

# CSE 311 Section 4

Set Theory & Structural Induction

# Announcements & Reminders

- Homework 4 due Monday 2/16 @ 11:00pm
  - Please use latex 🙏

# **Sets: Quick Review**



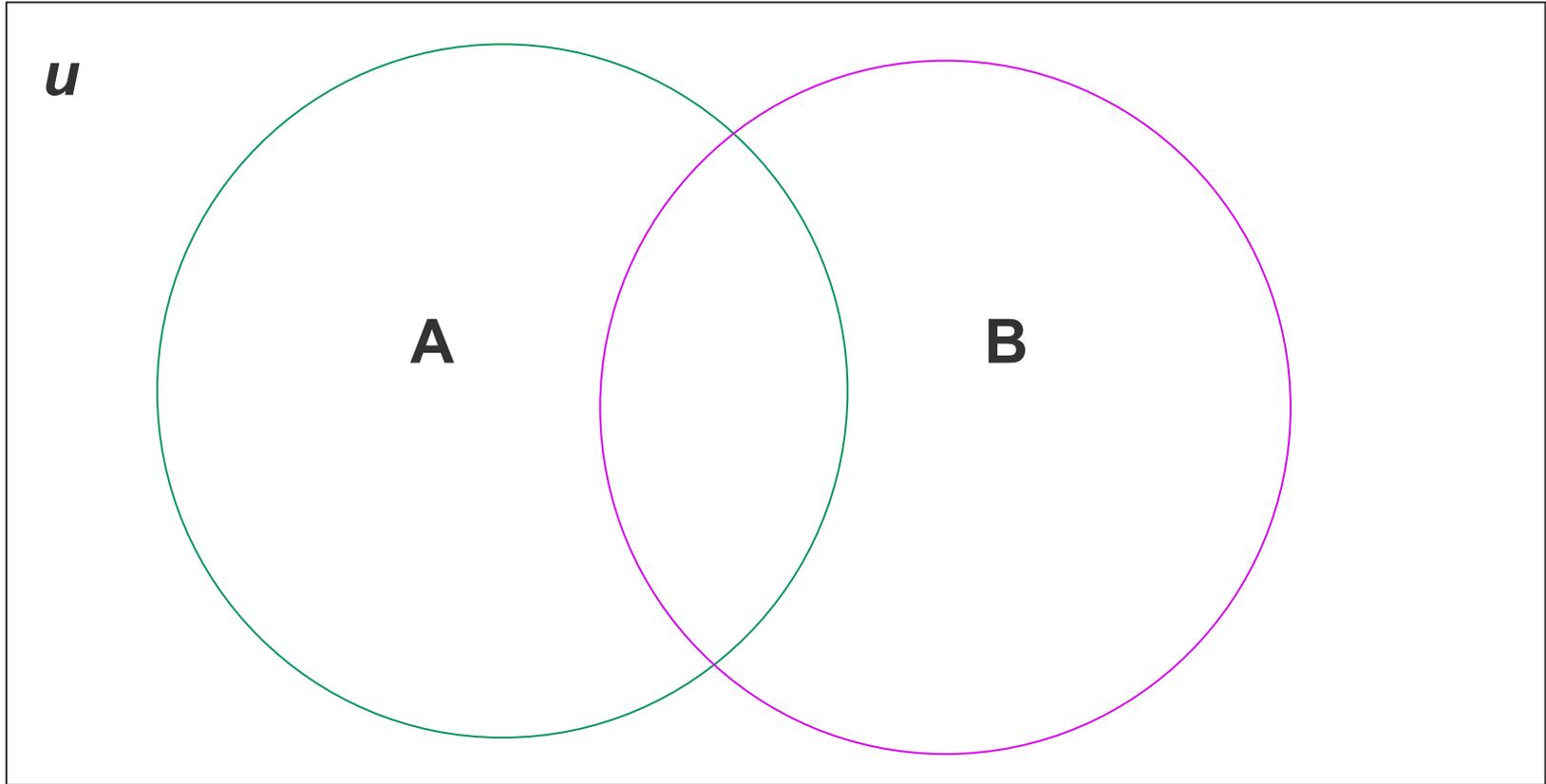
# Sets

- A set is an **unordered** group of **distinct** elements
  - Set variable names are capital letters, with lower-case letters for elements
- Set Notation:
  - $a \in A$ : “ $a$  is in  $A$ ” or “ $a$  is an element of  $A$ ”
  - $A \subseteq B$ : “ $A$  is a subset of  $B$ ”, every element of  $A$  is also in  $B$
  - $\emptyset$ : “empty set”, a unique set containing no elements
  - $\mathcal{P}(A)$ : “power set of  $A$ ”, the set of all subsets of  $A$  including the empty set and  $A$  itself

# Set Operators

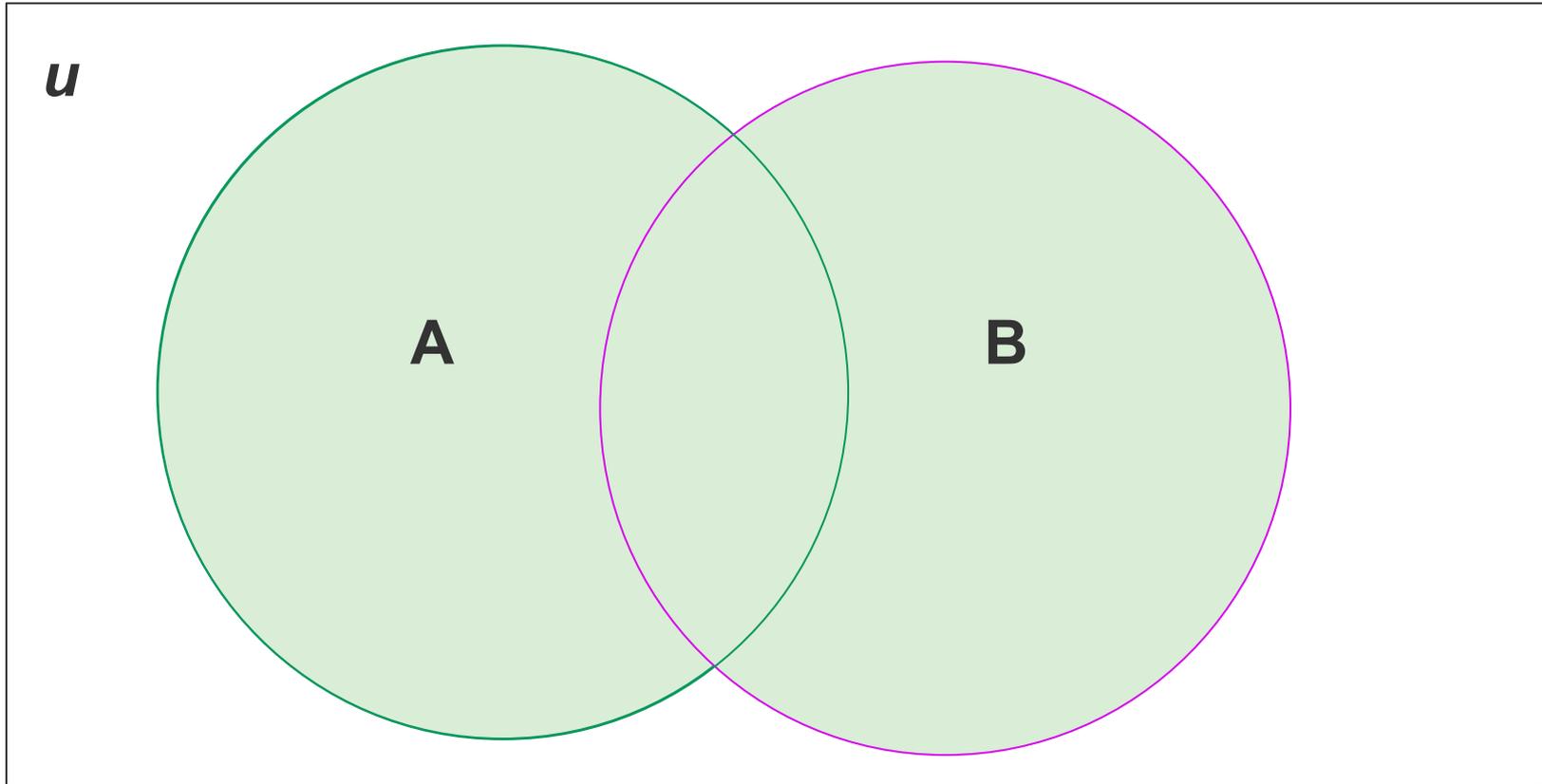
- Subset:  $A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
- Equality:  $A = B \equiv \forall x(x \in A \leftrightarrow x \in B) \equiv A \subseteq B \wedge B \subseteq A$
- Union:  $A \cup B = \{x: x \in A \vee x \in B\}$
- Intersection:  $A \cap B = \{x: x \in A \wedge x \in B\}$
- Complement:  $\overline{A} = \{x: x \notin A\}$
- Difference:  $A \setminus B = \{x: x \in A \wedge x \notin B\}$
- Cartesian Product:  $A \times B = \{(a, b): a \in A \wedge b \in B\}$

# Understand Sets Visually!



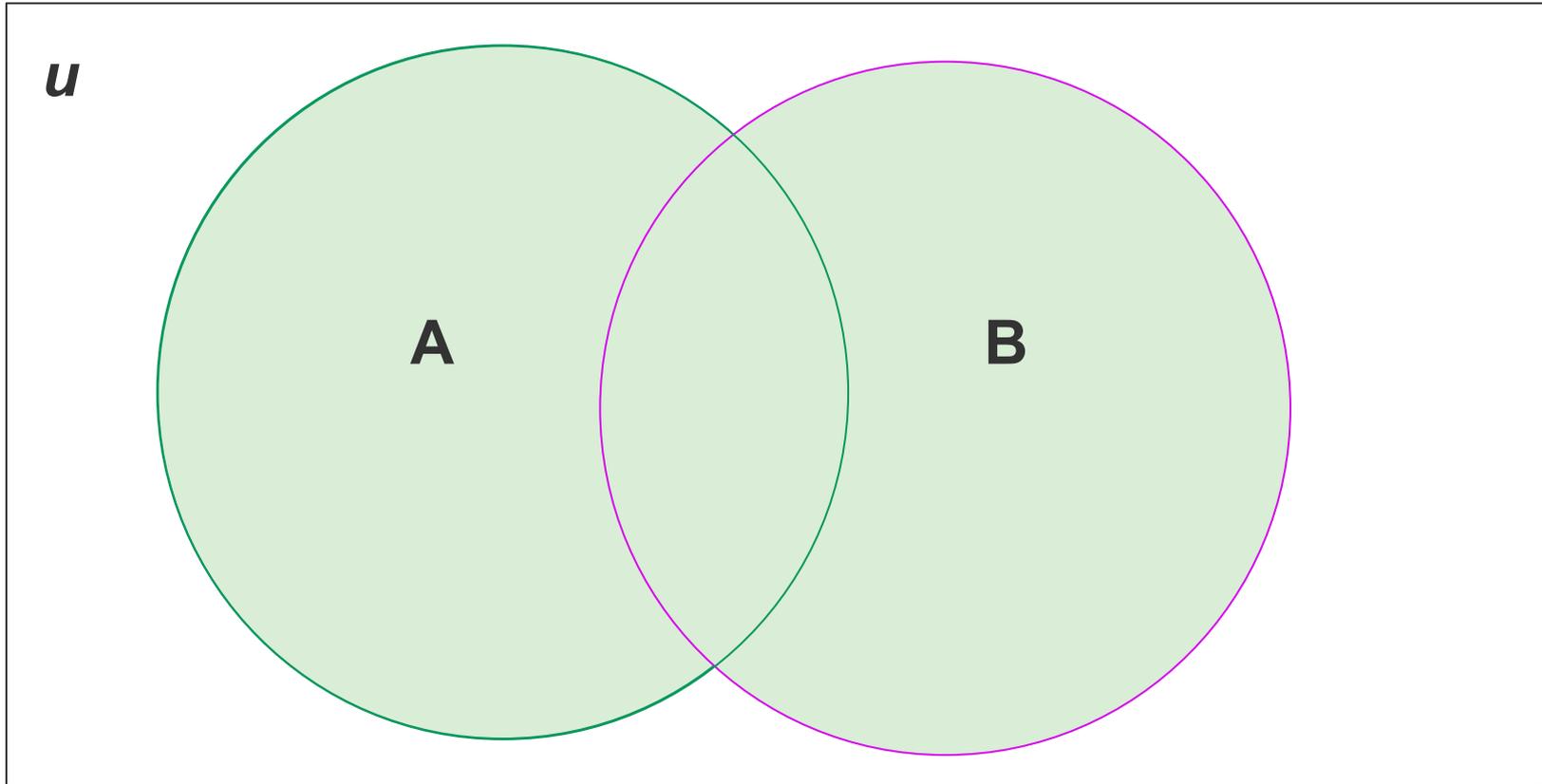
# Understand Sets Visually!

What Set Operation is this?



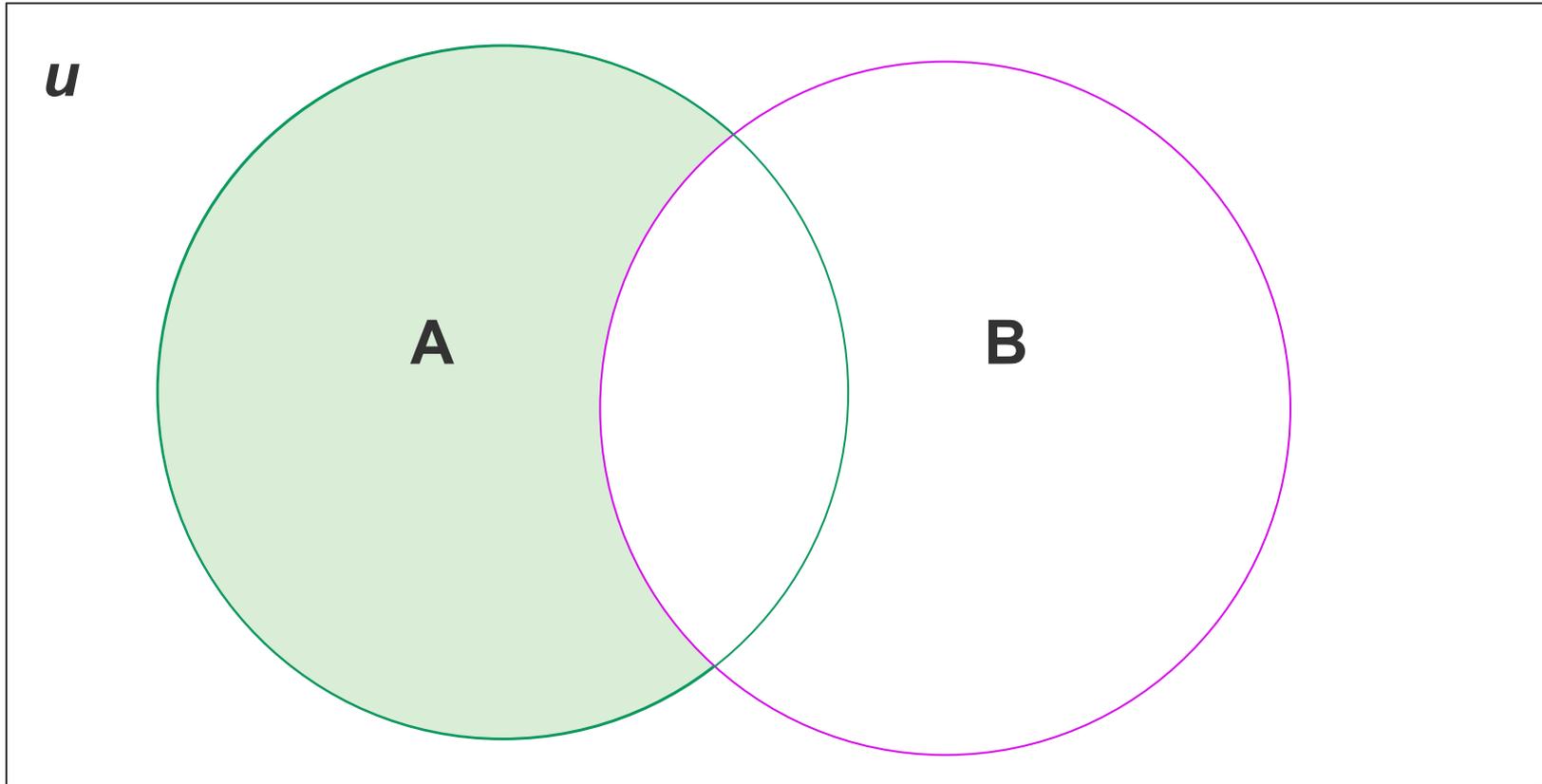
# Understand Sets Visually!

Union:  $A \cup B$



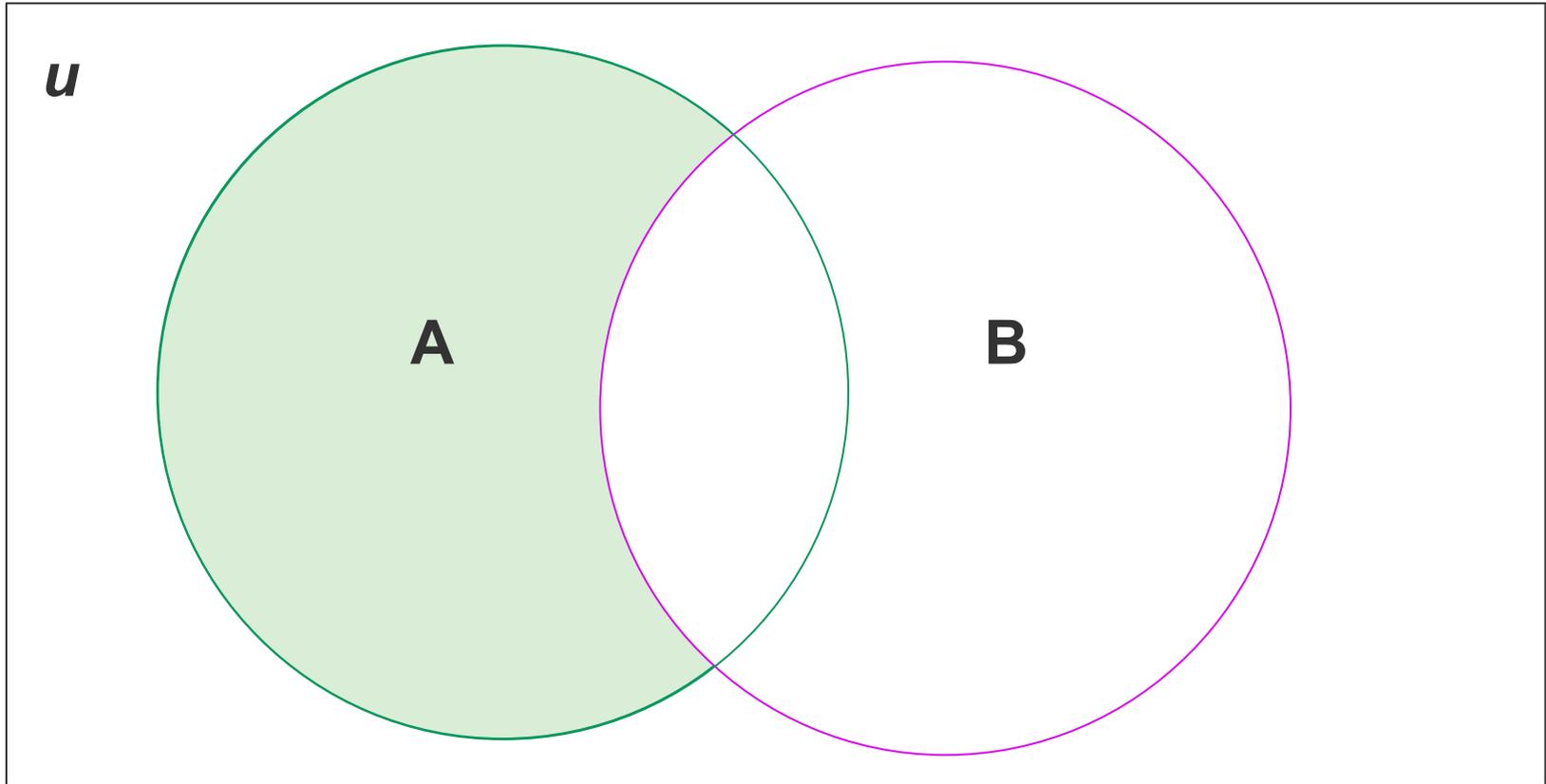
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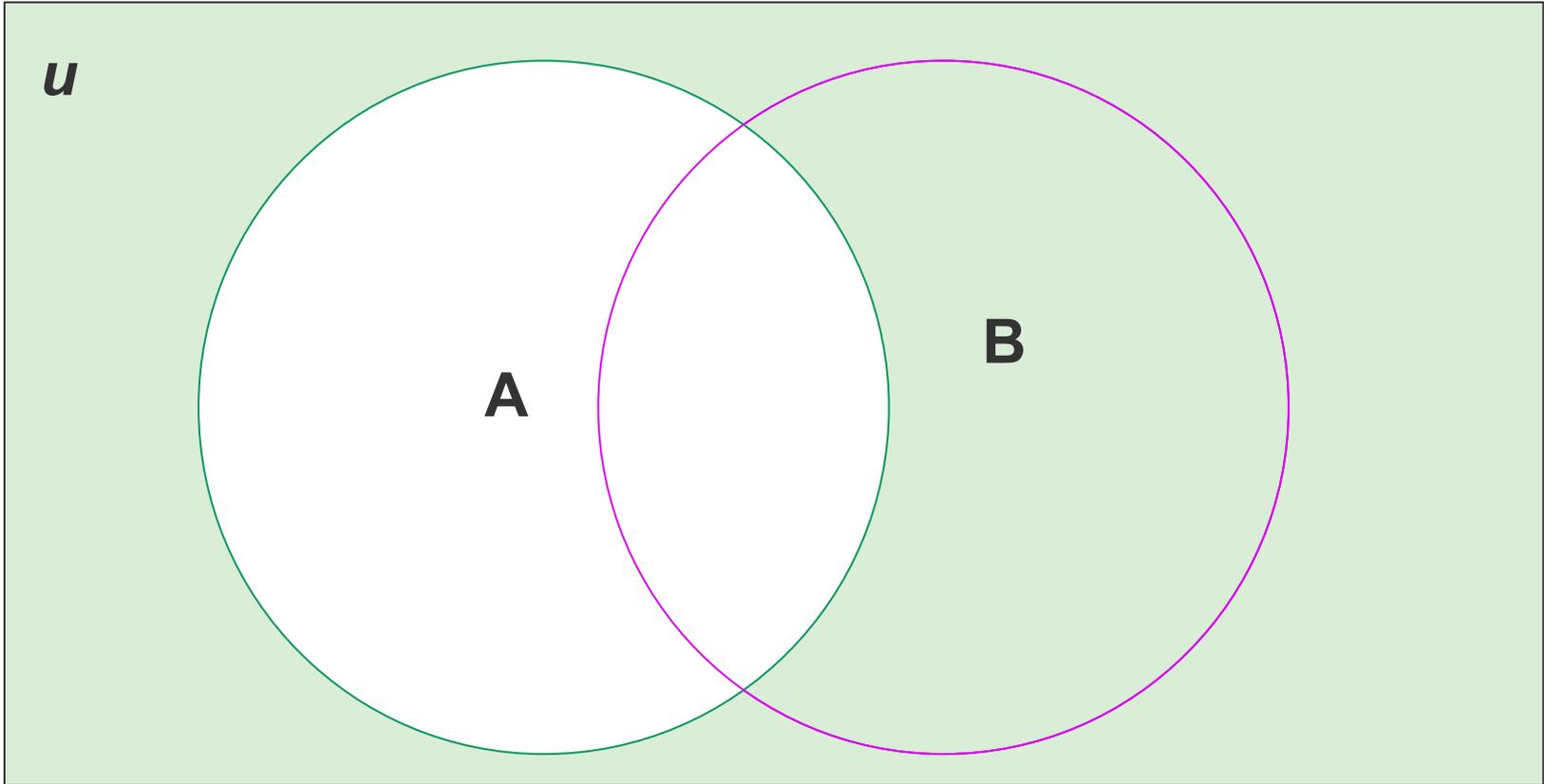
# Understand Sets Visually!

Difference:  $A \setminus B$



# Understand Sets Visually!

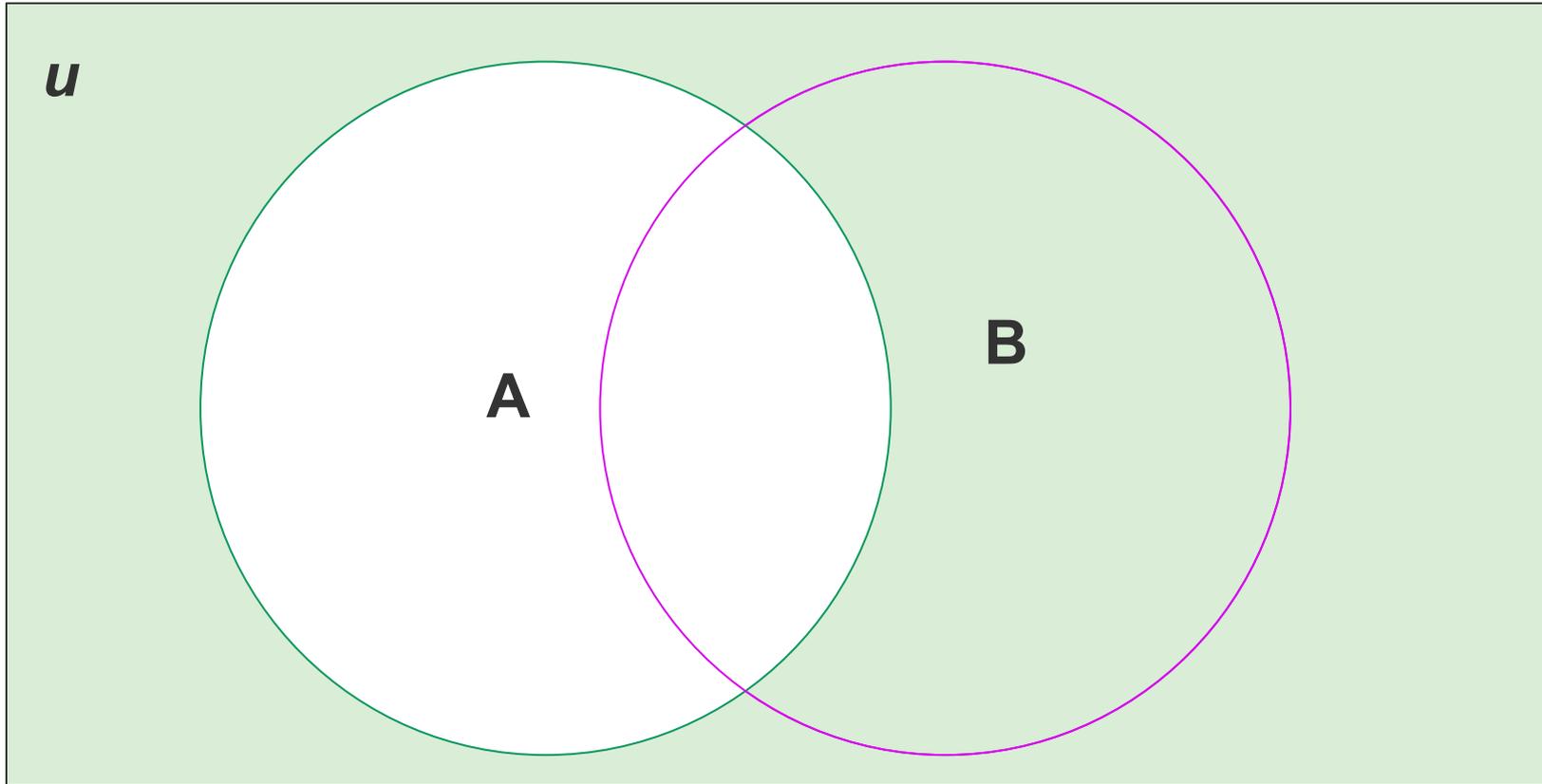
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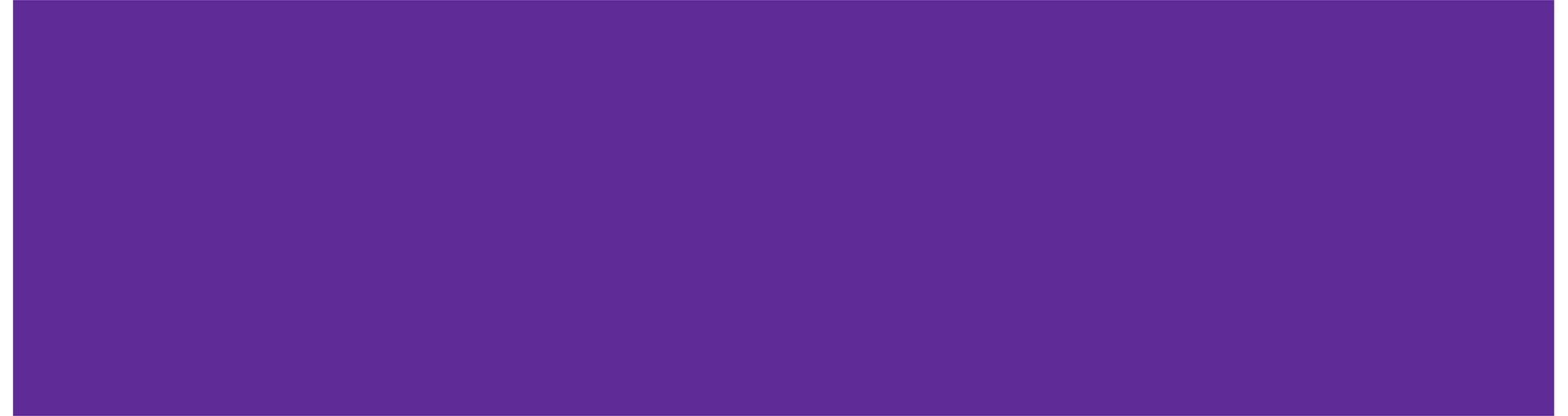
# Understand Sets Visually!

A complement:

$\bar{A}$



# Cartesian Product!



# Problem 1: Cartesian Product

Let  $A$ ,  $B$ ,  $C$ , and  $D$  be sets. Consider the following claim:

$$(A \cap B) \times C \subseteq A \times (C \cup D)$$

**a)** Suppose that  $A = \{1, 2\}$ ,  $B = \{1, 2, 3\}$ ,  $C = \{3, 4\}$ ,  $D = \{2\}$ .

Calculate the values of the sets  $(A \cap B) \times C$  and  $A \times (C \cup D)$ . Check whether the claim holds.

# Cartesian Product

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$$A \cap B = \{1, 2\}$$

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$$A \cap B = \{1, 2\}$$

$$(A \cap B) \times C = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

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$$A \cap B = \{1, 2\}$$

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$$C \cup D = \{2, 3, 4\}$$

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$$A \cap B = \{1, 2\}$$

$$(A \cap B) \times C = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

$$C \cup D = \{2, 3, 4\}$$

$$A \times (C \cup D) = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$$

# Cartesian Product

Let  $A$ ,  $B$ ,  $C$ , and  $D$  be sets. Consider the following claim:

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Calculate the values of the sets  $(A \cap B) \times C$  and  $A \times (C \cup D)$ . Check whether the claim holds.

$$A \cap B = \{1, 2\}$$

$$(A \cap B) \times C = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$$

$$C \cup D = \{2, 3, 4\}$$

$$A \times (C \cup D) = \{(1, 2), (1, 3), (1, 4), (2, 2), (2, 3), (2, 4)\}$$

We can see that  $(A \cap B) \times C \subseteq A \times (C \cup D)$ . The claim holds.

# Cartesian Product

$$(A \cap B) \times C \subseteq A \times (C \cup D)$$

- c) Write an **English proof** that the claim holds.

Follow the structure of our template for subset proofs.

**Note:** even though we want you to write your proof directly in English, it must still look like the translation of a formal proof. In particular, you must include all steps that would be required of a formal proof, excepting only those that we have explicitly said are okay to skip in English proofs.

# Cartesian Product

$$(A \cap B) \times C \subseteq A \times (C \cup D)$$

Let  $x$  be an **arbitrary** object.

Since  $x$  was **arbitrary**, we have shown that  $(A \cap B) \times C \subseteq A \times (C \cup D)$  by the definition of subset.

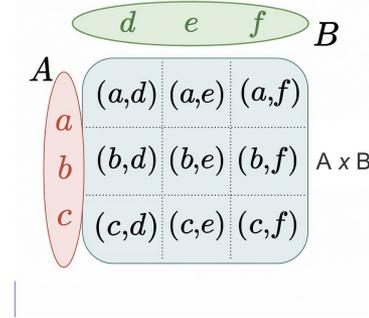
# Cartesian Product

$$(A \cap B) \times C \subseteq A \times (C \cup D)$$

Let  $x$  be an **arbitrary** object.

Suppose that  $x \in (A \cap B) \times C$ .

Remember,  $x$  is just a coordinate point!



Since  $x$  was **arbitrary**, we have shown that  $(A \cap B) \times C \subseteq A \times (C \cup D)$  by the definition of subset.

# Cartesian Product

$$(A \cap B) \times C \subseteq A \times (C \cup D)$$

Let  $x$  be an **arbitrary** object.

Suppose that  $x \in (A \cap B) \times C$

Then, by definition of Cartesian product, there is some  $y \in A \cap B$  and  $c \in C$  such that  $x = (y, c)$ .

Since  $x$  was **arbitrary**, we have shown that  $(A \cap B) \times C \subseteq A \times (C \cup D)$  by the definition of subset.

# Cartesian Product

$$(A \cap B) \times C \subseteq A \times (C \cup D)$$

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Suppose that  $x \in (A \cap B) \times C$

Then, by definition of Cartesian product, there is some  $y \in A \cap B$  and  $c \in C$  such that  $x = (y, c)$ .

Then, by the definition of intersection,  $y \in A$  and  $y \in B$ .

Since  $x$  was **arbitrary**, we have shown that  $(A \cap B) \times C \subseteq A \times (C \cup D)$  by the definition of subset.

# Cartesian Product

$$(A \cap B) \times C \subseteq A \times (C \cup D)$$

Work a step back!

Let  $x$  be an **arbitrary** object.

Suppose that  $x \in (A \cap B) \times C$

Then, by definition of Cartesian product, there is some  $y \in A \cap B$  and  $c \in C$  such that  $x = (y, c)$ .

Then, by the definition of intersection,  $y \in A$  and  $y \in B$ .

Since  $y \in A$  and  $c \in C \cup D$ , we can see that  $x = (y, c) \in A \times (C \cup D)$  by the definition of Cartesian product.

Since  $x$  was **arbitrary**, we have shown that  $(A \cap B) \times C \subseteq A \times (C \cup D)$  by the definition of subset.

# Cartesian Product

$$(A \cap B) \times C \subseteq A \times (C \cup D)$$

Let  $x$  be an **arbitrary** object.

Suppose that  $x \in (A \cap B) \times C$

Then, by definition of Cartesian product, there is some  $y \in A \cap B$  and  $c \in C$  such that  $x = (y, c)$ .

Then, by the definition of intersection,  $y \in A$  and  $y \in B$ .

Since  $c \in C$ , then  $c \in C$  or  $c \in D$ . Then, by definition of union,  $c \in C \cup D$ .

Since  $y \in A$  and  $c \in C \cup D$ , we can see that  $x = (y, c) \in A \times (C \cup D)$  by the definition of Cartesian product.

Since  $x$  was **arbitrary**, we have shown that  $(A \cap B) \times C \subseteq A \times (C \cup D)$  by the definition of subset.

# Structural Induction



# Idea of Structural Induction

Every **element is built up recursively...**

So to show  $P(s)$  for all  $s...$

Show  $P(b)$  for all base case elements  $b$ .

Show for an arbitrary element not in the base case, if  $P()$  holds for every named element in the recursive rule, then  $P()$  holds for the new element (each recursive rule will be a case of this proof).

# Structural Induction Template

Let  $P(x)$  be “<predicate>”. We show  $P(x)$  holds for all  $x$  by structural induction.

Base Case: Show  $P(x)$

[Do that for every base cases  $x$ .]

Inductive Hypothesis: Suppose  $P(x)$  for an arbitrary  $x$

[Do that for every  $x$  listed as in  $S$  in the recursive rules.]

Inductive Step: Show  $P(y)$  holds for  $y$ .

[You will need a separate case/step for every recursive rule.]

Therefore  $P(x)$  holds for all  $x$  by the principle of induction.

# Problem 4 - Structural Induction

Let  $P(L)$  be “”.

We show  $P(L)$  holds for all  $L \dots$  by structural induction on  $L$ .

Base Case: Show  $P(L)$  (for all  $L$  in the basis rules)

Inductive Hypothesis: Suppose  $P(L)$  (for all  $L$  in the recursive rules),  
i.e. (IH in terms of  $P(L)$ )

Inductive Step: Goal: Show that  $P(?)$  holds. (IS goal in terms of  $P(?)$ )

Conclusion: Therefore  $P(L)$  holds for all  $L \in S$  by the principle of induction.

# Problem 4 - Structural Induction

Let  $P(L)$  be “ $\text{len}(\text{echo-pos}(L)) \leq 2 \text{len}(L)$ ”.

We show  $P(L)$  holds for all lists  $L$  by structural induction on  $L$ .

Base Case: Show  $P(L)$  (for all  $L$  in the basis rules)

Inductive Hypothesis: Suppose  $P(L)$  (for all  $L$  in the recursive rules),  
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We show  $P(L)$  holds for all lists  $L$  by structural induction on  $L$ .

Base Case: Show  $P(\text{nil})$

$$\text{len}(\text{echo-pos}(\text{nil})) \leq \text{len}(\text{nil})$$

So  $P(\text{nil})$  holds.

Inductive Hypothesis: Suppose  $P(L)$  (for all  $L$  in the recursive rules),  
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Inductive Hypothesis: Suppose  $P(L)$  holds for an arbitrary  $L \in \text{List}$

Inductive Step: Goal: Show that  $P(a::L)$  holds.

Conclusion: Therefore  $P(L)$  holds for all  $L \in S$  by the principle of induction.

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Inductive Hypothesis: Suppose  $P(L)$  holds for an arbitrary  $L \in \text{List}$

Inductive Step: Goal: Show that  $P(a::L)$  holds. Let  $a \in Z$  be arbitrary.

Conclusion: Therefore  $P(L)$  holds for all  $L \in S$  by the principle of induction.

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So  $P(\text{nil})$  holds.

Inductive Hypothesis: Suppose  $P(L)$  holds for an arbitrary  $L \in \text{List}$

Inductive Step: Goal: Show that  $P(a::L)$  holds. Let  $a \in Z$  be arbitrary.

Suppose that  $a \leq 0$

**We can do casework!**

Suppose that  $a > 0$

Conclusion: Therefore  $P(L)$  holds for all  $L \in S$  by the principle of induction.

# Problem 4 - Structural Induction

Inductive Step: Goal: Show that  $P(a::L)$  holds. Let  $a \in \mathbb{Z}$  be arbitrary.

Suppose that  $a \leq 0$

Suppose that  $a > 0$

$$\begin{aligned}\text{len}(\text{nil}) &:= 0 \\ \text{len}(a :: L) &:= 1 + \text{len}(L) \quad \forall a \in \mathbb{Z}, \forall L \in \mathbf{List}\end{aligned}$$

The second function, echo-pos, which duplicates each positive number in the list, is defined by:

$$\begin{aligned}\text{echo-pos}(\text{nil}) &:= \text{nil} \\ \text{echo-pos}(a :: L) &:= a :: \text{echo-pos}(L) && \text{if } a \leq 0 \quad \forall a \in \mathbb{Z}, \forall L \in \mathbf{List} \\ \text{echo-pos}(a :: L) &:= a :: a :: \text{echo-pos}(L) && \text{if } a > 0 \quad \forall a \in \mathbb{Z}, \forall L \in \mathbf{List}\end{aligned}$$

# Problem 4 - Structural Induction

Inductive Step: Goal: Show that  $P(a::L)$  holds. Let  $a \in \mathbb{Z}$  be arbitrary.

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$$\text{len}(\text{echo-pos}(a::L)) =$$

Suppose that  $a > 0$

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# Problem 4 - Structural Induction

Inductive Step: Goal: Show that  $P(a::L)$  holds. Let  $a \in \mathbb{Z}$  be arbitrary.

Suppose that  $a \leq 0$

$$\text{len}(\text{echo-pos}(a::L)) = \text{len}(a::\text{echo-pos}(L)) \quad \text{Def of echo-pos (since } a \leq 0)$$

Suppose that  $a > 0$

$$\begin{aligned} \text{len}(\text{nil}) &:= 0 \\ \text{len}(a :: L) &:= 1 + \text{len}(L) \quad \forall a \in \mathbb{Z}, \forall L \in \mathbf{List} \end{aligned}$$

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Inductive Step: Goal: Show that  $P(a::L)$  holds. Let  $a \in \mathbb{Z}$  be arbitrary.

Suppose that  $a \leq 0$

$$\begin{aligned}\text{len}(\text{echo-pos}(a::L)) &= \text{len}(a::\text{echo-pos}(L)) \\ &= 1 + \text{len}(\text{echo-pos}(L))\end{aligned}$$

Def of echo-pos (since  $a \leq 0$ )

Def of len

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Suppose that  $a > 0$

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Suppose that  $a > 0$

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# Problem 4 - Structural Induction

Inductive Step: Goal: Show that  $P(a::L)$  holds. Let  $a \in Z$  be arbitrary.

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$$\begin{aligned} \text{len}(\text{echo-pos}(a::L)) &= \text{len}(a::\text{echo-pos}(L)) && \text{Def of echo-pos (since } a \leq 0) \\ &= 1 + \text{len}(\text{echo-pos}(L)) && \text{Def of len} \\ &\leq 1 + 2\text{len}(L) && \text{IH} \\ &\leq 2 + 2\text{len}(L) \\ &= 2(1 + \text{len}(L)) \\ &= 2(\text{len}(a::L)) && \text{Def of len} \end{aligned}$$

Suppose that  $a > 0$

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Since these cases are exhaustive,  $P(a::L)$  holds.

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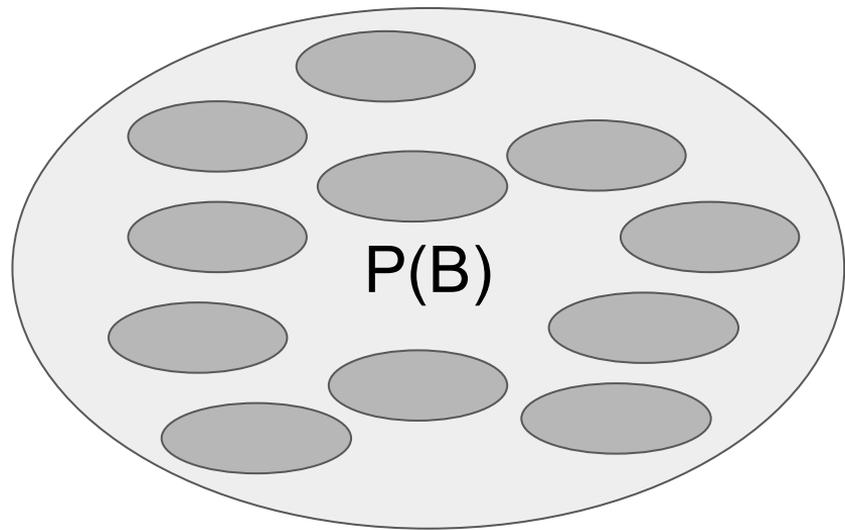
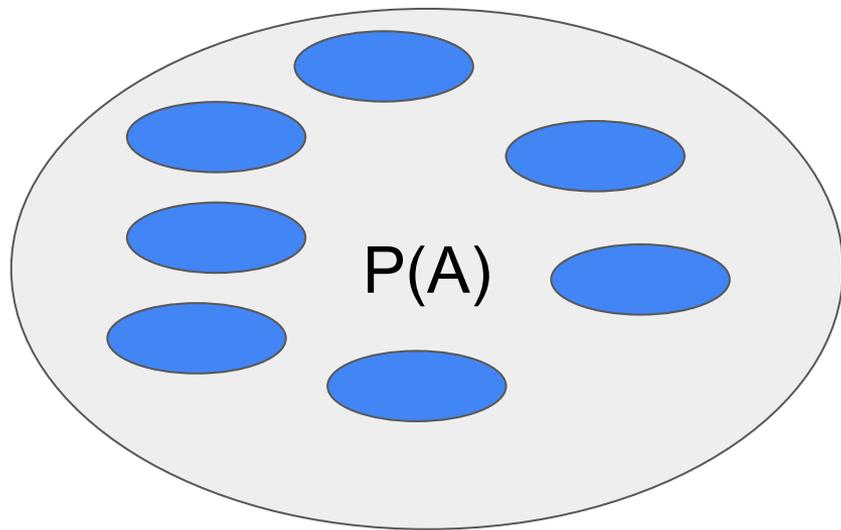
Since these cases are exhaustive,  $P(a::L)$  holds.

Conclusion: Therefore  $P(L)$  holds for all  $L \in \text{List}$  by the principle of induction.

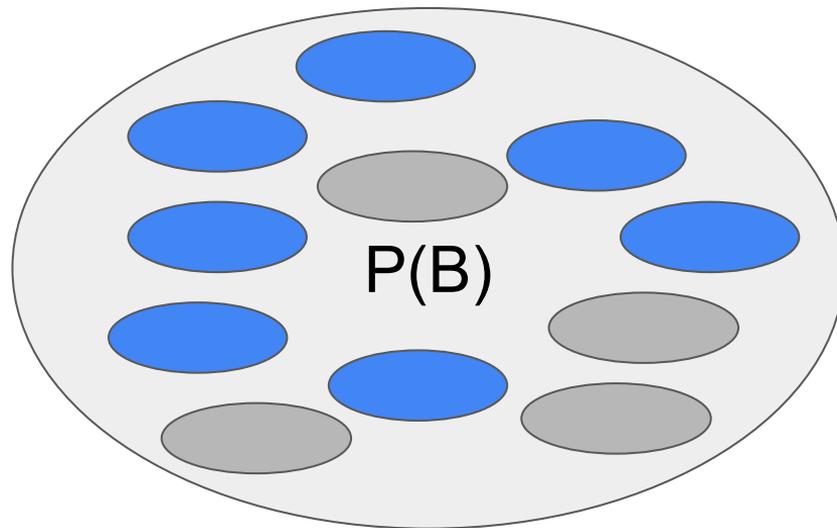
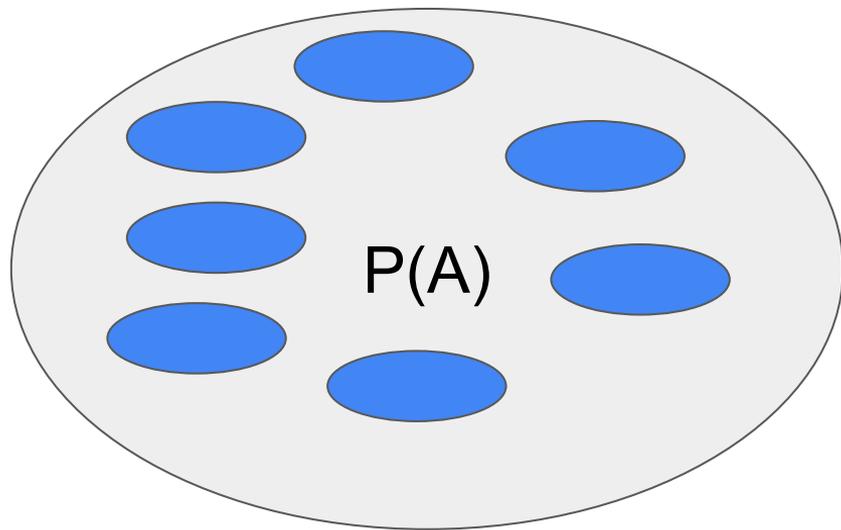
# PowerSet English Proof



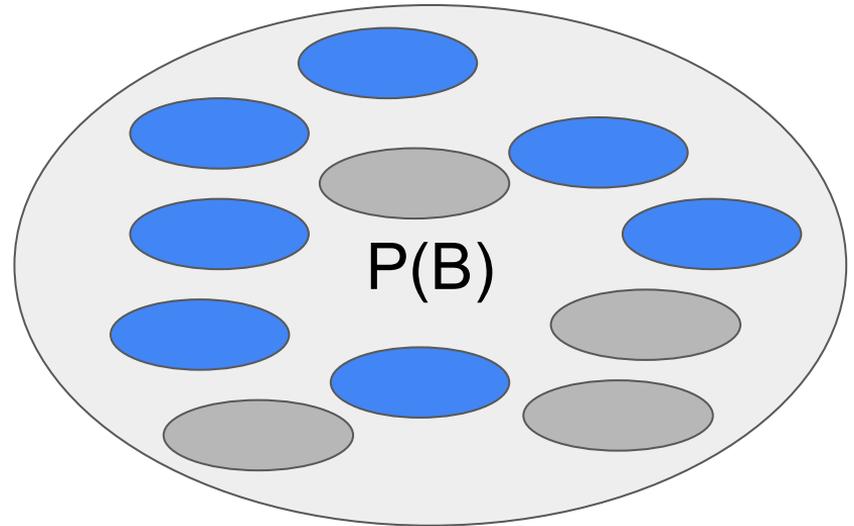
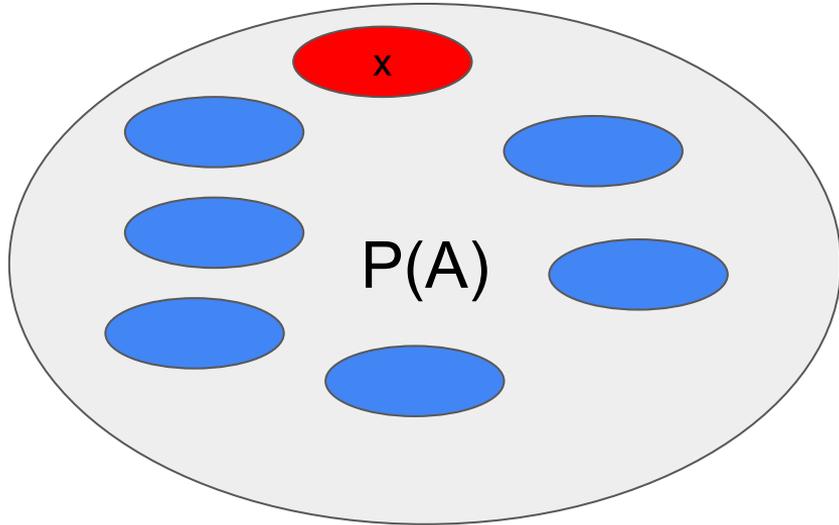
Given  $A \subseteq B$ , we want  $P(A) \subseteq P(B)$



To show  $P(A) \subseteq P(B)$ , show that the (set) elements of  $P(A)$  can be found in  $P(B)$

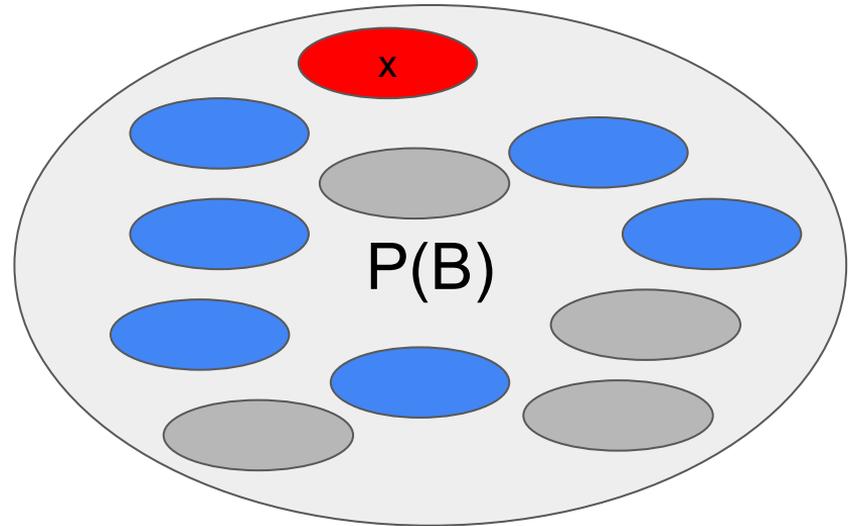
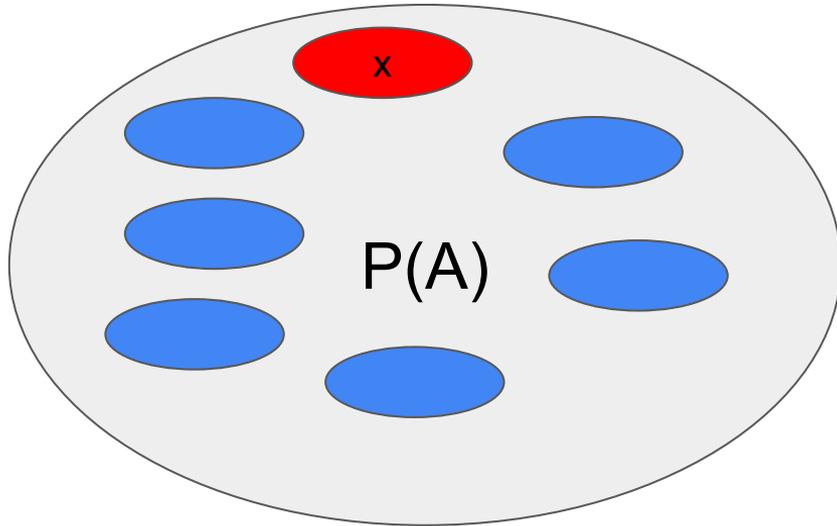


Subset proof strategy: take an arbitrary element  $x$  of  $P(A)$ ...

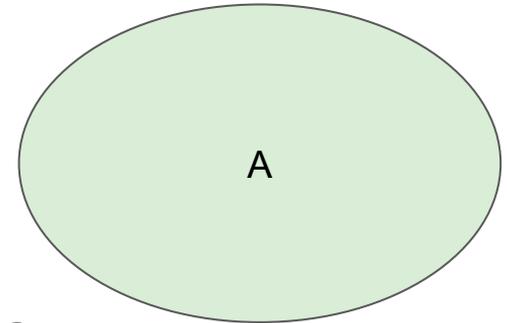
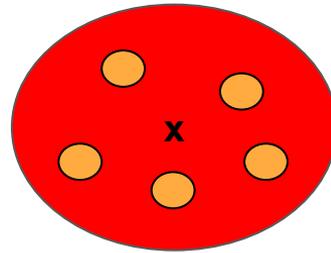
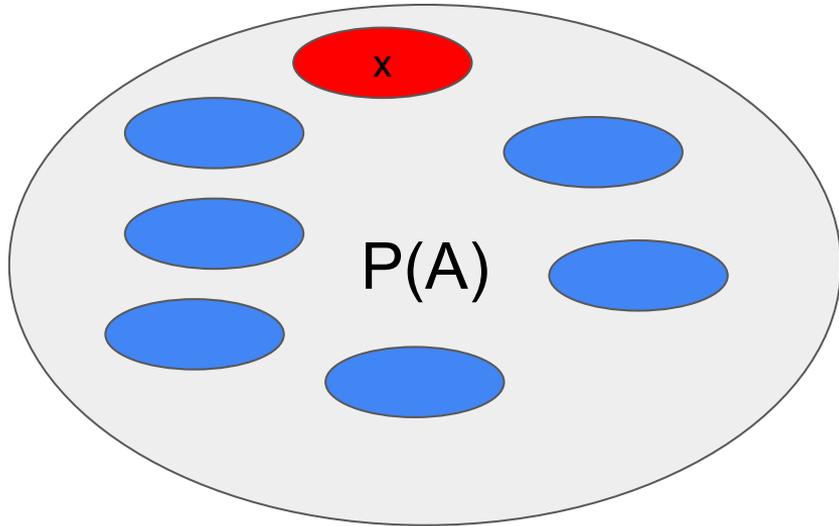


# Subset proof strategy: ... and show that it's in $P(B)$

How do we show  $x$  is in  $P(B)$ ?

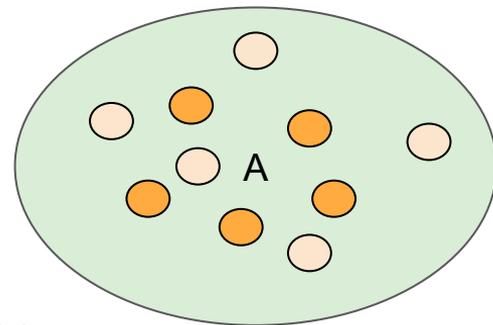
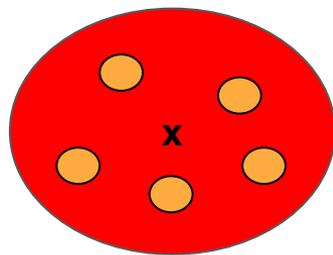
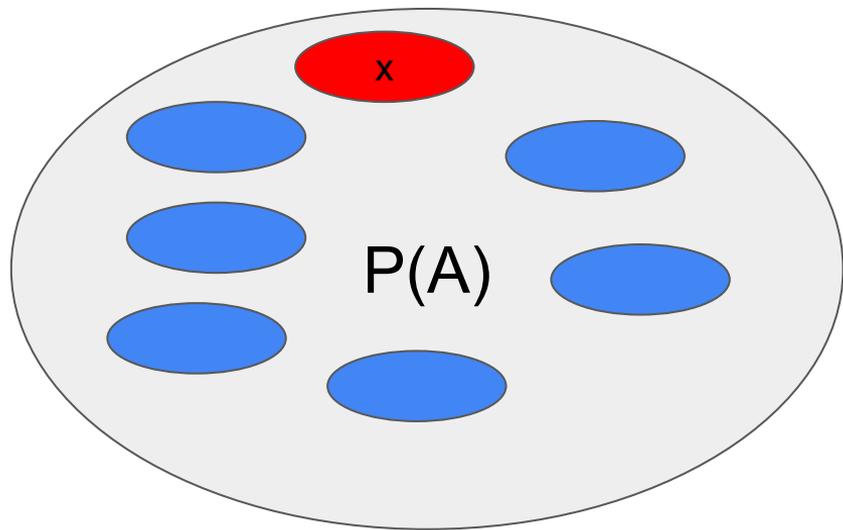


Well,  $x$  is in  $P(A)$ , so  $x \subseteq A$  by definition of powerset. Our target is showing  $x$  is in  $P(B)$ , i.e.,  $x \subseteq B$ .



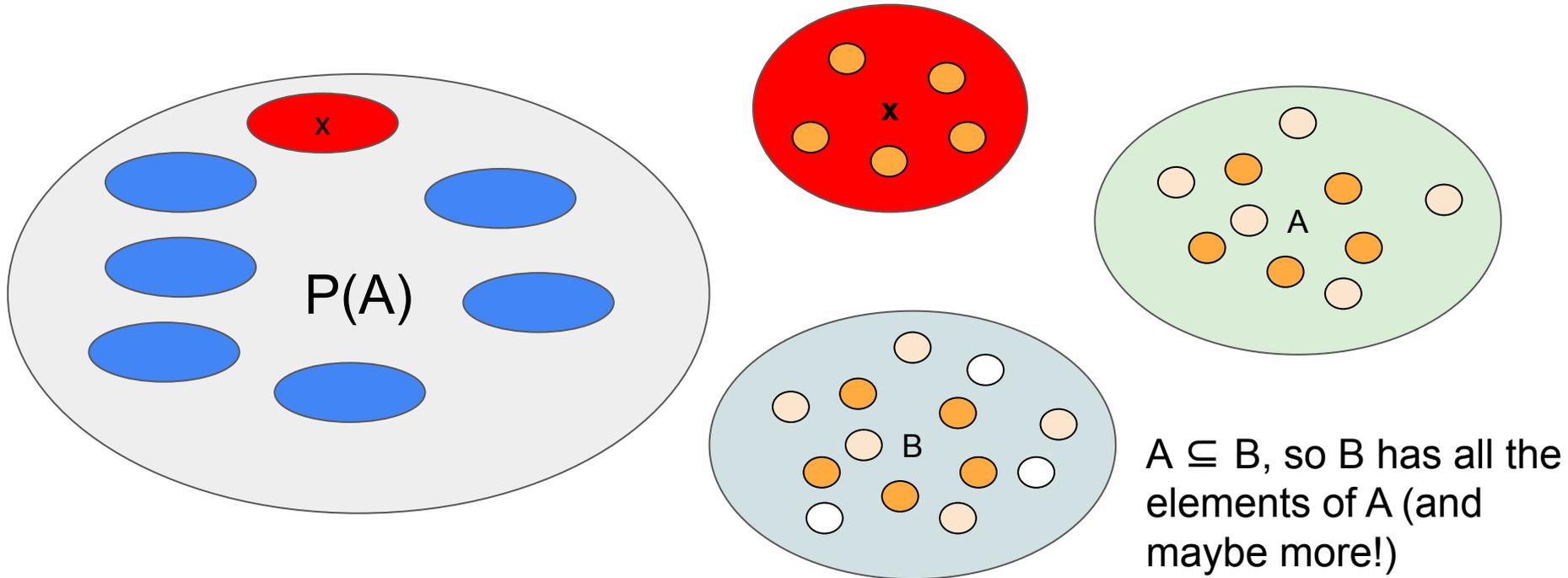
$L$  is a set, so it has elements in it!

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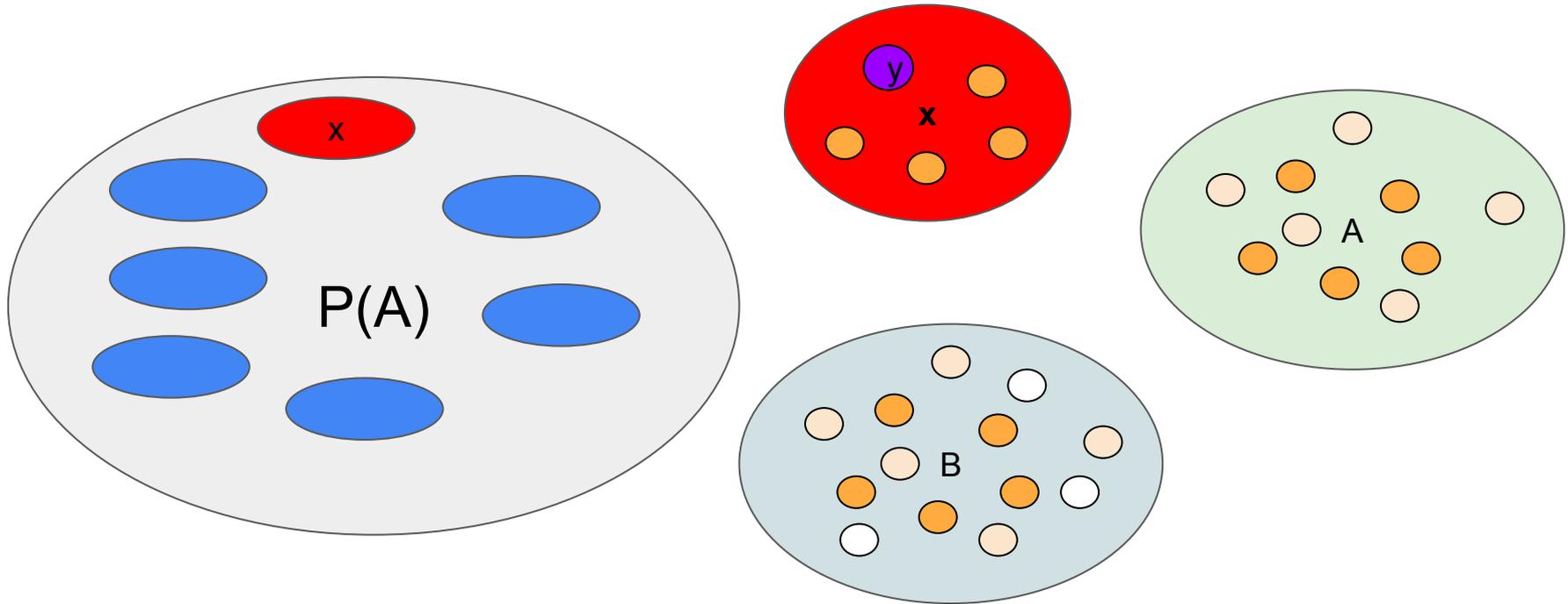


Since  $L \subseteq A$ ,  $A$  has those elements too (and maybe more stuff!)

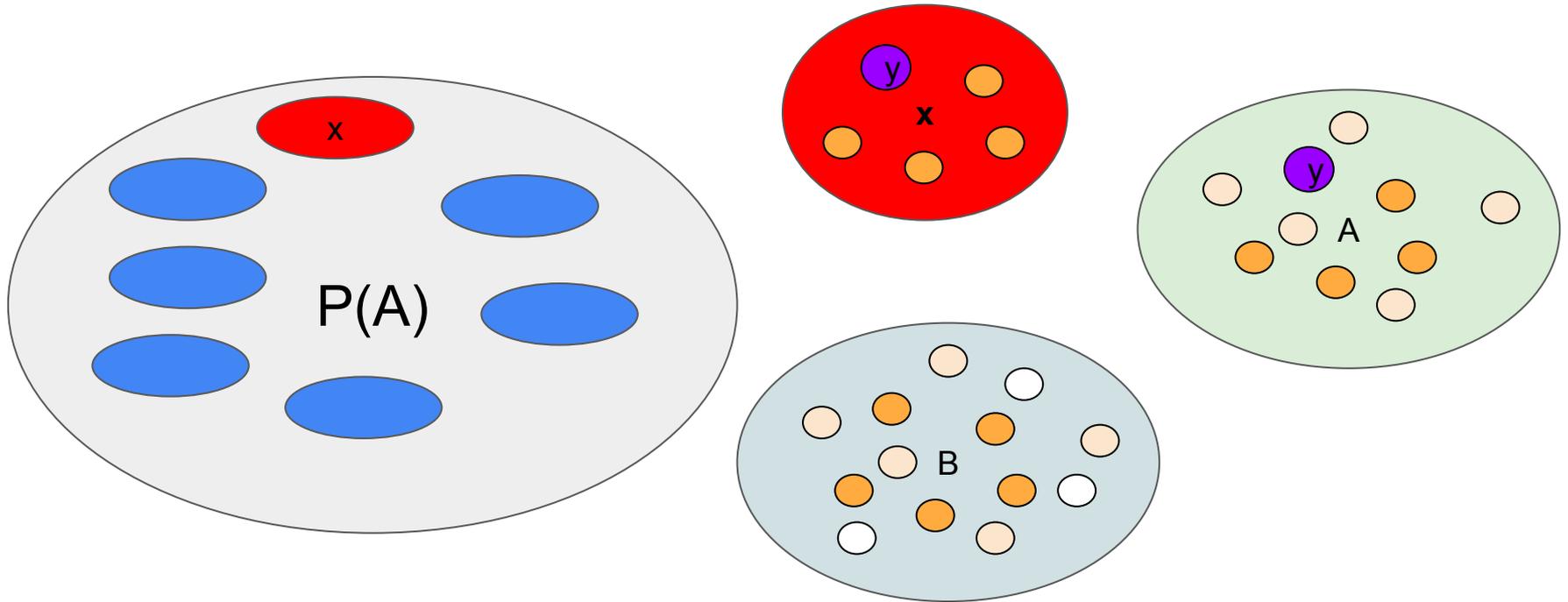
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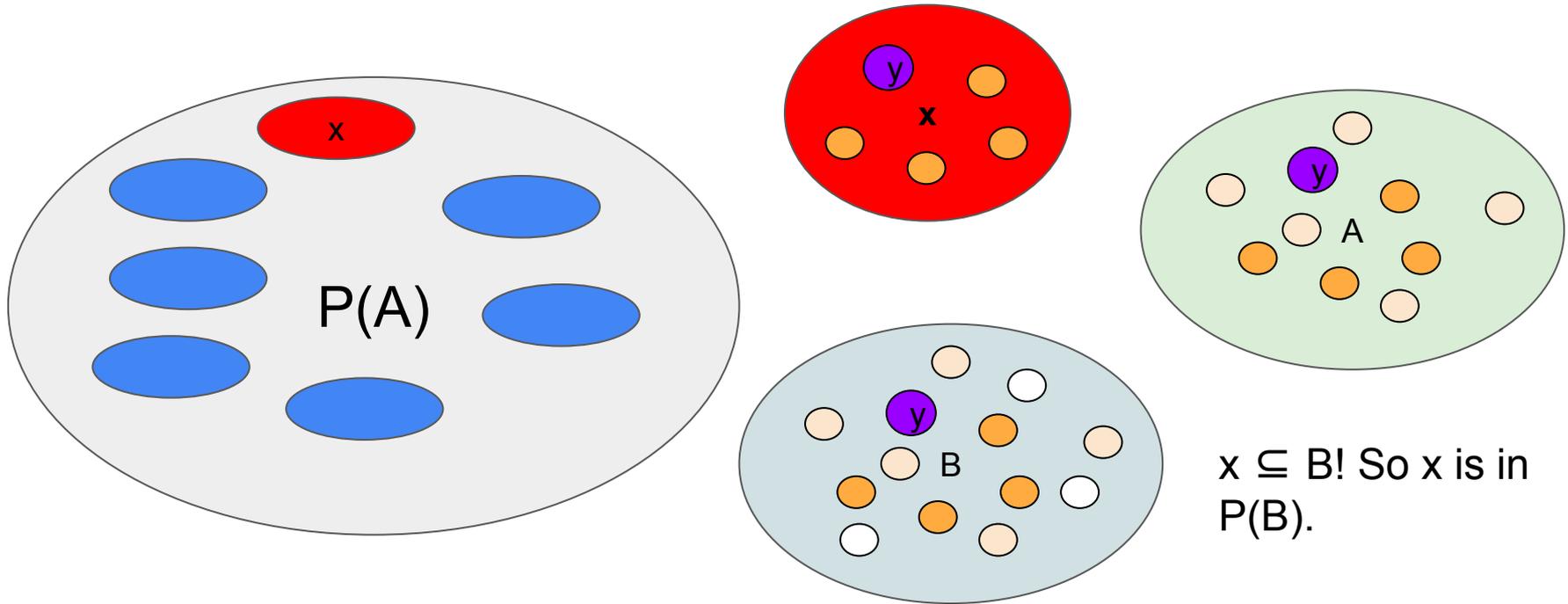
To show  $x \subseteq B$ , we do the subset strategy again: take an arbitrary  $y$  in  $x$ ...



To show  $x \subseteq B$ , we do the subset strategy again: Since  $x \subseteq A$ ,  $y$  is in  $A$ ...



To show  $x \subseteq B$ , we do the subset strategy again: And finally since  $A \subseteq B$ ,  $y$  is in  $B$ .



## Problem 2

Let  $A$  and  $B$  be sets. Prove that  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$  follows from  $A \subseteq B$ .

## Problem 4

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Let  $X$  be an arbitrary set in  $\mathcal{P}(A)$ .

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Let  $x$  be an arbitrary element of  $X$ . Since  $X \subseteq A$ , it must be the case that  $x \in A$ . We were given that  $A \subseteq B$ . By definition of subset, any element of  $A$  is an element of  $B$ . So, it must also be the case that  $x \in B$ .

Since  $x$  was arbitrary, we know any element of  $X$  is an element of  $B$ . By definition of subset,  $X \subseteq B$ . By definition of power set,  $X \in \mathcal{P}(B)$ .

Since  $X$  was an arbitrary set, any set in  $\mathcal{P}(A)$  is in  $\mathcal{P}(B)$ , or, by definition of subset,  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ . We have shown the claim.

**That's all Folks!**

