Quiz Section 3: Number Theory

Task 1 – This is Really Mod

Let n and m be positive integers.

Consider the following claim: for any integers a and b, if $a \equiv_m b$, then $a \equiv_n b$.

a) Write a **formal** proof that the claim holds, *given* that $n \mid m$.

Hint: the claim to prove is $\forall a\, \forall b\, ((a \equiv_m b) \rightarrow (a \equiv_n b))$. The fact we are given is that $n \mid m$.

b) Translate your formal proof to an **English** proof.

Task 2 – Extended Euclidean Algorithm Practice

a) Find the multiplicative inverse of y of 7 mod 33. That is, find y such that $7y \equiv 1 (\text{mod } 33)$, You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \leqslant y < 33$.

b) Now solve $7z \equiv 2$ (mod 33) for all of its integers solutions z.

Task 3 – Induction with Equality

For all $n \in \mathbb{N}$, prove by induction that $\displaystyle\sum_{i=0}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

Task 4 – Strong Induction

Consider the function $a(n)$ defined for $n \geqslant 1$ recursively as follows.

$$a(1) = 1$$

$$a(2) = 3$$

$$a(n) = 2a(n-1) - a(n-2) \text{ for } n \geqslant 3$$

Use strong induction to prove that $a(n) = 2n - 1$ for all integers $n \geqslant 1$.