are *dependent* events

   The dice are still not physically coupled, but "$D_1 + D_2 = 5$" couples them ***mathematically****: info about D_1 constrains D_2. (But dependence/independence not always intuitively obvious; "use the definition, Luke".)*

Two events E and F are *independent* if

$P(EF) = P(E)\, P(F)$

If $P(F)>0$, this is equivalent to: $P(E|F) = P(E)$

Otherwise, they are called *dependent*

Three events E, F, G are independent if

$P(EF) = P(E)\, P(F)$

$P(EG) = P(E)\, P(G)$ *and* $P(EFG) = P(E)\, P(F)\, P(G)$

$P(FG) = P(F)\, P(G)$

Example: Let X, Y be each $\{-1,1\}$ with equal prob

$E = \{X = 1\}, F = \{Y = 1\}, G = \{ XY = 1\}$

$P(EF) = P(E)P(F), P(EG) = P(E)P(G), P(FG) = P(F)P(G)$

but $P(EFG) = 1/4$!!! (because $P(G|EF) = 1$)

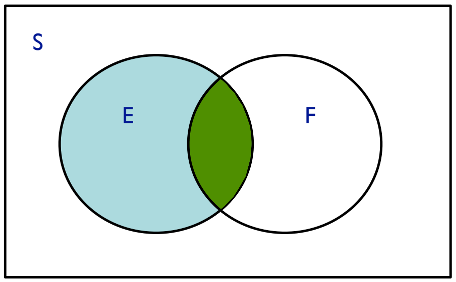
In general, events E_1, E_2, …, E_n are independent if for *every subset* S of {1,2,…, n}, we have

$$P\left(\bigcap_{i \in S} E_i\right) = \prod_{i \in S} P(E_i)$$

(Sometimes this property holds only for small subsets S. E.g., E, F, G on the previous slide are *pairwise* independent, but not fully independent.)

Theorem: E, F independent $\Rightarrow$ E, F^c independent

$E = EF \cup EF^c$



Proof: $P(EF^c) = P(E) - P(EF)$

$= P(E) - P(E)\, P(F)$

$= P(E)\, (1 - P(F))$

$= P(E)\, P(F^c)$

Theorem: if $P(E)>0$, $P(F)>0$, then

E, F independent $\Leftrightarrow P(E|F)=P(E) \Leftrightarrow P(F|E) = P(F)$

Proof: Note $P(EF) = P(E|F)\, P(F)$, regardless of in/dep.

Assume independent. Then

$$P(E)P(F) = P(EF) = P(E|F)\, P(F) \Rightarrow P(E|F)=P(E) \quad (\div \text{ by } P(F))$$

Conversely, $P(E|F)=P(E) \Rightarrow P(E)P(F) = P(EF)$ $\quad (\times \text{ by } P(F))$

Suppose a biased coin comes up heads with probability p, *independent* of other flips

P(n heads in n flips) $= p^n$

P(n tails in n flips) $= (1-p)^n$

P(exactly k heads in n flips) $= \binom{n}{k} p^k (1-p)^{n-k}$

Aside: note that the probability of *some* number of heads = $\sum_k \binom{n}{k} p^k (1-p)^{n-k} = (p + (1-p))^n = 1$ as it should, by the binomial theorem.

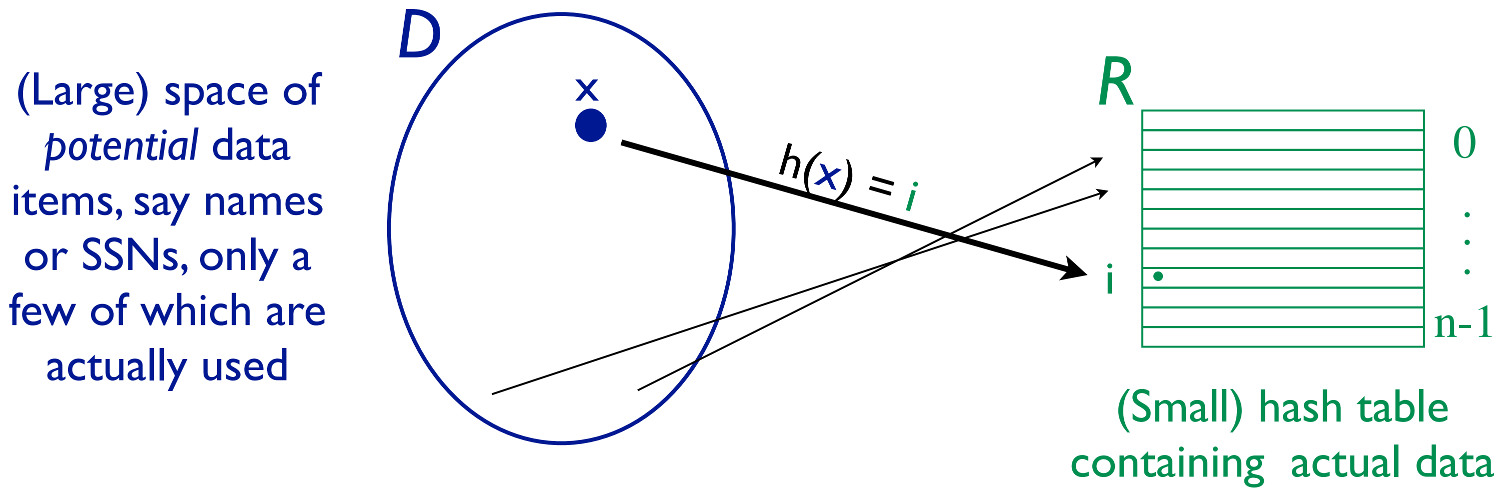
Suppose a biased coin comes up heads with probability p, *independent* of other flips

P(exactly k heads in n flips) $= \binom{n}{k} p^k (1-p)^{n-k}$

Note when p=1/2, this is the same result we would have gotten by considering n flips in the "equally likely outcomes" scenario. But $p \neq 1/2$ makes that inapplicable. Instead, the *independence* assumption allows us to conveniently assign a probability to each of the 2^n outcomes, e.g.:

Pr(HHTHTTT) = $p^2(1-p)p(1-p)^3 = p^{\#H}(1-p)^{\#T}$

A data structure problem: *fast* access to *small* subset of data drawn from a *large* space.



A solution: *hash function* h:D→{0,...,n-1} crunches/scrambles names from large space into small one. E.g., if x is integer:

$$h(x) = x \bmod n$$

Good hash functions *approximately* randomize placement.

m strings hashed (uniformly) into a table with n buckets

Each string hashed is an *independent* trial

E = at least one string hashed to first bucket

What is P(E) ?

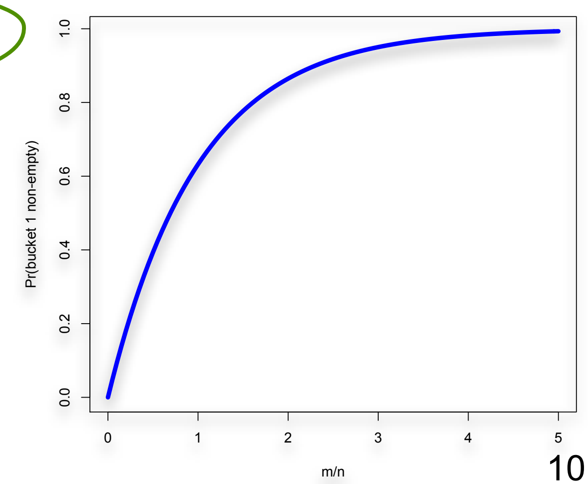
Solution:

F_i = string i *not* hashed into first bucket (i=1,2,…,m)

$P(F_i) = 1 - 1/n = (n-1)/n$ for all i=1,2,…,m

Event $(F_1\ F_2\ \dots\ F_m)$ = no strings hashed to first bucket

$$
\begin{aligned}
P(E) &= 1 - P(F_1\ F_2 \cdots F_m) \\
&\overset{\text{indp}}{=} 1 - P(F_1)\ P(F_2) \cdots P(F_m) \\
&= 1 - ((n-1)/n)^m \\
&\approx 1\text{-}\exp(-m/n)
\end{aligned}
$$



m strings hashed (non-uniformly) to table w/ n buckets

- Each string hashed is an *independent* trial, with probability p_i of getting hashed to bucket i
- E = At least 1 of buckets 1 to k gets ≥ 1 string

What is P(E) ?

Solution:

- F_i = at least one string hashed into i-th bucket
- $P(E) = P(F_1 \cup \cdots \cup F_k) = 1 - P((F_1 \cup \cdots \cup F_k)^c)$

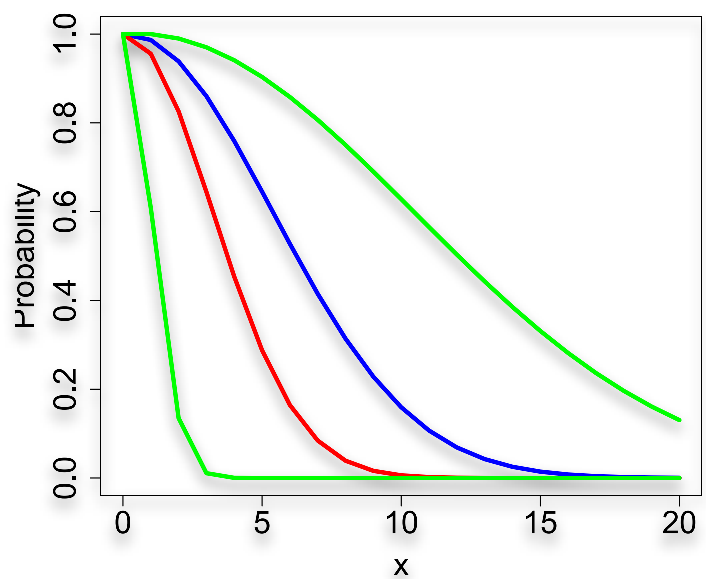
$$= 1 - P(F_1^c\, F_2^c \ldots F_k^c)$$
$$= 1 - P(\text{no strings hashed to buckets 1 to k})$$
$$= 1 - (1 - p_1 - p_2 - \cdots - p_k)^m$$

Let $D_0 \subseteq D$ be a fixed set of m strings, R = {0,...,n-1}. A hash function h:D→R is *perfect* for D_0 if $h:D_0$→R is injective (no collisions). How hard is it to find a perfect hash function?

1) Fix h; pick m elements of D_0 *independently* at random ∈ D

Suppose h maps ≈ (1/n)th of D to each element of R. This is like the birthday problem:

$$P(\text{h is perfect for } D_0) = \frac{n}{n}\frac{n-1}{n}\cdots\frac{n-m+1}{n}$$



caution; this analysis is heuristic, not rigorous, but still useful.

Let $D_0 \subseteq D$ be a fixed set of m strings, R = {0,...,n-1}. A hash function h:D→R is *perfect* for D_0 if $h:D_0 \to R$ is injective (no collisions). How hard is it to find a perfect hash function?

2) Fix D_0; pick h at random

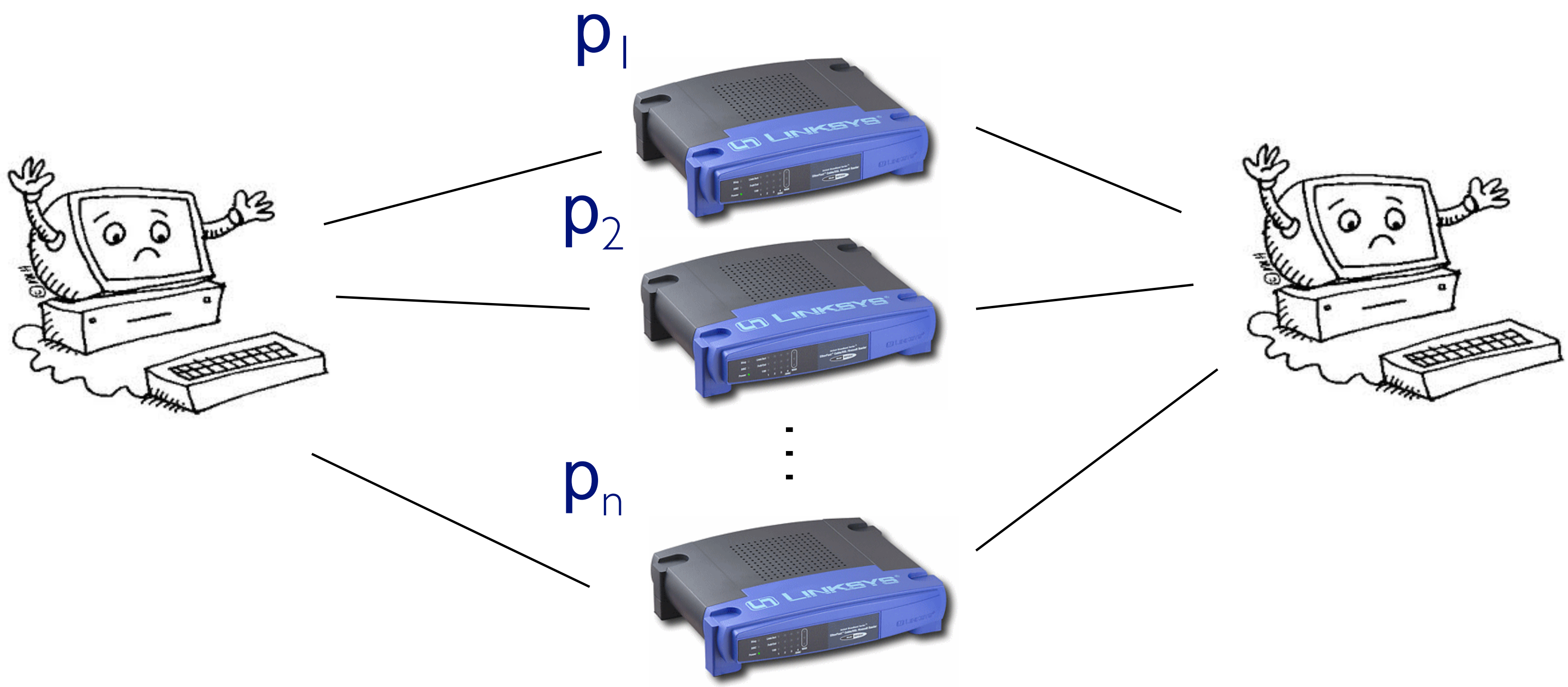
E.g., if m = $|D_0|$ = 23 and n = 365, then there is ~50% chance that h is perfect for this *fixed* D_0. If it isn't, pick h', h'', etc. With high probability, you'll quickly find a perfect one!

"Picking a random function h" is easier said than done, but, empirically, picking among a set of functions like

$$h(x) = (a \cdot x + b) \bmod n$$

where a, b are random 64-bit ints is a start.

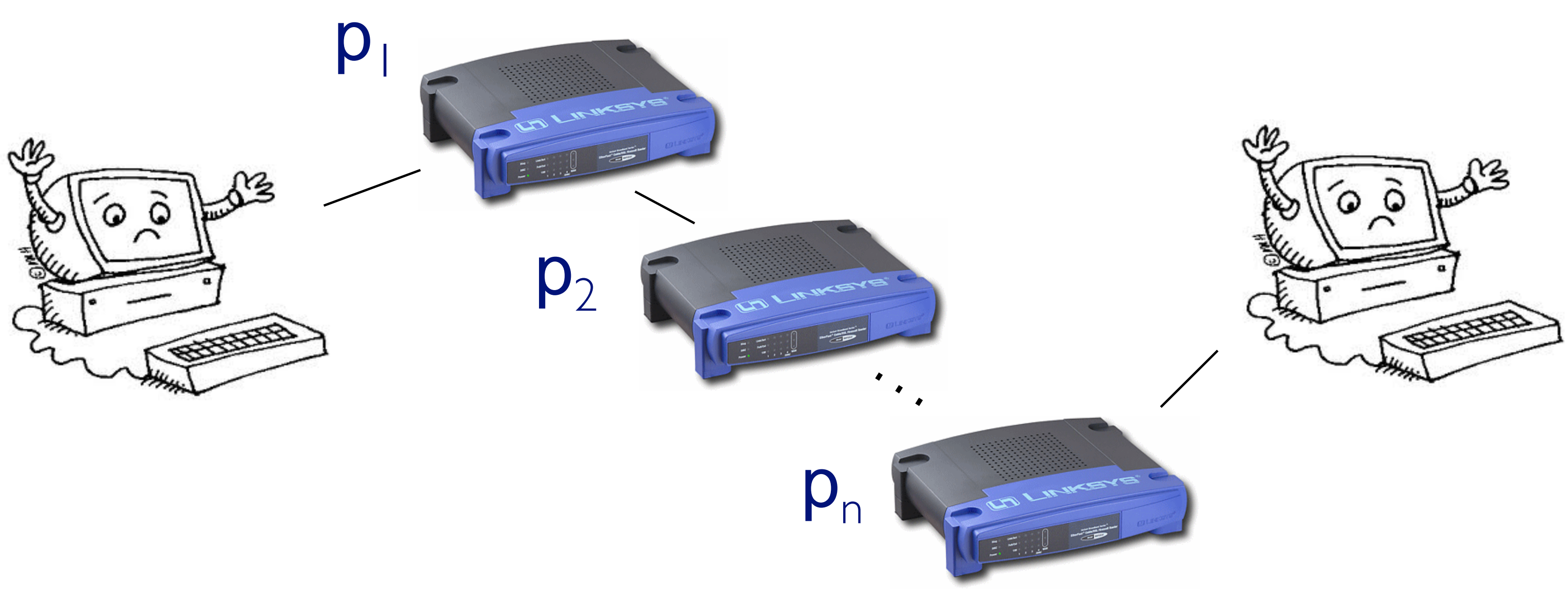
Consider the following parallel network



n routers, i^{th} has probability p_i of failing, independently

$$P(\text{there is functional path}) = 1 - P(\text{all routers fail})$$
$$= 1 - p_1 p_2 \cdots p_n$$

Contrast: a *series* network



n routers, i^{th} has probability p_i of failing, independently

P(there is functional path) =

P(*no* routers fail) = $(1 - p_1)(1 - p_2) \cdots (1 - p_n)$

Recall: Two events E and F are independent if

$P(EF) = P(E)\, P(F)$

If E & F are independent, does that tell us anything about

$P(EF|G), P(E|G), P(F|G),$

when G is an arbitrary event? In particular, is

$P(EF|G) = P(E|G)\, P(F|G)$?

In general, *no*.

Roll two 6-sided dice, yielding values D_1 and D_2

$E = \{ D_1 = 1 \}$

$F = \{ D_2 = 6 \}$

$G = \{ D_1 + D_2 = 7 \}$

E and F are independent

$P(E|G) = 1/6$

$P(F|G) = 1/6$, but

$P(EF|G) = 1/6$, *not* $1/36$

so E|G and F|G are not independent!

Two events E and F are called *conditionally independent given G*, if

$P(EF|G) = P(E|G)\ P(F|G)$

Or, equivalently (assuming $P(F)>0$, $P(G)>0$),

$P(E|FG) = P(E|G)$

do CSE majors get fewer A's?

Say you are in a dorm with 100 students

10 are CS majors: $P(C) = 0.1$

30 get straight A's: $P(A) = 0.3$

3 are CS majors who get straight A's

$P(CA) = 0.03$

$P(CA) = P(C)\,P(A)$, so C and A *independent*

At faculty night, only CS majors and A students show up

So 37 students arrive

Of 37 students, 10 are CS $\Rightarrow$

$P(C \mid C \text{ or } A) = 10/37 = 0.27 < .3 = P(A)$

Seems CS major *lowers* your chance of straight A's ☹

Weren't they supposed to be independent?

In fact, CS and A are *conditionally dependent* at fac night

conditioning can also break DEPENDENCE

Randomly choose a day of the week

A = { It is not a Monday }

B = { It is a Saturday }

C = { It is the weekend }

A and B are dependent events

P(A) = 6/7, P(B) = 1/7, P(AB) = 1/7.

Now condition both A and B on C:

P(A|C) = 1, P(B|C) = ½, P(AB|C) = ½

P(AB|C) = P(A|C) P(B|C) ⇒ A|C and B|C independent

Dependent events can become independent by conditioning on additional information!

Another reason why conditioning is so useful

Events E & F are *independent* if

P(EF) = P(E) P(F), or, equivalently P(E|F) = P(E) (if p(E)>0)

More than 2 events are indp if, for *all subsets*, joint probability = product of separate event probabilities

Independence can greatly simplify calculations

For fixed G, conditioning on G gives a probability measure, P(E|G)

But "conditioning" and "independence" are orthogonal:

Events E & F that are (unconditionally) independent may become dependent when conditioned on G

Events that are (unconditionally) dependent may become independent when conditioned on G

gamblers ruin

2 Gamblers: Alice & Bob.

A has i dollars; B has (N-i)

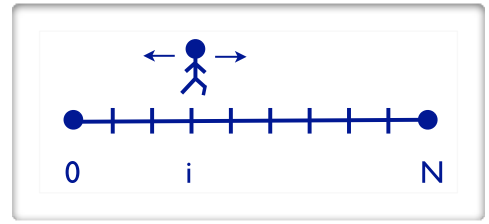
Flip a coin. Heads – A wins \$1; Tails – B wins \$1

Repeat until A or B has all N dollars

What is P(A wins)?

Let E_i = event that A wins starting with \$i

Approach: Condition on 1st flip; H = heads



aka "Drunkard's Walk"

nice example of the utility of conditioning: future decomposed into two crisp cases instead of being a blurred superposition thereof

$$
\begin{aligned}
p_i &= P(E_i) = P(E_i \mid H)P(H) + P(E_i \mid T)P(T) \\
p_i &= \frac{1}{2}(p_{i+1} + p_{i-1}) \\
2p_i &= p_{i+1} + p_{i-1} \\
p_{i+1} - p_i &= p_i - p_{i-1} \\
p_2 - p_1 &= p_1 - p_0 = p_1, \text{ since } p_0 = 0
\end{aligned}
$$

So:
$$
\begin{aligned}
p_2 &= 2p_1 \\
&\cdots \\
p_i &= ip_1 \\
p_N &= Np_1 = 1 \\
\boxed{p_i} &\boxed{= i/N}
\end{aligned}
$$