CSE 311 Section 2

Quantifiers and Proofs

Administrivia & Introductions

Announcements & Reminders

- HW1 out
  - If you think something was graded incorrectly, submit a regrade request!
  - Regrades generally will be open for a week

- HW2 is due Monday 1/26 on Gradescope
  - Use a late day if you need to!
  - Gradescope: Make sure you select the pages for each question correctly
  - !! Selecting the pages after the deadline won't mark it as late

Formal Proofs

Inference Rules to Remember

Direct Proof

$$\frac{A \Rightarrow B}{\therefore \quad A \rightarrow B}$$

Modus Ponens

$$\frac{A \qquad A \rightarrow B}{\therefore \qquad B}$$

Tautology

$$\frac{A \equiv \top}{\therefore \quad A}$$

Intro ∧

$$\frac{A \qquad B}{\therefore \quad A \wedge B}$$

Elim ∧

$$\frac{A \wedge B}{\therefore \ A \qquad B}$$

Equivalent

$$\frac{A \equiv B \qquad B}{\therefore \qquad A}$$

Intro ∨

$$\frac{A}{\therefore \ A \vee B \quad B \vee A}$$

Elim ∨

$$\frac{A \vee B \qquad \neg A}{\therefore \qquad B}$$

Proof By Cases

$$\frac{A \vee B \quad A \rightarrow C \quad B \rightarrow C}{\therefore \qquad C}$$

Relevant Rules for Problem 1b

Elim $\wedge$

$$\frac{A \wedge B}{\therefore\ A \qquad\qquad B}$$

Proof By Cases

$$\frac{A \vee B \quad A \rightarrow C \quad B \rightarrow C}{\therefore\qquad\qquad C}$$

Problem 1b - Formal Proof

Given $(a \rightarrow b)$, $(c \rightarrow b)$, $a \vee (c \wedge d)$, show that b holds.

Lets get setup:

Problem 1b - Formal Proof

Given $(a \rightarrow b)$, $(c \rightarrow b)$, $a \vee (c \wedge d)$, show that b holds.

Lets get setup:

1.	$a \rightarrow b$	Given
2.	$c \rightarrow b$	Given
3.	$a \vee (c \wedge d)$	Given
	b	

Problem 1b - Formal Proof

Given $(a \to b)$, $(c \to b)$, $a \vee (c \wedge d)$, show that b holds.

Initial observation:

1.	$a \to \boxed{b}$	Given
2.	$c \to \boxed{b}$	Given
3.	$a \vee (c \wedge d)$	Given

$\boxed{b}$

Problem 1b - Formal Proof

Given $(a \rightarrow b)$, $(c \rightarrow b)$, $a \vee (c \wedge d)$, show that b holds.

Initial observation:
if we get a or c,
then we can get to b

1.	$a \rightarrow b$	Given
2.	$c \rightarrow b$	Given
3.	$a \vee (c \wedge d)$	Given

b

Problem 1b - Formal Proof

Given $(a \to b)$, $(c \to b)$, $a \lor (c \land d)$, show that b holds.

We can work a step back!

1. $a \to b$ Given
2. $c \to b$ Given
3. $a \lor (c \land d)$ Given

$(a \lor c)$

b

Problem 1b - Formal Proof

Given $(a \to b)$, $(c \to b)$, $a \vee (c \wedge d)$, show that b holds.

What is this called?

1.	$a \to b$	Given
2.	$c \to b$	Given
3.	$a \vee (c \wedge d)$	Given

$(a \vee c)$

b

Problem 1b - Formal Proof

Given $(a \to b)$, $(c \to b)$, $a \vee (c \wedge d)$, show that b holds.

What is this called?

1.	$a \to b$	Given
2.	$c \to b$	Given
3.	$a \vee (c \wedge d)$	Given
	$(a \vee c)$	
	b	Cases: x 1, 2

Problem 1b - Formal Proof

Given $(a \to b)$, $(c \to b)$, $a \vee (c \wedge d)$, show that b holds.

How can we get

$(a \vee c)$

1.	$a \to b$	Given
2.	$c \to b$	Given
3.	$a \vee (c \wedge d)$	Given
	$(a \vee c)$	
	b	Cases: x 1, 2

Problem 1b - Formal Proof

Given $(a \rightarrow b)$, $(c \rightarrow b)$, $a \vee (c \wedge d)$, show that b holds.

Distributivity!

1.	$a \rightarrow b$	Given
2.	$c \rightarrow b$	Given
3.	$a \vee (c \wedge d)$	Given
4.	$(a \vee c) \wedge (a \vee d)$	Distributivity: 3
5.	$(a \vee c)$	
6.	b	Cases: x 1, 2

Problem 1b - Formal Proof

Given $(a \rightarrow b)$, $(c \rightarrow b)$, $a \vee (c \wedge d)$, show that b holds.

What's missing?

1.	$a \rightarrow b$	Given
2.	$c \rightarrow b$	Given
3.	$a \vee (c \wedge d)$	Given
4.	$(a \vee c) \wedge (a \vee d)$	Distributivity: 3
5.	$(a \vee c)$	
6.	b	Cases: x 1, 2

Problem 1b - Formal Proof

Given $(a \to b)$, $(c \to b)$, $a \lor (c \land d)$, show that b holds.

We did it!

1.	$a \to b$	Given
2.	$c \to b$	Given
3.	$a \lor (c \land d)$	Given
4.	$(a \lor c) \land (a \lor d)$	Distributivity: 3
5.	$(a \lor c)$	Elim $\land$: 4
6.	b	Cases: 5, 1, 2

Problem 1b - Formal Proof

Given $(a \rightarrow b)$, $(c \rightarrow b)$, $a \vee (c \wedge d)$, show that b holds.

1.	$a \rightarrow b$	Given
2.	$c \rightarrow b$	Given
3.	$a \vee (c \wedge d)$	Given
4.	$(a \vee c) \wedge (a \vee d)$	Distributivity: 3
5.	$(a \vee c)$	Elim $\wedge$: 4
6.	b	Cases: 5, 1, 2

Direct Proofs

Direct Proof

$$\frac{A \Rightarrow B}{\therefore A \rightarrow B}$$

Introduce an assumption like a new variable when you are conducting an experiment…

You will typically need this new assumption because your Givens alone are not sufficient

Problem 2b

Show that $r \to p$ follows from $p \vee \neg q$, $(r \vee s) \to (q \vee s)$, and $\neg s$.

Problem 2b

Show that $r \to p$ follows from $p \vee \neg q$, $(r \vee s) \to (q \vee s)$, and $\neg s$.

Just the setup:

1.	$p \vee \neg q$	[Given]
2.	$(r \vee s) \to (q \vee s)$	[Given]
3.	$\neg s$	[Given]

$r \to p$

Problem 2b

Show that $r \to p$ follows from $p \vee \neg q$, $(r \vee s) \to (q \vee s)$, and $\neg s$.

How do we conclude if r then p?

r does not exist alone…

(2) contains r but we cannot elim or here…

1.	$p \vee \neg q$	[Given]
2.	$(r \vee s) \to (q \vee s)$	[Given]
3.	$\neg s$	[Given]

$r \to p$

Problem 2b

Show that $r \rightarrow p$ follows from $p \vee \neg q$, $(r \vee s) \rightarrow (q \vee s)$, and $\neg s$.

1.	$p \vee \neg q$	[Given]
2.	$(r \vee s) \rightarrow (q \vee s)$	[Given]
3.	$\neg s$	[Given]

How do we conclude if r then p?

r does not exist alone…

Could we assume r?

$$r \rightarrow p$$

Problem 2b

Show that $r \rightarrow p$ follows from $p \vee \neg q$, $(r \vee s) \rightarrow (q \vee s)$, and $\neg s$.

1.	$p \vee \neg q$	[Given]
2.	$(r \vee s) \rightarrow (q \vee s)$	[Given]
3.	$\neg s$	[Given]

How do we conclude
if r then p?

r does not exist alone…

Could we assume r?
Yes! Let's use **direct proof rule**!

$$r \rightarrow p$$

Problem 2b

Show that $r \rightarrow p$ follows from $p \vee \neg q$, $(r \vee s) \rightarrow (q \vee s)$, and $\neg s$.

How do we conclude p ?

1.	$p \vee \neg q$	[Given]
2.	$(r \vee s) \rightarrow (q \vee s)$	[Given]
3.	$\neg s$	[Given]
4.1.	r	[Assumption]
4	p	
	$r \rightarrow p$	[Direct Proof Rule]

Problem 2b

Show that $r \to p$ follows from $p \vee \neg q$, $(r \vee s) \to (q \vee s)$, and $\neg s$.

How do we conclude p ?

Since we have r, can we use line 2?

1.	$p \vee \neg q$	[Given]
2.	$(r \vee s) \to (q \vee s)$	[Given]
3.	$\neg s$	[Given]
4.1.	r	[Assumption]
4	p	
	$r \to p$	[Direct Proof Rule]

Problem 2b

Show that $r \to p$ follows from $p \vee \neg q$, $(r \vee s) \to (q \vee s)$, and $\neg s$.

How do we conclude p ?

Since we have r, can we use line 2?
Almost! Let's create the left hand side of line 2

1.	$p \vee \neg q$	[Given]
2.	$(r \vee s) \to (q \vee s)$	[Given]
3.	$\neg s$	[Given]
4.1.	r	[Assumption]
4.2.	$r \vee s$	[$\vee$intro, 4.1]
4	p	
	$r \to p$	[Direct Proof Rule]

Problem 2b

Show that $r \to p$ follows from $p \vee \neg q$, $(r \vee s) \to (q \vee s)$, and $\neg s$.

How do we conclude p ?

Next: Modus Ponens!

1.	$p \vee \neg q$		[Given]
2.	$(r \vee s) \to (q \vee s)$		[Given]
3.	$\neg s$		[Given]
4.1.	r	[Assumption]	
4.2.	$r \vee s$	[$\vee$intro, 4.1]	
4	p		
	$r \to p$		[Direct Proof Rule]

Problem 2b

Show that $r \to p$ follows from $p \vee \neg q$, $(r \vee s) \to (q \vee s)$, and $\neg s$.

How do we conclude p ?

Next: Modus Ponens!

1. $p \vee \neg q$ [Given]
2. $(r \vee s) \to (q \vee s)$ [Given]
3. $\neg s$ [Given]

4.1. r [Assumption]

4.2. $r \vee s$ [$\vee$intro, 4.1]

4.3. $q \vee s$ [MP 4.2, 2]

4 p

$r \to p$ [Direct Proof Rule]

Problem 2b

Show that $r \rightarrow p$ follows from $p \vee \neg q$, $(r \vee s) \rightarrow (q \vee s)$, and $\neg s$.

How do we conclude p ?

We should use **q** to get to p…
How can we get **q** alone?

1. $p \vee \neg q$ [Given]
2. $(r \vee s) \rightarrow (q \vee s)$ [Given]
3. $\neg s$ [Given]

4.1. r [Assumption]
4.2. $r \vee s$ [$\vee$intro, 4.1]
4.3. $q \vee s$ [MP 4.2, 2]

4 p [$\vee$elim, 4.5, 1]

$r \rightarrow p$ [Direct Proof Rule]

Problem 2b

Show that $r \to p$ follows from $p \vee \neg q$, $(r \vee s) \to (q \vee s)$, and $\neg s$.

How do we conclude p ?

We should use **q** to get to p…

use elim or!

1.	$p \vee \neg q$	[Given]
2.	$(r \vee s) \to (q \vee s)$	[Given]
3.	$\neg s$	[Given]
4.1.	r	[Assumption]
4.2.	$r \vee s$	[$\vee$intro, 4.1]
4.3.	$q \vee s$	[MP 4.2, 2]
4.4.	q	[$\vee$elim, 4.3, 3]
4	p	
	$r \to p$	[Direct Proof Rule]

Problem 2b

Show that $r \to p$ follows from $p \vee \neg q$, $(r \vee s) \to (q \vee s)$, and $\neg s$.

How do we conclude p ?

We should use **q** to get to p…

use double negation!

1.	$p \vee \neg q$	[Given]
2.	$(r \vee s) \to (q \vee s)$	[Given]
3.	$\neg s$	[Given]
4.1.	r	[Assumption]
4.2.	$r \vee s$	[$\vee$intro, 4.1]
4.3.	$q \vee s$	[MP 4.2, 2]
4.4.	q	[$\vee$elim, 4.3, 3]
4.5.	$\neg\neg q$	[double negation, 4.4]
4	p	
	$r \to p$	[Direct Proof Rule]

Problem 2b

Show that $r \to p$ follows from $p \lor \neg q$, $(r \lor s) \to (q \lor s)$, and $\neg s$.

How do we conclude p ?

We should use **q** to get to p…

now we can use line 1!

1.	$p \lor \neg q$	[Given]
2.	$(r \lor s) \to (q \lor s)$	[Given]
3.	$\neg s$	[Given]
4.1.	r	[Assumption]
4.2.	$r \lor s$	[$\lor$intro, 4.1]
4.3.	$q \lor s$	[MP 4.2, 2]
4.4.	q	[$\lor$elim, 4.3, 3]
4.5.	$\neg\neg q$	[double negation, 4.4]
4.6.	p	[$\lor$elim, 4.5, 1]
	$r \to p$	[Direct Proof Rule]

Problem 2b

1.	$p \vee \neg q$	[Given]
2.	$(r \vee s) \rightarrow (q \vee s)$	[Given]
3.	$\neg s$	[Given]
4.1.	r	[Assumption]
4.2.	$r \vee s$	[$\vee$intro, 4.1]
4.3.	$q \vee s$	[MP 4.2, 2]
4.4.	q	[$\vee$elim, 4.3, 3]
4.5.	$\neg\neg q$	[double negation, 4.4]
4.6.	p	[$\vee$elim, 4.5, 1]
4.	$r \rightarrow p$	[Direct Proof Rule]

Predicate Logic Proofs

Predicate Logic Inference Rules

Elim $\forall$

$$\frac{\forall x, P(x)}{\therefore\ P(a) \text{ for \textit{any} object } a}$$

Intro $\forall$

$$\frac{\text{Let } a \text{ be arbitrary} \Rightarrow P(a)}{\therefore \qquad \forall x, P(x)}$$

Elim $\exists$

$$\frac{\exists x, P(x)}{\therefore\ P(c) \text{ for a \textit{new} name } c}$$

Intro $\exists$

$$\frac{P(c) \text{ for some } c}{\therefore \qquad \exists x, P(x)}$$

Problem 3a

Given $\forall x\,(T(x) \to M(x))$, we wish to prove $(\exists x\, T(x)) \to (\exists y\, M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof.

1. $\forall x\,(T(x) \to M(x))$ ____________________

    2.1. $\exists x\, T(x)$ ____________________

    2.2. $T(c)$ ____________________

    2.3. $T(c) \to M(c)$ ____________________

    2.4. $M(c)$ ____________________

    2.5. $\exists y\, M(y)$ ____________________

2. $(\exists x\, T(x)) \to (\exists y\, M(y))$ ____________________

Problem 3a

Given $\forall x\,(T(x) \rightarrow M(x))$, we wish to prove $(\exists x\, T(x)) \rightarrow (\exists y\, M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof.

1. $\forall x\,(T(x) \rightarrow M(x))$ ______ Given ______

    2.1. $\exists x\, T(x)$ ____________

    2.2. $T(c)$ ____________

    2.3. $T(c) \rightarrow M(c)$ ____________

    2.4. $M(c)$ ____________

    2.5. $\exists y\, M(y)$ ____________

2. $(\exists x\, T(x)) \rightarrow (\exists y\, M(y))$ ____________

Problem 3a

Given $\forall x\,(T(x) \to M(x))$, we wish to prove $(\exists x\, T(x)) \to (\exists y\, M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof.

1. $\forall x\,(T(x) \to M(x))$ ______ Given

    2.1. $\exists x\, T(x)$ ______ Assumption

    2.2. $T(c)$ ______

    2.3. $T(c) \to M(c)$ ______

    2.4. $M(c)$ ______

    2.5. $\exists y\, M(y)$ ______

2. $(\exists x\, T(x)) \to (\exists y\, M(y))$ ______ Direct Proof: 2.1-2.5

Problem 3a

Given $\forall x\,(T(x) \to M(x))$, we wish to prove $(\exists x\,T(x)) \to (\exists y\,M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof.

1.	$\forall x\,(T(x) \to M(x))$	Given
2.1.	$\exists x\,T(x)$	Assumption
2.2.	$T(c)$	Elim $\exists$: 2.1 (c: specific)
2.3.	$T(c) \to M(c)$	
2.4.	$M(c)$	
2.5.	$\exists y\,M(y)$	
2.	$(\exists x\,T(x)) \to (\exists y\,M(y))$	Direct Proof: 2.1-2.5

Problem 3a

Given $\forall x\,(T(x) \rightarrow M(x))$, we wish to prove $(\exists x\, T(x)) \rightarrow (\exists y\, M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof.

1.	$\forall x\,(T(x) \rightarrow M(x))$	Given
2.1.	$\exists x\, T(x)$	Assumption
2.2.	$T(c)$	Elim ∃ : 2.1 (c: specific)
2.3.	$T(c) \rightarrow M(c)$	Elim ∀ : 1
2.4.	$M(c)$	
2.5.	$\exists y\, M(y)$	
2.	$(\exists x\, T(x)) \rightarrow (\exists y\, M(y))$	Direct Proof: 2.1-2.5

Problem 3a

Given $\forall x\,(T(x) \rightarrow M(x))$, we wish to prove $(\exists x\,T(x)) \rightarrow (\exists y\,M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof.

1.	$\forall x\,(T(x) \rightarrow M(x))$	Given
2.1.	$\exists x\,T(x)$	Assumption
2.2.	$T(c)$	Elim $\exists$: 2.1 (c: specific)
2.3.	$T(c) \rightarrow M(c)$	Elim $\forall$: 1
2.4.	$M(c)$	Modus Ponens: 2.2, 2.3
2.5.	$\exists y\,M(y)$	
2.	$(\exists x\,T(x)) \rightarrow (\exists y\,M(y))$	Direct Proof: 2.1-2.5

Problem 3a

Given $\forall x\,(T(x) \rightarrow M(x))$, we wish to prove $(\exists x\,T(x)) \rightarrow (\exists y\,M(y))$.

The following formal proof does this, but it is missing explanations for each line. Fill in the blanks with inference rules or equivalences to apply (as well as the line numbers) to complete the proof.

1.	$\forall x\,(T(x) \rightarrow M(x))$	Given
2.1.	$\exists x\,T(x)$	Assumption
2.2.	$T(c)$	Elim $\exists$: 2.1 (c: specific)
2.3.	$T(c) \rightarrow M(c)$	Elim $\forall$: 1
2.4.	$M(c)$	Modus Ponens: 2.2, 2.3
2.5.	$\exists y\,M(y)$	Intro $\exists$: 2.4
2.	$(\exists x\,T(x)) \rightarrow (\exists y\,M(y))$	Direct Proof: 2.1-2.5

English Proofs

Writing a Proof (symbolically or in English)

- Don't just jump right in!

1. Look at the **claim**, and make sure you know:
   - What every word in the claim means
   - What the claim as a whole means

2. Translate the claim in predicate logic.

3. Next, write down the **Proof Skeleton**:
   - Where to **start**
   - What your **target** is
   -

4. Then once you know what claim you are proving and your starting point and ending point, you can finally write the proof!

Helpful Tips for English Proofs

- Start by introducing your assumptions
  - Introduce variables with "let"
    - "Let x be an arbitrary prime number…"
  - Introduce assumptions with "suppose"
    - "Suppose that $y \in A \wedge y \notin B$…"

- When you supply a value for an existence proof, use "Consider"
  - "Consider $x = 2$…"

- **ALWAYS** state what type your variable is (integer, set, etc.)

- Universal Quantifier means variable must be arbitrary

- Existential Quantifier means variable can be specific

Problem 4

Let domain of discourse be the integers. Consider the following claim:

$$\forall x\, \forall y\, ((\mathsf{Even}(x) \wedge \mathsf{Odd}(y)) \rightarrow \mathsf{Odd}(x+y))$$

In English, this says that, for any even integer x and odd integer y, the integer $x+y$ is odd.

a) Write a **formal proof** that the claim holds.

b) Translate your formal proof to an **English proof**.

Keep in mind that your proof will be read by a *human*, not a computer, so you should explain the algebra steps in more detail, whereas some of the predicate logic steps (e.g., Elim $\exists$) can be skipped.

Problem 4a

a) Write a formal proof that $\forall x \; \forall y \; ((\text{Even}(x) \land \text{Odd}(y)) \rightarrow \text{Odd}(x + y))$

Problem 4a

a) Write a formal proof that $\forall x\ \forall y\ ((\text{Even}(x) \wedge \text{Odd}(y)) \rightarrow \text{Odd}(x + y))$

Let x and y be arbitrary.

1. $\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x+y)$ Intro $\forall$

Problem 4a

a) Write a formal proof that $\forall x\ \forall y\ ((\text{Even}(x) \wedge \text{Odd}(y)) \rightarrow \text{Odd}(x + y))$

Let x and y be arbitrary.

1.1.1. $\text{Even}(x) \wedge \text{Odd}(y)$ Assumption

1.1. $\text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$ Direct Proof

1. $\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$ Intro $\forall$

Problem 4a

a) Write a formal proof that $\forall x\ \forall y\ ((\text{Even}(x) \wedge \text{Odd}(y)) \rightarrow \text{Odd}(x + y))$

Let x and y be arbitrary.

1.1.1. $\text{Even}(x) \wedge \text{Odd}(y)$ — Assumption

1.1.2. $\text{Even}(x)$ — Elim $\wedge$: 1.1.1

1.1. $\text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$ — Direct Proof

1. $\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$ — Intro $\forall$

Problem 4a

a) Write a formal proof that ∀x ∀y ((Even(x) ∧ Odd(y)) → Odd(x + y))

Let x and y be arbitrary.

1.1.1.	$\text{Even}(x) \wedge \text{Odd}(y)$	Assumption
1.1.2.	$\text{Even}(x)$	Elim $\wedge$: 1.1.1
1.1.3.	$\text{Odd}(y)$	Elim $\wedge$: 1.1.1

1.1.	$\text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x+y)$	Direct Proof
1.	$\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x+y)$	Intro $\forall$

Problem 4a

a) Write a formal proof that ∀x ∀y ((Even(x) ∧ Odd(y)) → Odd(x + y))

Let x and y be arbitrary.

1.1.1.	$\text{Even}(x) \land \text{Odd}(y)$	Assumption
1.1.2.	$\text{Even}(x)$	Elim ∧: 1.1.1
1.1.3.	$\text{Odd}(y)$	Elim ∧: 1.1.1
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2

1.1.	$\text{Even}(x) \land \text{Odd}(y) \rightarrow \text{Odd}(x+y)$	Direct Proof
1.	$\forall x, \forall y, \text{Even}(x) \land \text{Odd}(y) \rightarrow \text{Odd}(x+y)$	Intro ∀

Problem 4a

a) Write a formal proof that ∀x ∀y ((Even(x) ∧ Odd(y)) → Odd(x + y))

Let x and y be arbitrary.

1.1.1.	Even(x) $\wedge$ Odd(y)	Assumption
1.1.2.	Even(x)	Elim $\wedge$: 1.1.1
1.1.3.	Odd(y)	Elim $\wedge$: 1.1.1
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2
1.1.5.	$x = 2a$	Elim $\exists$: 1.1.4

1.1.	Even(x) $\wedge$ Odd(y) $\rightarrow$ Odd($x + y$)	Direct Proof
1.	$\forall x, \forall y$, Even(x) $\wedge$ Odd(y) $\rightarrow$ Odd($x + y$)	Intro $\forall$

Problem 4a

a) Write a formal proof that ∀x ∀y ((Even(x) ∧ Odd(y)) → Odd(x + y))

Let x and y be arbitrary.

1.1.1.	Even(x) ∧ Odd(y)	Assumption
1.1.2.	Even(x)	Elim ∧: 1.1.1
1.1.3.	Odd(y)	Elim ∧: 1.1.1
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2
1.1.5.	$x = 2a$	Elim ∃: 1.1.4
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3

1.1.	Even(x) ∧ Odd(y) → Odd($x + y$)	Direct Proof
1.	$\forall x, \forall y$, Even(x) ∧ Odd(y) → Odd($x + y$)	Intro ∀

Problem 4a

a) Write a formal proof that ∀x ∀y ((Even(x) ∧ Odd(y)) → Odd(x + y))

Let x and y be arbitrary.

1.1.1.	Even(x) $\wedge$ Odd(y)	Assumption
1.1.2.	Even(x)	Elim $\wedge$: 1.1.1
1.1.3.	Odd(y)	Elim $\wedge$: 1.1.1
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2
1.1.5.	$x = 2a$	Elim $\exists$: 1.1.4
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3
1.1.7.	$y = 2b + 1$	Elim $\exists$: 1.1.6

1.1.	Even(x) $\wedge$ Odd(y) $\rightarrow$ Odd($x + y$)	Direct Proof
1.	$\forall x, \forall y$, Even(x) $\wedge$ Odd(y) $\rightarrow$ Odd($x + y$)	Intro $\forall$

Problem 4a

a) Write a formal proof that ∀x ∀y ((Even(x) ∧ Odd(y)) → Odd(x + y))

Let x and y be arbitrary.

1.1.1.	$\text{Even}(x) \wedge \text{Odd}(y)$	Assumption
1.1.2.	$\text{Even}(x)$	Elim ∧: 1.1.1
1.1.3.	$\text{Odd}(y)$	Elim ∧: 1.1.1
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2
1.1.5.	$x = 2a$	Elim ∃: 1.1.4
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3
1.1.7.	$y = 2b + 1$	Elim ∃: 1.1.6
1.1.8.	$x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7

1.1.	$\text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Direct Proof
1.	$\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Intro ∀

Problem 4a

a) Write a formal proof that $\forall x\ \forall y\ ((\text{Even}(x) \wedge \text{Odd}(y)) \rightarrow \text{Odd}(x + y))$

Let x and y be arbitrary.

1.1.1.	$\text{Even}(x) \wedge \text{Odd}(y)$	Assumption
1.1.2.	$\text{Even}(x)$	Elim $\wedge$: 1.1.1
1.1.3.	$\text{Odd}(y)$	Elim $\wedge$: 1.1.1
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2
1.1.5.	$x = 2a$	Elim $\exists$: 1.1.4
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3
1.1.7.	$y = 2b + 1$	Elim $\exists$: 1.1.6
1.1.8.	$x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7
1.1.9.	$\exists k, x + y = 2k + 1$	Intro $\exists$: 1.1.8
1.1.	$\text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Direct Proof
1.	$\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Intro $\forall$

Problem 4a

a) Write a formal proof that ∀x ∀y ((Even(x) ∧ Odd(y)) → Odd(x + y))

Let x and y be arbitrary.

1.1.1.	Even(x) ∧ Odd(y)	Assumption
1.1.2.	Even(x)	Elim ∧: 1.1.1
1.1.3.	Odd(y)	Elim ∧: 1.1.1
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2
1.1.5.	$x = 2a$	Elim ∃: 1.1.4
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3
1.1.7.	$y = 2b + 1$	Elim ∃: 1.1.6
1.1.8.	$x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7
1.1.9.	$\exists k, x + y = 2k + 1$	Intro ∃: 1.1.8
1.1.10.	Odd($x + y$)	Undef Odd: 1.1.9
1.1.	Even(x) ∧ Odd(y) → Odd($x + y$)	Direct Proof
1.	$\forall x, \forall y$, Even(x) ∧ Odd(y) → Odd($x + y$)	Intro ∀

Problem 4a

a) Write a formal proof that ∀x ∀y ((Even(x) ∧ Odd(y)) → Odd(x + y))

Let x and y be arbitrary.

1.1.1.	Even(x) ∧ Odd(y)	Assumption
1.1.2.	Even(x)	Elim ∧: 1.1.1
1.1.3.	Odd(y)	Elim ∧: 1.1.1
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2
1.1.5.	$x = 2a$	Elim ∃: 1.1.4
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3
1.1.7.	$y = 2b + 1$	Elim ∃: 1.1.6
1.1.8.	$x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7
1.1.9.	$\exists k, x + y = 2k + 1$	Intro ∃: 1.1.8
1.1.10.	Odd$(x + y)$	Undef Odd: 1.1.9
1.1.	Even(x) ∧ Odd(y) → Odd$(x + y)$	Direct Proof
1.	$\forall x, \forall y,$ Even(x) ∧ Odd(y) → Odd$(x + y)$	Intro ∀

Problem 4b

b) Translate the formal proof into an English proof.

	Let x and y be arbitrary.		Let x and y be arbitrary integers
1.1.1.	$\mathsf{Even}(x) \land \mathsf{Odd}(y)$	Assumption	
1.1.2.	$\mathsf{Even}(x)$	Elim $\land$: 1.1.1	
1.1.3.	$\mathsf{Odd}(y)$	Elim $\land$: 1.1.1	
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2	
1.1.5.	$x = 2a$	Elim $\exists$: 1.1.4	
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3	
1.1.7.	$y = 2b + 1$	Elim $\exists$: 1.1.6	
1.1.8.	$x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7	
1.1.9.	$\exists k, x + y = 2k + 1$	Intro $\exists$: 1.1.8	
1.1.10.	$\mathsf{Odd}(x + y)$	Undef Odd: 1.1.9	
1.1.	$\mathsf{Even}(x) \land \mathsf{Odd}(y) \rightarrow \mathsf{Odd}(x + y)$	Direct Proof	
1.	$\forall x, \forall y, \mathsf{Even}(x) \land \mathsf{Odd}(y) \rightarrow \mathsf{Odd}(x + y)$	Intro $\forall$	Since x and y were arbitrary, the claim holds

Problem 4b

b) Translate the formal proof into an English proof.

	Let x and y be arbitrary.		Let x and y be arbitrary integers
1.1.1.	$\text{Even}(x) \wedge \text{Odd}(y)$	Assumption	Suppose x is even and y is odd
1.1.2.	$\text{Even}(x)$	Elim $\wedge$: 1.1.1	
1.1.3.	$\text{Odd}(y)$	Elim $\wedge$: 1.1.1	
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2	
1.1.5.	$x = 2a$	Elim $\exists$: 1.1.4	
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3	
1.1.7.	$y = 2b + 1$	Elim $\exists$: 1.1.6	
1.1.8.	$x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7	
1.1.9.	$\exists k, x + y = 2k + 1$	Intro $\exists$: 1.1.8	
1.1.10.	$\text{Odd}(x + y)$	Undef Odd: 1.1.9	
1.1.	$\text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Direct Proof	
1.	$\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Intro $\forall$	Since x and y were arbitrary, the claim holds

Problem 4b

b) Translate the formal proof into an English proof.

Let x and y be arbitrary.		Let x and y be arbitrary integers
1.1.1. $\mathsf{Even}(x) \wedge \mathsf{Odd}(y)$	Assumption	Suppose x is even and y is odd
1.1.2. $\mathsf{Even}(x)$	Elim $\wedge$: 1.1.1	
1.1.3. $\mathsf{Odd}(y)$	Elim $\wedge$: 1.1.1	
1.1.4. $\exists m, x = 2m$	Def of Even: 1.1.2	
1.1.5. $x = 2a$	Elim $\exists$: 1.1.4	
1.1.6. $\exists n, y = 2n + 1$	Def of Odd: 1.1.3	
1.1.7. $y = 2b + 1$	Elim $\exists$: 1.1.6	
1.1.8. $x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7	
1.1.9. $\exists k, x + y = 2k + 1$	Intro $\exists$: 1.1.8	
1.1.10. $\mathsf{Odd}(x + y)$	Undef Odd: 1.1.9	
1.1. $\mathsf{Even}(x) \wedge \mathsf{Odd}(y) \to \mathsf{Odd}(x + y)$	Direct Proof	
1. $\forall x, \forall y, \mathsf{Even}(x) \wedge \mathsf{Odd}(y) \to \mathsf{Odd}(x + y)$	Intro $\forall$	Since x and y were arbitrary, the claim holds

Problem 4b

b) Translate the formal proof into an English proof.

Let x and y be arbitrary.			Let x and y be arbitrary integers
1.1.1.	$\text{Even}(x) \wedge \text{Odd}(y)$	Assumption	Suppose x is even and y is odd
1.1.2.	$\text{Even}(x)$	Elim $\wedge$: 1.1.1	
1.1.3.	$\text{Odd}(y)$	Elim $\wedge$: 1.1.1	
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2	By def of even, x = 2a for some integer a
1.1.5.	$x = 2a$	Elim $\exists$: 1.1.4	
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3	
1.1.7.	$y = 2b + 1$	Elim $\exists$: 1.1.6	
1.1.8.	$x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7	
1.1.9.	$\exists k, x + y = 2k + 1$	Intro $\exists$: 1.1.8	
1.1.10.	$\text{Odd}(x + y)$	Undef Odd: 1.1.9	
1.1.	$\text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Direct Proof	
1.	$\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Intro $\forall$	Since x and y were arbitrary, the claim holds

Problem 4b

b) Translate the formal proof into an English proof.

Let x and y be arbitrary.			Let x and y be arbitrary integers
1.1.1.	$\text{Even}(x) \wedge \text{Odd}(y)$	Assumption	Suppose x is even and y is odd
1.1.2.	$\text{Even}(x)$	Elim $\wedge$: 1.1.1	
1.1.3.	$\text{Odd}(y)$	Elim $\wedge$: 1.1.1	
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2	By def of even, x = 2a for some integer a
1.1.5.	$x = 2a$	Elim $\exists$: 1.1.4	
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3	By def of odd, y = 2b + 1 for some integer b
1.1.7.	$y = 2b + 1$	Elim $\exists$: 1.1.6	
1.1.8.	$x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7	
1.1.9.	$\exists k, x + y = 2k + 1$	Intro $\exists$: 1.1.8	
1.1.10.	$\text{Odd}(x + y)$	Undef Odd: 1.1.9	
1.1.	$\text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Direct Proof	
1.	$\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Intro $\forall$	Since x and y were arbitrary, the claim holds

Problem 4b

b) Translate the formal proof into an English proof.

Let x and y be arbitrary.		Let x and y be arbitrary integers
1.1.1. $\text{Even}(x) \wedge \text{Odd}(y)$	Assumption	Suppose x is even and y is odd
1.1.2. $\text{Even}(x)$	Elim $\wedge$: 1.1.1	
1.1.3. $\text{Odd}(y)$	Elim $\wedge$: 1.1.1	
1.1.4. $\exists m, x = 2m$	Def of Even: 1.1.2	By def of even, x = 2a for some integer a
1.1.5. $x = 2a$	Elim $\exists$: 1.1.4	
1.1.6. $\exists n, y = 2n + 1$	Def of Odd: 1.1.3	By def of odd, y = 2b + 1 for some integer b
1.1.7. $y = 2b + 1$	Elim $\exists$: 1.1.6	
1.1.8. $x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7	Thus, x + y = 2 (a + b) + 1
1.1.9. $\exists k, x + y = 2k + 1$	Intro $\exists$: 1.1.8	
1.1.10. $\text{Odd}(x + y)$	Undef Odd: 1.1.9	
1.1. $\text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Direct Proof	
1. $\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Intro $\forall$	Since x and y were arbitrary, the claim holds

Problem 4b

b) Translate the formal proof into an English proof.

	Let x and y be arbitrary.		Let x and y be arbitrary integers
1.1.1.	$\text{Even}(x) \wedge \text{Odd}(y)$	Assumption	Suppose x is even and y is odd
1.1.2.	$\text{Even}(x)$	Elim $\wedge$: 1.1.1	
1.1.3.	$\text{Odd}(y)$	Elim $\wedge$: 1.1.1	
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2	By def of even, x = 2a for some integer a
1.1.5.	$x = 2a$	Elim $\exists$: 1.1.4	
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3	By def of odd, y = 2b + 1 for some integer b
1.1.7.	$y = 2b + 1$	Elim $\exists$: 1.1.6	
1.1.8.	$x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7	Thus, x + y = 2 (a + b) + 1
1.1.9.	$\exists k, x + y = 2k + 1$	Intro $\exists$: 1.1.8	Since integers are closed under addition, a + b is an integer
1.1.10.	$\text{Odd}(x + y)$	Undef Odd: 1.1.9	
1.1.	$\text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Direct Proof	
1.	$\forall x, \forall y, \text{Even}(x) \wedge \text{Odd}(y) \rightarrow \text{Odd}(x + y)$	Intro $\forall$	Since x and y were arbitrary, the claim holds

Problem 4b

b) Translate the formal proof into an English proof.

	Let x and y be arbitrary.		Let x and y be arbitrary integers
1.1.1.	$\mathsf{Even}(x) \land \mathsf{Odd}(y)$	Assumption	Suppose x is even and y is odd
1.1.2.	$\mathsf{Even}(x)$	Elim $\land$: 1.1.1	
1.1.3.	$\mathsf{Odd}(y)$	Elim $\land$: 1.1.1	
1.1.4.	$\exists m, x = 2m$	Def of Even: 1.1.2	By def of even, x = 2a for some integer a
1.1.5.	$x = 2a$	Elim $\exists$: 1.1.4	
1.1.6.	$\exists n, y = 2n + 1$	Def of Odd: 1.1.3	By def of odd, y = 2b + 1 for some integer b
1.1.7.	$y = 2b + 1$	Elim $\exists$: 1.1.6	
1.1.8.	$x + y = 2(a + b) + 1$	Algebra: 1.1.5 1.1.7	Thus, x + y = 2 (a + b) + 1
1.1.9.	$\exists k, x + y = 2k + 1$	Intro $\exists$: 1.1.8	Since integers are closed under addition, a + b is an integer
1.1.10.	$\mathsf{Odd}(x + y)$	Undef Odd: 1.1.9	By def of odd, x + y is odd
1.1.	$\mathsf{Even}(x) \land \mathsf{Odd}(y) \rightarrow \mathsf{Odd}(x + y)$	Direct Proof	
1.	$\forall x, \forall y, \mathsf{Even}(x) \land \mathsf{Odd}(y) \rightarrow \mathsf{Odd}(x + y)$	Intro $\forall$	Since x and y were arbitrary, the claim holds

Problem 4b

b) Translate the formal proof into an English proof.

Let x and y be arbitrary integers. Suppose x is even and y is odd.

By the definition of even, $x = 2a$ for some integer a, and by the definition of odd, $y = 2b + 1$ for some integer b. Thus, $x + y = 2a + 2b + 1 = 2(a + b) + 1$. As integers are closed under addition, $a + b$ is an integer such that $x + y$ is odd, by definition of odd.

Since x and y were arbitrary, we have proven the desired result.

That's All, Folks!

Thanks for coming to section this week!
Any questions?