CSE 312 Foundations II

2. Counting

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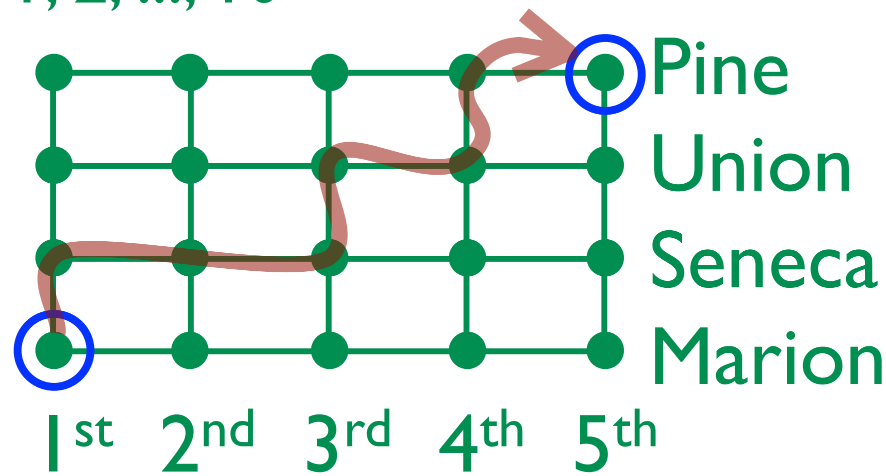
How many ways are there to do X?

E.g., X = "choose an integer 1, 2, ..., 10"

E.g., X = "Walk from 1st & Marion to 5th & Pine, going only North or East at each intersection."



The Point:

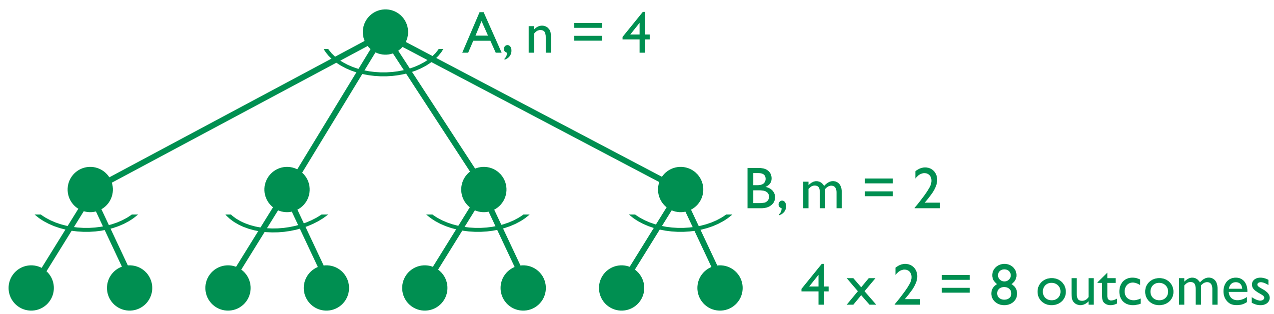
Counting gets hard when numbers are large, implicit and/or constraints are complex.

Systematic approaches help.

If there are

n outcomes for some event A,
sequentially followed by *m* outcomes for event B,

then there are *n•m* outcomes overall.



aka "The Product Rule"
Easily generalized to more events

examples

Q. How many n-bit numbers are there?

A. $\overbrace{2 \cdot 2 \cdot \ldots \cdot 2}^{n} = 2^n$

Q. How many subsets of a set of size n are there?

A. 1st member in or out; 2nd member in or out,... $\Rightarrow 2^n$

Q. How many 4-character passwords are there, if each character must be one of a, b, ..., z, 0, 1, ..., 9 ?

A. 36 • 36 • 36 • 36 = 1,679,616 ≈ 1.7 million

Q. Ditto, but no character may be repeated?

A. 36 • 35 • 34 • 33 = 1,413,720 ≈ 1.4 million

(And a non-mathematical question: why do security experts generally prefer schemes such as the second, even though it offers fewer choices?)

How many arrangements of 1, 2, 3 are possible (each used once, no repeat, order matters)?

1 2 3	2 1 3	3 1 2
1 3 2	2 3 1	3 2 1

More generally: How many arrangements of n distinct items are possible?

n	choices for 1st
(n-1)	choices for 2nd
(n-2)	choices for 3rd
...	...
1	choices for last

n • (n-1) • (n-1) • ... • 1 = n!   (n factorial)

Q. How many permutations of DOGIE are there?

A. $5! = 120$

Q. How many of DOGGY ?

A. $5!/2! = 60$

$DOG_1G_2Y = DOG_2G_1Y$

$ODG_1YG_2 = ODG_2YG_1$

...

Q. How many of GODOGGY ?

A. $$\frac{7!}{3!2!1!1!} = 420$$

Q. Your elf-lord avatar can carry 3 objects chosen from

1. sword
2. knife
3. staff
4. water jug
5. iPad w/magic WiFi

How many ways can you equip him/her?

A. $\frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3! \cdot 2!} = 10$

Combinations: number ways to choose r things from n

"n choose r" $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ aka binomial coefficients

Important special case:

how many (unordered) pairs from n objects

$$\binom{n}{2} = \frac{n(n-1)}{2} = \Theta(n^2)$$

Many Identities. E.g.:

$$\binom{n}{r} = \binom{n}{n-r}$$ ← by symmetry of definition

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$ ← 1st object either in or out

$$\binom{n}{r} = \frac{n}{r}\binom{n-1}{r-1}$$ ← by definition + algebra

$$(x+y)^n = \sum_k \binom{n}{k} x^k y^{n-k}$$

proof 1: induction ...

proof 2: counting –

(x+y) • (x+y) • (x+y) • ... • (x+y)

pick either x or y from 1st binomial factor

pick either x or y from 2nd binomial factor

...

pick either x or y from nth binomial factor

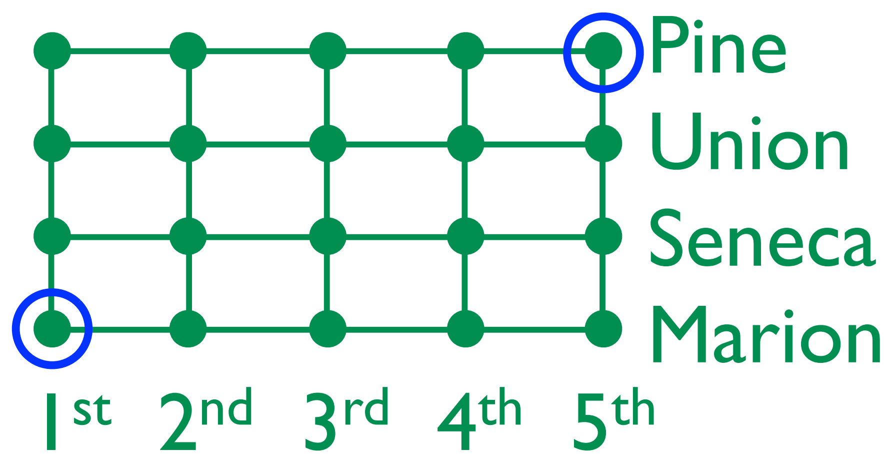
How many ways did you get exactly k x's? $\binom{n}{k}$

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Proof:

$$\sum_{k=0}^{n} \binom{n}{k} = \sum_{k=0}^{n} \binom{n}{k} 1^k 1^{n-k} = (1+1)^n = 2^n$$

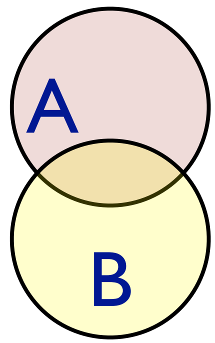
Q. How many ways are there to walk from 1st & Marion to 5th & Pine, going only North or East?

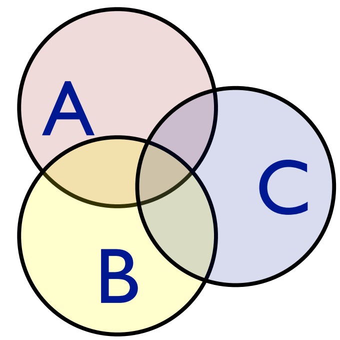


A: 7 choose 3 = 35:

Changing the visualization often helps. Instead of tracing paths on the grid above, list choices. You walk 7 blocks; at each intersection choose N or E; must choose N exactly 3 times.

NNNEEEE
NNENEEE
NNEENEE
...
EEEENNN





$|A \cup B| =$
$|A|+|B|-|A \cap B|$

$|A \cup B \cup C| =$
$|A| + |B| + |C|$
$-|A \cap B|-|A \cap C|-|B \cap C|$
$+ |A \cap B \cap C|$

General: + singles - pairs + triples - quads + ...

pigeonhole principle

If there are n pigeons in k holes and $n > k$, then some hole contains more than one pigeon.
More precisely, some hole contains at least $\lceil n/k \rceil$ pigeons.

There are two people in London who have the same number of hairs on their head.

Typical head ~ 150,000 hairs

Let's say max-hairy-head ~ 1,000,000 hairs

Since there are more than 1,000,000 people in London…