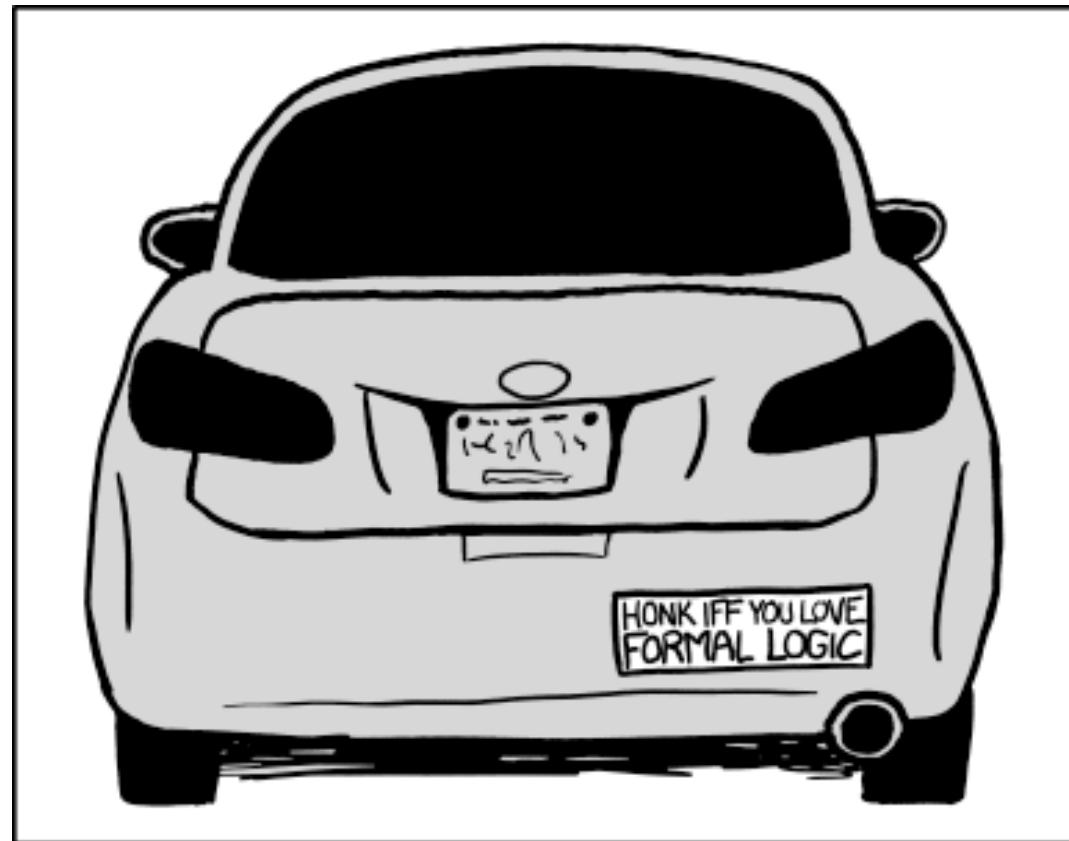


# CSE 311: Foundations of Computing I

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## Topic 1: Formal Logic



## 390z

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- **1 credit workshop meeting Tuesdays at 3:30pm**
  - credit / no credit
- **Available to anyone who wants more practice**
  - we will move very quickly in 311
- **More info at [tinyurl.com/CSE390Z-WI26](http://tinyurl.com/CSE390Z-WI26)**

# What is logic and why do we need it?

---

Logic is a language, like English or Java, with its own

- words and rules for combining words into sentences  
([syntax](#))
- ways to assign meaning to words and sentences  
([semantics](#))

Compared to English, Logic is more

- concise (useful)
- precise (critical!)

Importantly, Logic comes with its own [toolkit](#)

# Why not use English?

---

- Turn right here...

Does “right” mean the direction or now?

- We saw her duck

Does “duck” mean the animal or crouch down?

- Buffalo buffalo Buffalo buffalo buffalo  
buffalo Buffalo buffalo

This means “Bison from Buffalo, that bison from Buffalo bully, themselves  
bully bison from Buffalo.”

**Natural languages can be unclear / imprecise**

# Propositions: building blocks of logic

---

A ***proposition*** is a statement that

- is “well-formed” (syntactically valid)
- is either true or false

# Propositions: building blocks of logic

---

A ***proposition*** is a statement that

- is “well-formed”
- is either true or false

Garfield is a mammal and Garfield is a cat

true



Odie is a mammal and Odie is a cat

false



# Are These Propositions?

---

**2 + 2 = 5**

This is a proposition. It's okay for propositions to be false.

**$x + 2 = 5389$ , where  $x$  is my PIN number**

This is a proposition. We don't need to know what  $x$  is.

**Akjsdf!**

Not a proposition because it's gibberish.

**Who are you?**

This is a question which means it doesn't have a truth value.

**Every positive even integer can be written as the sum of two primes.**

This is a proposition. We don't know if it's true or false, but we know it's one of them!

# Propositions

---

We need a way of talking about *arbitrary* ideas...

Propositional Variables:  $p, q, r, s, \dots$

Constant truth values:

- **T** for **true**
- **F** for **false**

# Familiar from Java

---

Java boolean represents a truth value

- constants true and false
- variables hold *unknown* values

Operators calculate new values from given ones

- unary: not (!)
- binary: and (&&), or (||)

# Logical Connectives

---

**Negation (not)**  $\neg p$

**Conjunction (and)**  $p \wedge q$

**Disjunction (or)**  $p \vee q$

**con** with  $p$  with  $q$  (i.e., both)

**dis-** apart from not necessarily both

# Logical Connectives

---

**Negation (not)**  $\neg p$

**Conjunction (and)**  $p \wedge q$

**Disjunction (or)**  $p \vee q$

**Exclusive Or**  $p \oplus q$

$p \vee q$  at least one of  $p$  or  $q$

$p \oplus q$  exactly one of  $p$  or  $q$

Logic forces us to distinguish  $\vee$  from  $\oplus$

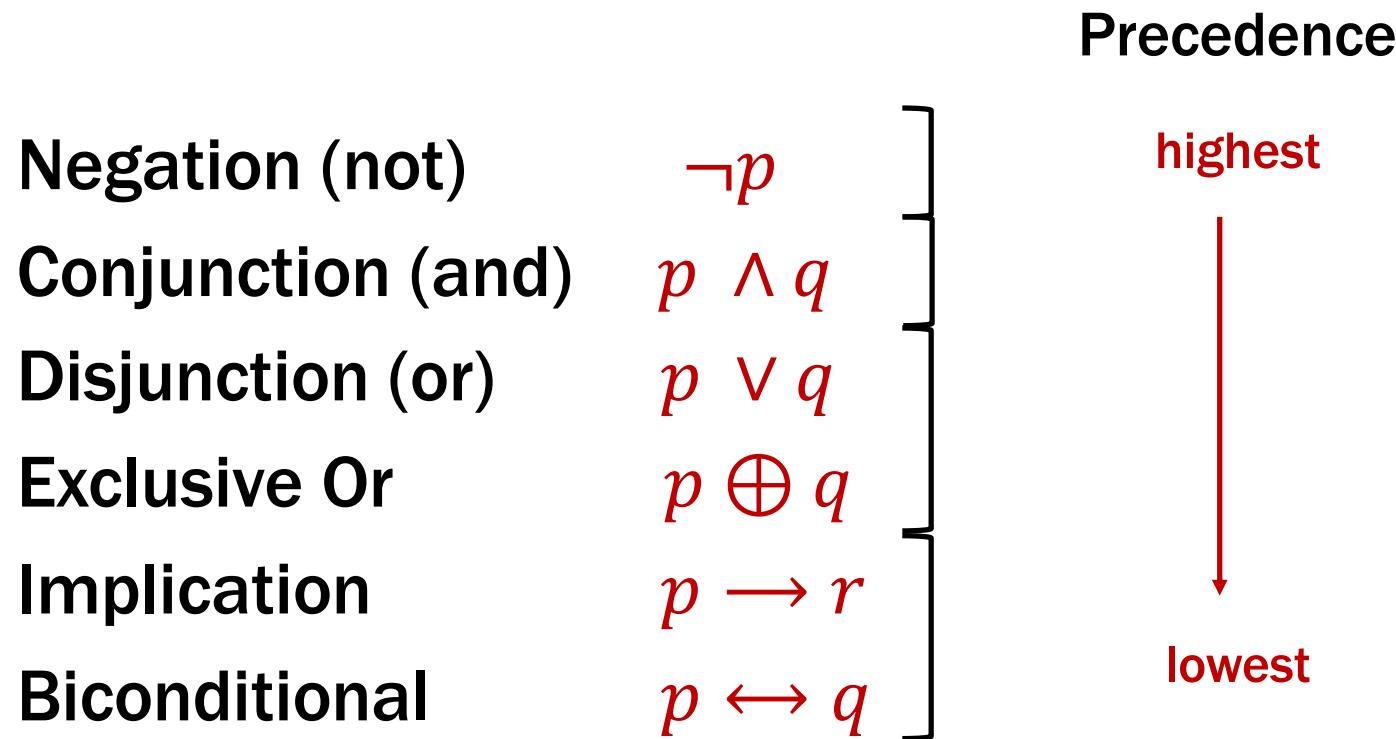
# Logical Connectives

---

<b>Negation (not)</b>	$\neg p$
<b>Conjunction (and)</b>	$p \wedge q$
<b>Disjunction (or)</b>	$p \vee q$
<b>Exclusive Or</b>	$p \oplus q$
<b>Implication</b>	$p \rightarrow r$
<b>Biconditional</b>	$p \leftrightarrow q$

# Syntax of Logical Connectives

---



$p \vee q \wedge r \rightarrow t$  means  $(p \vee (q \wedge r)) \rightarrow t$

# Syntax of Logical Connectives

---

Associativity		
<b>Conjunction (and)</b>	$p \wedge q$	left-to-right
<b>Disjunction (or)</b>	$p \vee q$	left-to-right
<b>Exclusive Or</b>	$p \oplus q$	
<b>Implication</b>	$p \rightarrow r$	right-to-left
<b>Biconditional</b>	$p \leftrightarrow q$	

$p \vee q \vee r \vee t$  means  $((p \vee q) \vee r) \vee t$

$p \rightarrow q \rightarrow r$  means  $p \rightarrow (q \rightarrow r)$

# Some Truth Tables

---

$p$	$\neg p$
T	
F	

$p$	$q$	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

$p$	$q$	$p \vee q$
T	T	
T	F	
F	T	
F	F	

$p$	$q$	$p \oplus q$
T	T	
T	F	
F	T	
F	F	

# Some Truth Tables

---

$p$	$\neg p$
T	F
F	T

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$p$	$q$	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

# Another Truth Table

---

$p$	$r$	$p \rightarrow r$
T	T	
T	F	
F	T	
F	F	

With implication ( $\rightarrow$ ),  $p$  is called the "premise" and  $r$  is called the "conclusion".

The implication is true when  $p$  and  $r$  are true.

The implication is true ("vacuously") when  $p$  is false.

# Another Truth Table

---

$p$	$r$	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

With implication ( $\rightarrow$ ),  $p$  is called the "**premise**" and  $r$  is called the "**conclusion**".

The implication is true when  $p$  and  $r$  are true.

The implication is true ("vacuously") when  $p$  is **false**.

# Implication

---

*“If it was raining, then I had my umbrella”*

*It's useful to think of implications as promises. That is “Was I **wrong**?”*

$p$	$r$	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

	It's raining	It's not raining
I have my umbrella		
I do not have my umbrella		

# Implication

---

*“If it was raining, then I had my umbrella”*

*It's useful to think of implications as promises. That is “Was I **wrong**?”*

$p$	$r$	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

	It's raining	It's not raining
I have my umbrella	No	No
I do not have my umbrella	Yes	No

*I am only wrong when:*

- (a) It's raining AND*
- (b) I don't have my umbrella*

# Implication

---

***“If the Seahawks won,  
then I was at the game.”***

**In what scenario was I wrong?**

$p$	$r$	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

	I was at the game	I wasn't at the game
Seahawks won		
Seahawks lost		

# Implication

---

***“If the Seahawks won,  
then I was at the game.”***

**In what scenario was I wrong?**

$p$	$r$	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

	I was at the game	I wasn't at the game
Seahawks won	Ok	<b>Doh!</b>
Seahawks lost	Ok	Ok

# Implication

---

*“If it’s raining, then I have my umbrella”*

*Are these true?*

$p$	$r$	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

*$2 + 2 = 4 \rightarrow \text{earth is a planet}$*

The fact that these are unrelated doesn’t make the statement false! “ $2 + 2 = 4$ ” is true; “earth is a planet” is true.  $T \rightarrow T$  is true. So, the statement is true.

*$2 + 2 = 5 \rightarrow 26 \text{ is prime}$*

Again, these statements may or may not be related. “ $2 + 2 = 5$ ” is false; so, the implication is true. (Whether 26 is prime or not is irrelevant).

*Implication is not a causal relationship!*

$$p \rightarrow r$$

---

- (1) ***“I have collected all 151 Pokémons if I am a Pokémons master”***
- (2) ***“I have collected all 151 Pokémons only if I am a Pokémons master”***

In English, the “if” can be written at the end of the sentence rather than at the beginning of the sentence (followed by a “,”).

$$p \rightarrow r$$

---

- (1) *"I have collected all 151 Pokémons if I am a Pokémons master"*
- (2) *"I have collected all 151 Pokémons only if I am a Pokémons master"*

The implications are:

- (1) *If I am a Pokémons master, then I have collected all 151 Pokémons.*
- (2) *If I have collected all 151 Pokémons, then I am a Pokémons master.*

$$p \rightarrow r$$

---

## Implication:

- $p$  implies  $r$
- whenever  $p$  is true,  $r$  must be true
- if  $p$ , then  $r$
- $r$  if  $p$
- $p$  only if  $r$
- $p$  is sufficient for  $r$
- $r$  is necessary for  $p$

$p$	$r$	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

## Biconditional: $p \leftrightarrow q$

---

- $p$  if and only if  $q$
- $p$  “iff”  $q$ 
  - $p$  and  $q$  have the same value truth value

$p$	$q$	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

# A Compound Proposition (Practical Example)

---

**“Show the notification to the user if its their second login or they’ve used it for two weeks and haven’t tried the feature X unless they did use the feature Y.”**

**Not at all clear what exactly this means!**

**Can use logic to understand exactly when to show it**

# A Compound Proposition (Silly Example)

---

“Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna”

We'd like to *understand* what this proposition means.



# A Compound Proposition

---

“Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna”

We'd like to *understand* what this proposition means.

First find the simplest (**atomic**) propositions:

- q* “Garfield has black stripes”
- r* “Garfield is an orange cat”
- s* “Garfield likes lasagna”

$(q \text{ if } (r \text{ and } s)) \text{ and } (r \text{ or } (\text{not } s))$



# Logical Connectives

---

**Negation (not)**  $\neg p$

**Conjunction (and)**  $p \wedge q$

**Disjunction (or)**  $p \vee q$

**Exclusive Or**  $p \oplus q$

**Implication**  $p \rightarrow r$

**Biconditional**  $p \leftrightarrow q$

$q$	“Garfield has black stripes”
$r$	“Garfield is an orange cat”
$s$	“Garfield likes lasagna”

“Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna”



(q if (r and s)) and (r or (not s))

# Logical Connectives

---

**Negation (not)**  $\neg p$

**Conjunction (and)**  $p \wedge q$

**Disjunction (or)**  $p \vee q$

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$q$	“Garfield has black stripes”
$r$	“Garfield is an orange cat”
$s$	“Garfield likes lasagna”

“Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna”



$(q \text{ if } (r \text{ and } s)) \text{ and } (r \text{ or } (\text{not } s))$



$((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$

# Analyzing the Garfield Sentence with a Truth Table

---

$q$	$r$	$s$	$((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$
F	F	F	
F	F	T	
F	T	F	
F	T	T	
T	F	F	
T	F	T	
T	T	F	
T	T	T	

subexpressions are not (yet)  
columns in this table

we will always include  
all subexpressions  
(easiest to verify)

# Analyzing the Garfield Sentence with a Truth Table

---

$q$	$r$	$s$	$r \vee \neg s$	$(r \wedge s) \rightarrow q$	$((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$
F	F	F			
F	F	T			
F	T	F			
F	T	T			
T	F	F			
T	F	T			
T	T	F			
T	T	T			

# Analyzing the Garfield Sentence with a Truth Table

---

$q$	$r$	$s$	$\neg s$	$r \vee \neg s$	$r \wedge s$	$(r \wedge s) \rightarrow q$	$((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$
F	F	F					
F	F	T					
F	T	F					
F	T	T					
T	F	F					
T	F	T					
T	T	F					
T	T	T					

# Analyzing the Garfield Sentence with a Truth Table

---

$q$	$r$	$s$	$\neg s$	$r \vee \neg s$	$r \wedge s$	$(r \wedge s) \rightarrow q$	$((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$
F	F	F	T	T	F	T	T
F	F	T	F	F	F	T	F
F	T	F	T	T	F	T	T
F	T	T	F	T	T	F	F
T	F	F	T	T	F	T	T
T	F	T	F	F	F	T	F
T	T	F	T	T	F	T	T
T	T	T	F	T	T	T	T

# Understanding Garfield Claim

---

“Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna”



Black Stripes	Orange	Likes Lasagna	Claim
F	F	F	T
F	F	T	F
F	T	F	T
F	T	T	F
T	F	F	T
T	F	T	F
T	T	F	T
T	T	T	T

Propositional Logic makes clear exactly what is being claimed.

# Understanding Garfield Claim

---

Black Stripes	Orange	Likes Lasagna	Claim
F	F	F	T
...	...	...	...
T	T	T	T

Consistent with



but also



# Converse, Contrapositive

---

Implication:

$$p \rightarrow r$$

Converse:

$$r \rightarrow p$$

Consider  
 $p$ : 6 is divisible by 2  
 $r$ : 6 is divisible by 4

Contrapositive:

$$\neg r \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg r$$

$p \rightarrow r$	
$r \rightarrow p$	
$\neg r \rightarrow \neg p$	
$\neg p \rightarrow \neg r$	

# Converse, Contrapositive

---

Implication:

$$p \rightarrow r$$

Converse:

$$r \rightarrow p$$

Consider  
 $p$ : 6 is divisible by 2  
 $r$ : 6 is divisible by 4

Contrapositive:

$$\neg r \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg r$$

$p \rightarrow r$	F
$r \rightarrow p$	T
$\neg r \rightarrow \neg p$	F
$\neg p \rightarrow \neg r$	T

# Converse, Contrapositive

---

Implication:

$$p \rightarrow r$$

Converse:

$$r \rightarrow p$$

Contrapositive:

$$\neg r \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg r$$

How do these relate to each other?

$p$	$r$	$p \rightarrow r$	$r \rightarrow p$	$\neg p$	$\neg r$	$\neg p \rightarrow \neg r$	$\neg r \rightarrow \neg p$
T	T						
T	F						
F	T						
F	F						

# Converse, Contrapositive

---

Implication:

$$p \rightarrow r$$

Converse:

$$r \rightarrow p$$

Contrapositive:

$$\neg r \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg r$$

An **implication** and its **contrapositive**  
have the same truth value!

$p$	$r$	$p \rightarrow r$	$r \rightarrow p$	$\neg p$	$\neg r$	$\neg p \rightarrow \neg r$	$\neg r \rightarrow \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

# Converse, Contrapositive

---

Implication:

$$p \rightarrow r$$

Converse:

$$r \rightarrow p$$

Contrapositive:

$$\neg r \rightarrow \neg p$$

Inverse:

$$\neg p \rightarrow \neg r$$

An **implication** and its **inverse**  
**do not have the same truth value!**

$p$	$r$	$p \rightarrow r$	$r \rightarrow p$	$\neg p$	$\neg r$	$\neg p \rightarrow \neg r$	$\neg r \rightarrow \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

# Equivalence

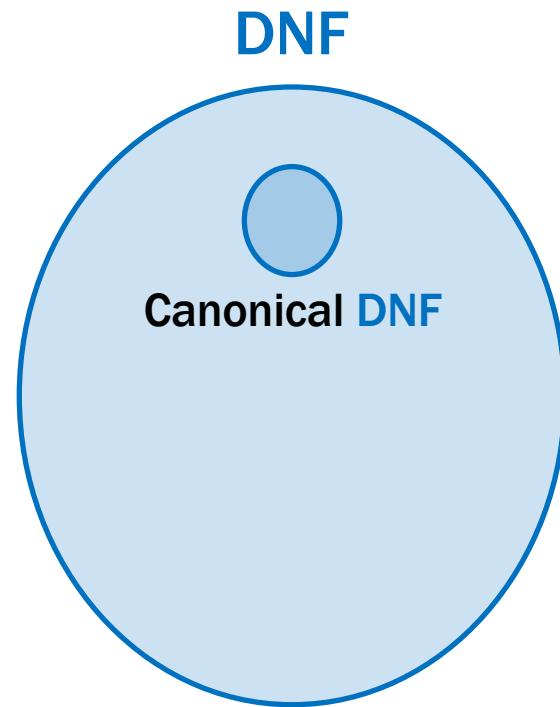
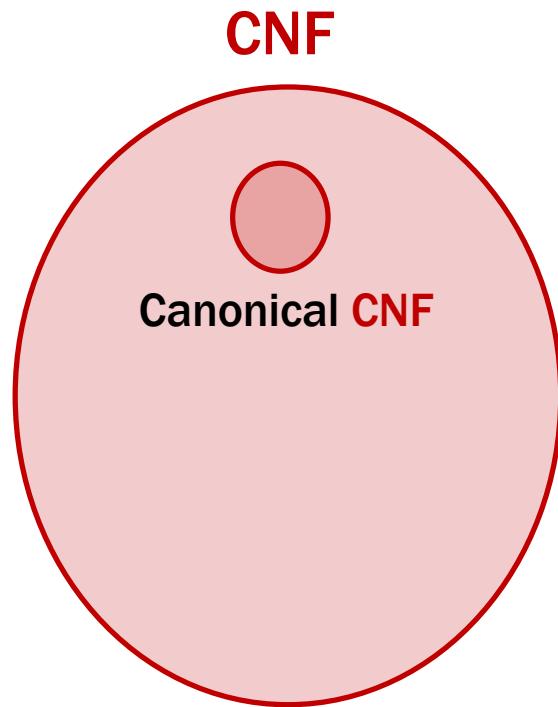
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- Propositional Logic expressions with the same truth table are called "**equivalent**"
- Examples:
  - implication and its contrapositive are equivalent  
e.g.,  $(p \vee q) \rightarrow (q \wedge r)$  is equivalent to  $\neg(q \wedge r) \rightarrow \neg(p \vee q)$
  - implication and its inverse are **not** equivalent  
e.g.,  $(p \vee q) \rightarrow (q \wedge r)$  is **not** equivalent to  $\neg(p \vee q) \rightarrow \neg(q \wedge r)$   
assuming they are the same is the "fallacy of the inverse"
- Greatly expand on equivalence next week
  - prove equivalence without a truth table

# Canonical Forms

# CNF & DNF

---



All Logic Expressions

# Canonical Forms

---

- Canonical is from Latin "canon" (ruler)
  - compare against to see if equivalent
- We saw one way to do this already: **truth table**
- Canonical forms are a **second way...**

# Canonical DNF

---

- 1 Find the T rows in the truth table

Suppose  $F$  is an expression  
using the variables  $a, b, c$

a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

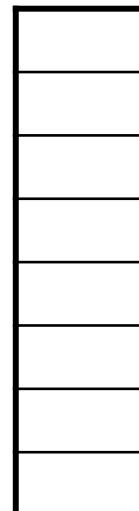
# Canonical DNF

---

- ① Find the T rows in the truth table
- ② For each T row, write an expression that is T in that row but *no others* ("min term")



a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T



$$\neg a \wedge \neg b \wedge c$$

This is only T if a = F, b = F, and c = T  
(AND requires *all* arguments to be T)

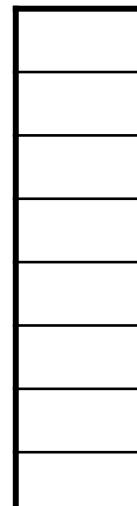
# Canonical DNF

---

- ① Find the T rows in the truth table
- ② For each T row, write an expression that is T in that row but *no others* ("min term")



a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T



$$\neg a \wedge \neg b \wedge c$$

$$\neg a \wedge b \wedge c$$

This is only T if a = F, b = T, and c = T

# Canonical DNF

---

- ① Find the T rows in the truth table
- ② For each T row, write an expression that is T in that row but *no others* ("min term")

a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T



$$\neg a \wedge \neg b \wedge c$$

$$\neg a \wedge b \wedge c$$

$$a \wedge \neg b \wedge c$$

$$a \wedge b \wedge \neg c$$

$$a \wedge b \wedge c$$

A **min term** includes every variable exactly once, either negated or unnegated, AND-ed together

# Canonical DNF

---

a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

- 1 Find the T rows in the truth table
- 2 For each T row, write an expression that is T in that row but *no others* ("min term")

$$\neg a \wedge \neg b \wedge c$$

$$\neg a \wedge b \wedge c$$

$$a \wedge \neg b \wedge c$$

$$a \wedge b \wedge \neg c$$

$$a \wedge b \wedge c$$

- 3 Form the disjunction (OR) of the min terms

$$(\neg a \wedge \neg b \wedge c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge c) \vee (a \wedge b \wedge \neg c) \vee (a \wedge b \wedge c)$$

# DNF: Canonical and Non

---

- Stands for "**Disjunctive Normal Form**"
  - **outermost** operation is disjunction (OR)
  - operands are conjunctions (ANDs) of variables or their negations

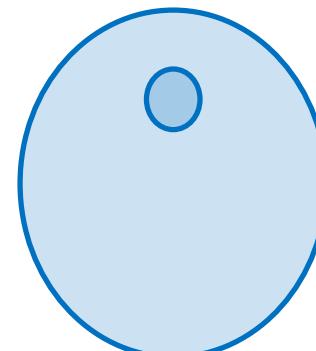
$$(a \wedge c) \vee (\neg a) \vee (\neg a \wedge \neg b)$$

non-canonical DNF

$$(a \wedge b \wedge \neg c) \vee (\neg a \wedge b \wedge c) \vee (a \wedge \neg b \wedge \neg c)$$

canonical DNF

(every disjunct is a min term)



# Canonical CNF

---

① Find the F rows in the truth table

a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

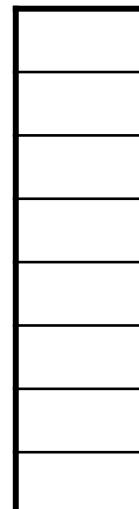
# Canonical CNF

---

- ① Find the F rows in the truth table
- ② For each F row, write an expression that is T in every row *but that one* ("max term")



a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T



$$a \vee b \vee c$$

This is only F if a = F, b = F, and c = F  
(OR is T if *any* arguments is a T)

# Canonical CNF

---

- ① Find the F rows in the truth table
- ② For each F row, write an expression that is T in every row *but that one* ("max term")



a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T


$$a \vee b \vee c$$

$$a \vee \neg b \vee c$$

This is only F if a = F, b = T, and c = F

# Canonical CNF

---

- ① Find the F rows in the truth table
- ② For each F row, write an expression that is T in every row *but that one* ("max term")



a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T


$a \vee b \vee c$

$a \vee \neg b \vee c$

$\neg a \vee b \vee c$

This is only F if a = T, b = F, and c = F

# Canonical CNF

---

- ① Find the F rows in the truth table
- ② For each F row, write an expression that is T in every row *but that one* ("max term")

a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

$$a \vee b \vee c$$

$$a \vee \neg b \vee c$$

$$\neg a \vee b \vee c$$

- ③ Form the conjunction (AND) of the max terms

$$(a \vee b \vee c) \wedge (a \vee \neg b \vee c) \wedge (\neg a \vee b \vee c)$$

# CNF: Canonical and Non

---

- Stands for "**Conjunctive Normal Form**"
  - **outermost** operation is conjunction (AND)
  - operands (conjuncts) are disjunctions (ORs) of variables or their negations

$(a \vee b \vee \neg c) \wedge (\neg a \vee b \vee c) \wedge (a \vee \neg b \vee \neg c)$       **canonical CNF**

$(a \vee c) \wedge (\neg a) \wedge (\neg a \vee \neg b)$       **non-canonical CNF**

# Comparing DNF and CNF

---

	<b>DNF</b>	<b>CNF</b>
operation	disjunction (OR)	conjunction (AND)
operands	conjunctions (ANDs) <i>(of only variables or their negations)</i>	disjunctions (ORs)
canonical iff	all conjunctions are <b>min terms</b>	all disjunctions are <b>max terms</b>

# Comparing **Min** and **Max** Terms

---

- Min/Max term if *every variable appears exactly once*

	<b>Min Term</b>	<b>Max Term</b>
operation	conjunction (AND) <i>(of every variables or its negation)</i>	disjunction (OR)
result	only one T row	only one F row

# Important Corollary of DNF Construction

---

$\neg, \wedge, \vee$  can implement any Boolean function!

no need for anything else

**Why? Because this construction only uses  $\neg, \wedge, \vee$**

DNF conversion works for any boolean function

# Digital Circuits

# Application: Digital Circuits

---

## Computing With Logic

- **T** corresponds to **1** or “high” voltage
- **F** corresponds to **0** or “low” voltage

## Gates

- Take inputs and produce outputs (functions)
- Several kinds of gates
- Correspond to propositional connectives

# AND, OR, NOT Gates

---

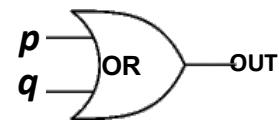
## AND Gate



$p$	$q$	OUT
1	1	1
1	0	0
0	1	0
0	0	0

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

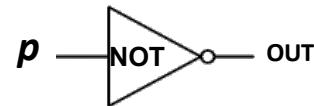
## OR Gate



$p$	$q$	OUT
1	1	1
1	0	1
0	1	1
0	0	0

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

## NOT Gate

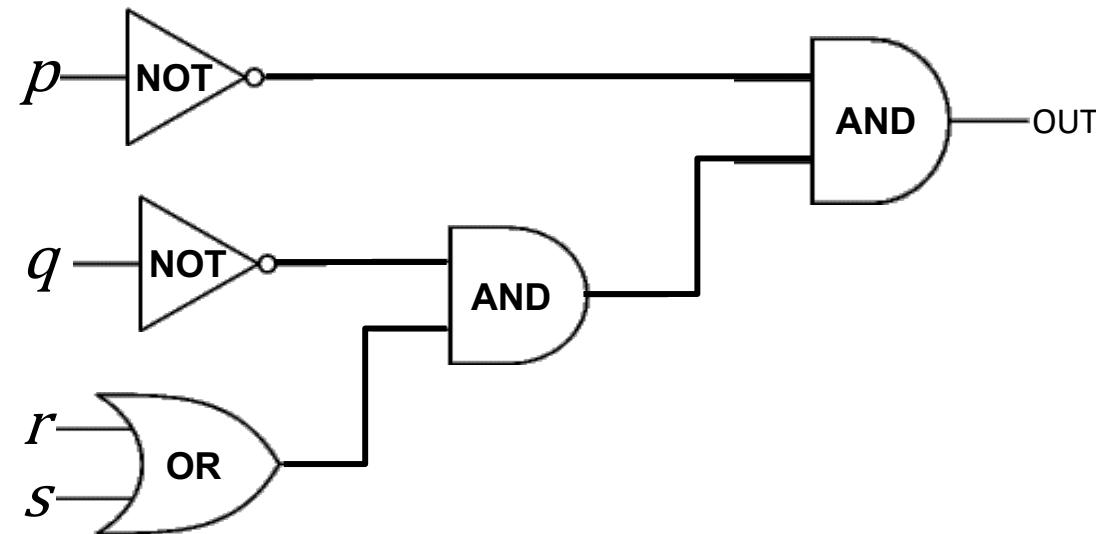


$p$	OUT
1	0
0	1

$p$	$\neg p$
T	F
F	T

# Combinational Logic Circuits

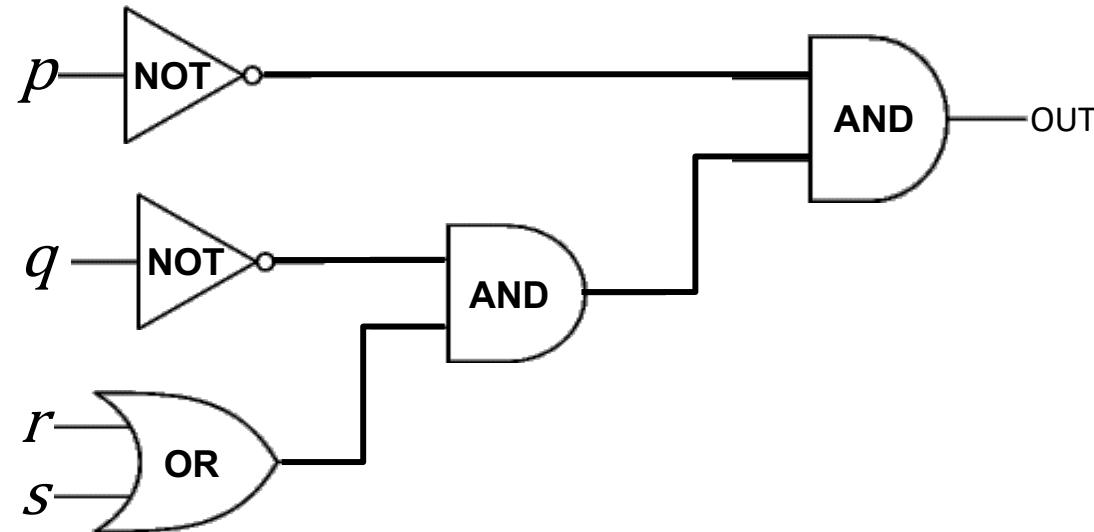
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Values get sent along wires connecting gates

# Combinational Logic Circuits

---

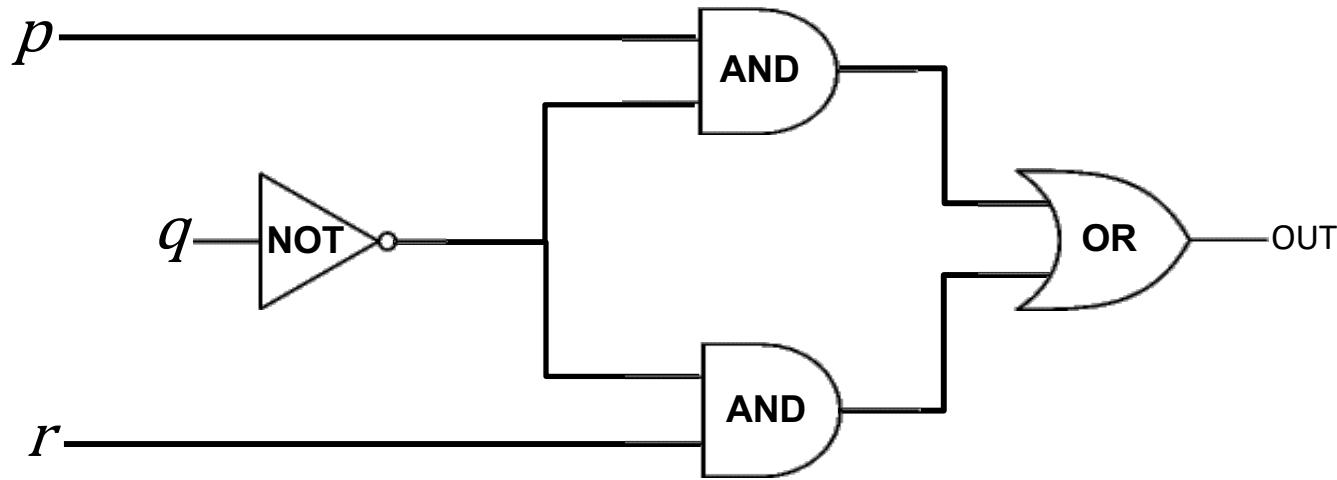


Values get sent along wires connecting gates

$$\neg p \wedge (\neg q \wedge (r \vee s))$$

# Combinational Logic Circuits

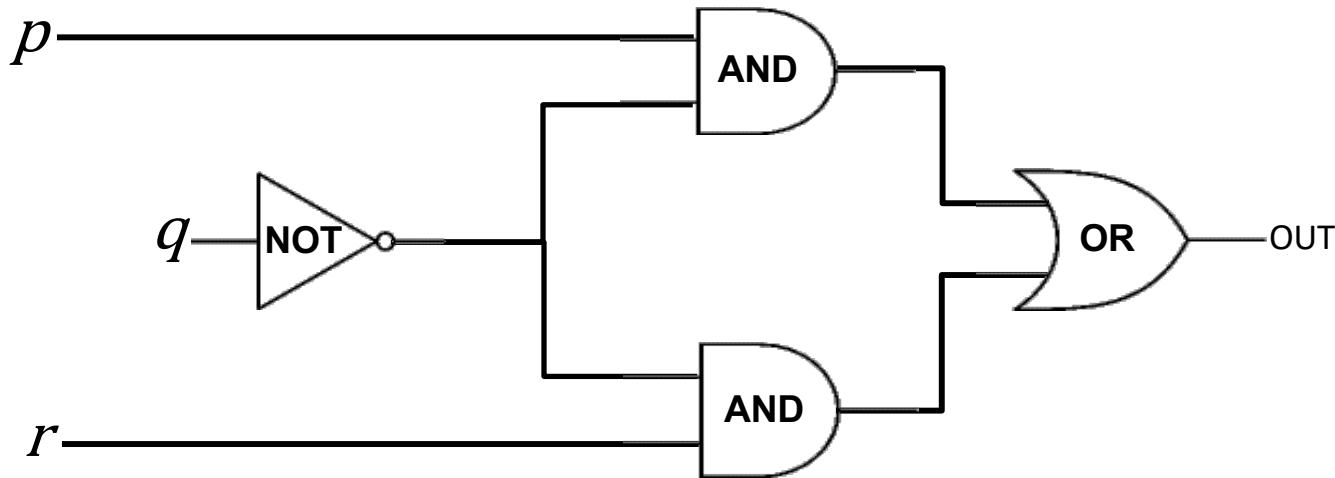
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Wires can send one value to multiple gates!

# Combinational Logic Circuits

---



Wires can send one value to multiple gates!

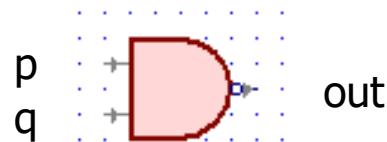
$$(p \wedge \neg q) \vee (\neg q \wedge r)$$

# Other Useful Gates

---

**NAND**

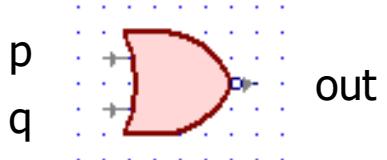
$$\neg(p \wedge q)$$



p	q	out
0	0	1
0	1	1
1	0	1
1	1	0

**NOR**

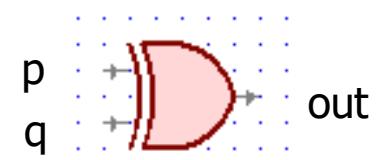
$$\neg(p \vee q)$$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	0

**XOR**

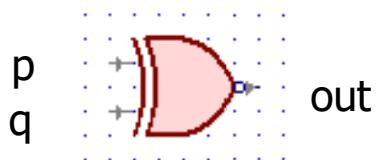
$$p \oplus q$$



p	q	out
0	0	0
0	1	1
1	0	1
1	1	0

**XNOR**

$$p \leftrightarrow q$$

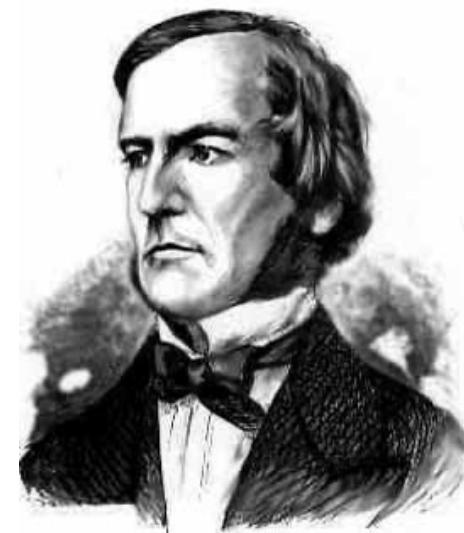


p	q	out
0	0	1
0	1	0
1	0	0
1	1	1

# Boolean Algebra

---

- Usual notation used in circuit design
- Boolean algebra
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { a' } or {  $\bar{a}$  }



Write these in Boolean Algebra:

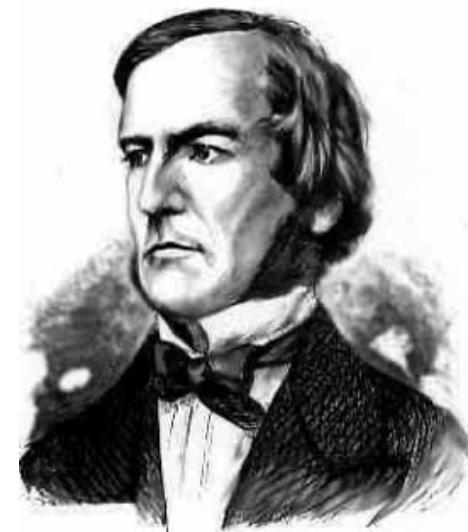
$$\neg p \wedge (\neg q \wedge (r \vee s))$$

$$(p \wedge \neg q) \vee (\neg q \wedge r)$$

# Boolean Algebra

---

- Usual notation used in circuit design
- Boolean algebra
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { a' } or {  $\bar{a}$  }



Write these in Boolean Algebra:

$$\neg p \wedge (\neg q \wedge (r \vee s)) \quad (p \wedge \neg q) \vee (\neg q \wedge r)$$

$$p'q'(r + s)$$

$$pq' + q'r$$

# A Combinational Logic Example

---

## Sessions of Class:

We would like to compute the number of lectures or quiz sections remaining *at the start* of a given day of the week.

- **Inputs:** Day of the Week, Lecture/Section flag
- **Output:** Number of sessions left

Examples: Input: (Wednesday, Lecture) Output: **2**

Input: (Monday, Section)      Output: **1**

# Implementation in Software

---

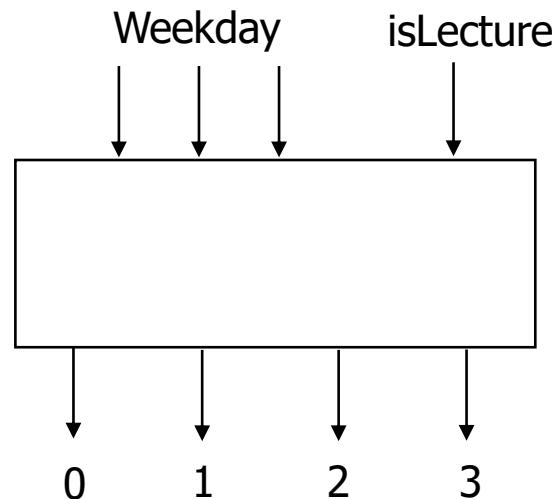
```
public int classesLeftInMorning(int weekday, boolean isLecture) {  
    switch (weekday) {  
        case SUNDAY:  
        case MONDAY:  
            return isLecture ? 3 : 1;  
        case TUESDAY:  
        case WEDNESDAY:  
            return isLecture ? 2 : 1;  
        case THURSDAY:  
            return isLecture ? 1 : 1;  
        case FRIDAY:  
            return isLecture ? 1 : 0;  
        case SATURDAY:  
            return isLecture ? 0 : 0;  
    }  
}
```

# Implementation with Hardware

---

## Encoding:

- How many bits for each input/output?
- Binary number for weekday
- One bit for each possible output



# Defining Our Inputs!

---

## Weekday Input:

- Binary number for weekday
- Sunday = 0, Monday = 1, ...
- We care about these in binary:

Weekday	Number	Binary
Sunday	0	000
Monday	1	001
Tuesday	2	010
Wednesday	3	011
Thursday	4	100
Friday	5	101
Saturday	6	110

# Converting to a Truth Table!

---

```
case SUNDAY or MONDAY:  
    return isLecture ? 3 : 1;  
case TUESDAY or WEDNESDAY:  
    return isLecture ? 2 : 1;  
case THURSDAY:  
    return isLecture ? 1 : 1;  
case FRIDAY:  
    return isLecture ? 1 : 0;  
case SATURDAY:  
    return isLecture ? 0 : 0;
```

	<b>Weekday</b>	<b>isLecture</b>	<b><math>c_0</math></b>	<b><math>c_1</math></b>	<b><math>c_2</math></b>	<b><math>c_3</math></b>
	SUN	000	0			
	SUN	000	1			
	MON	001	0			
	MON	001	1			
	TUE	010	0			
	TUE	010	1			
	WED	011	0			
	WED	011	1			
	THU	100	-			
	FRI	101	0			
	FRI	101	1			
	SAT	110	-			

# Converting to a Truth Table!

---

```
case SUNDAY or MONDAY:  
    return isLecture ? 3 : 1;  
case TUESDAY or WEDNESDAY:  
    return isLecture ? 2 : 1;  
case THURSDAY:  
    return isLecture ? 1 : 1;  
case FRIDAY:  
    return isLecture ? 1 : 0;  
case SATURDAY:  
    return isLecture ? 0 : 0;
```

	<b>Weekday</b>	<b>isLecture</b>	<b><math>c_0</math></b>	<b><math>c_1</math></b>	<b><math>c_2</math></b>	<b><math>c_3</math></b>
	SUN 000	0	0	1	0	0
	SUN 000	1	0	0	0	1
	MON 001	0	0	1	0	0
	MON 001	1	0	0	0	1
	TUE 010	0	0	1	0	0
	TUE 010	1	0	0	1	0
	WED 011	0	0	1	0	0
	WED 011	1	0	0	1	0
	THU 100	-	0	1	0	0
	FRI 101	0	1	0	0	0
	FRI 101	1	0	1	0	0
	SAT 110	-	1	0	0	0

# Truth Table to Logic

---

$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0
SUN	000	1	0	0	1
MON	001	0	0	1	0
MON	001	1	0	0	1
TUE	010	0	0	1	0
TUE	010	1	0	0	1
WED	011	0	0	1	0
WED	011	1	0	0	1
THU	100	-	0	1	0
FRI	101	0	1	0	0
FRI	101	1	0	1	0
SAT	110	-	1	0	0

Let's begin by finding an expression for  $c_3$ . To do this, we look at the rows where  $c_3 = 1$  (true).

# Truth Table to Logic

---

$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN 000	0	0	1	0	0
SUN 000	1	0	0	0	1
MON 001	0	0	1	0	0
MON 001	1	0	0	0	1
TUE 010	0	0	1	0	0
TUE 010	1	0	0	1	0
WED 011	0	0	1	0	0
WED 011	1	0	0	1	0
THU 100	-	0	1	0	0
FRI 101	0	1	0	0	0
FRI 101	1	0	1	0	0
SAT 110	-	1	0	0	0

DAY == SUN && L == 1

DAY == MON && L == 1

# Truth Table to Logic

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

$d_2d_1d_0 == 000 \&& L == 1$

$d_2d_1d_0 == 001 \&& L == 1$

Substituting DAY for the binary representation.

# Truth Table to Logic

$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN 000	0	0	1	0	0
SUN 000	1	0	0	0	1
MON 001	0	0	1	0	0
MON 001	1	0	0	0	1
TUE 010	0	0	1	0	0
TUE 010	1	0	0	1	0
WED 011	0	0	1	0	0
WED 011	1	0	0	1	0
THU 100	-	0	1	0	0
FRI 101	0	1	0	0	0
FRI 101	1	0	1	0	0
SAT 110	-	1	0	0	0

$d_2 == 0 \&\& d_1 == 0 \&\& d_0 == 0 \&\& L == 1$   
 $d_2 == 0 \&\& d_1 == 0 \&\& d_0 == 1 \&\& L == 1$

Splitting up the bits of the day;  
so, we can write a formula.

# Truth Table to Logic

$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN 000	0	0	1	0	0
SUN 000	1	0	0	0	1
MON 001	0	0	1	0	0
MON 001	1	0	0	0	1
TUE 010	0	0	1	0	0
TUE 010	1	0	0	1	0
WED 011	0	0	1	0	0
WED 011	1	0	0	1	0
THU 100	-	0	1	0	0
FRI 101	0	1	0	0	0
FRI 101	1	0	1	0	0
SAT 110	-	1	0	0	0

Replacing with  
Boolean Algebra...

$$d_2' \cdot d_1' \cdot d_0' \cdot L$$

$$d_2' \cdot d_1' \cdot d_0 \cdot L$$

# Truth Table to Logic

$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN 000	0	0	1	0	0
SUN 000	1	0	0	0	1
MON 001	0	0	1	0	0
MON 001	1	0	0	0	1
TUE 010	0	0	1	0	0
TUE 010	1	0	0	1	0
WED 011	0	0	1	0	0
WED 011	1	0	0	1	0
THU 100	-	0	1	0	0
FRI 101	0	1	0	0	0
FRI 101	1	0	1	0	0
SAT 110	-	1	0	0	0

$$d_2' \cdot d_1' \cdot d_0' \cdot L$$

$$d_2' \cdot d_1' \cdot d_0 \cdot L$$

How do we combine them?

# Truth Table to Logic

$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN 000	0	0	1	0	0
SUN 000	1	0	0	0	1
MON 001	0	0	1	0	0
MON 001	1	0	0	0	1
TUE 010	0	0	1	0	0
TUE 010	1	0	0	1	0
WED 011	0	0	1	0	0
WED 011	1	0	0	1	0
THU 100	-	0	1	0	0
FRI 101	0	1	0	0	0
FRI 101	1	0	1	0	0
SAT 110	-	1	0	0	0

$$d_2' \cdot d_1' \cdot d_0' \cdot L$$

$$d_2' \cdot d_1' \cdot d_0 \cdot L$$

Either situation causes  $c_3$  to be true. So, we “or” them.

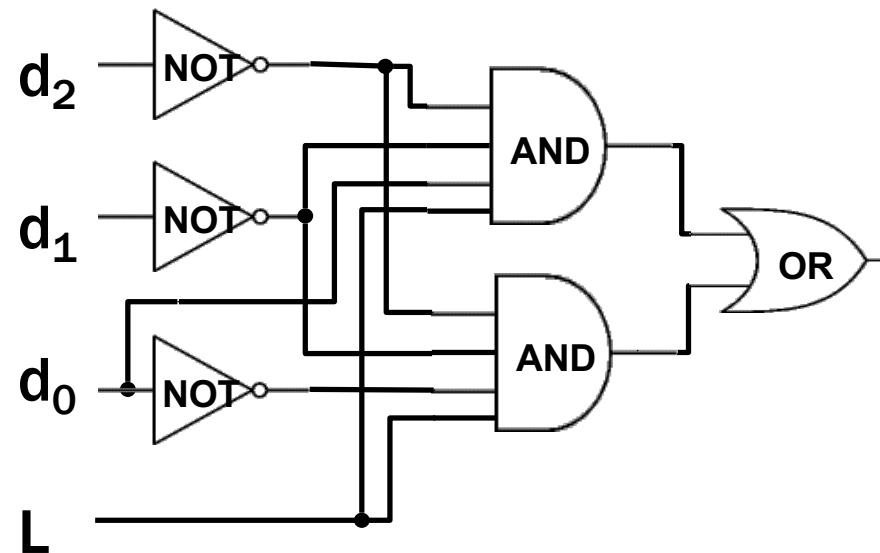
$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + \\ d_2' \cdot d_1' \cdot d_0 \cdot L$$

# Truth Table to Logic

---

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot \bar{L}$$

Here's  $c_3$  as a circuit:



# Truth Table to Logic (Part 2)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE 010 1			0	0	1	0
WED 011 0			0	1	0	0
WED 011 1			0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

Now, we do  $c_2$ .

# Truth Table to Logic (Part 3)

$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$	
SUN 000	0	0	1	0	0	
SUN 000	1	0	0	0	1	→ $d_2 + d_1 + d_0 + L'$
MON 001	0	0	1	0	0	
MON 001	1	0	0	0	1	→
TUE 010	0	0	1	0	0	
TUE 010	1	0	0	1	0	→
WED 011	0	0	1	0	0	
WED 011	1	0	0	1	0	→
THU 100	-	0	1	0	0	
FRI 101	0	1	0	0	0	→
FRI 101	1	0	1	0	0	→
SAT 110	-	1	0	0	0	→

For  $c_1$ , let's look at the 0s:

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

# Truth Table to Logic (Part 3)

$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$	
SUN 000	0	0	1	0	0	
SUN 000	1	0	0	0	1	$d_2 + d_1 + d_0 + L'$
MON 001	0	0	1	0	0	
MON 001	1	0	0	0	1	$d_2 + d_1 + d_0' + L'$
TUE 010	0	0	1	0	0	
TUE 010	1	0	0	1	0	$d_2 + d_1' + d_0 + L'$
WED 011	0	0	1	0	0	
WED 011	1	0	0	1	0	$d_2 + d_1' + d_0' + L'$
THU 100	-	0	1	0	0	
FRI 101	0	1	0	0	0	$d_2' + d_1 + d_0' + L$
FRI 101	1	0	1	0	0	
SAT 110	-	1	0	0	0	???

For  $c_1$ , let's look at the 0s:

# Truth Table to Logic (Part 3)

$d_2d_1d_0$	$L$	$c_0$	$c_1$	$c_2$	$c_3$	
SUN 000	0	0	1	0	0	
SUN 000	1	0	0	0	1	$d_2 + d_1 + d_0 + L'$
MON 001	0	0	1	0	0	
MON 001	1	0	0	0	1	$d_2 + d_1 + d_0' + L'$
TUE 010	0	0	1	0	0	
TUE 010	1	0	0	1	0	$d_2 + d_1' + d_0 + L'$
WED 011	0	0	1	0	0	
WED 011	1	0	0	1	0	$d_2 + d_1' + d_0' + L'$
THU 100	-	0	1	0	0	
FRI 101	0	1	0	0	0	$d_2' + d_1 + d_0' + L$
FRI 101	1	0	1	0	0	
SAT 110	-	1	0	0	0	$d_2' + d_1' + d_0$

For  $c_1$ , let's look at the 0s:

$$d_2 + d_1 + d_0 + L'$$

$$d_2 + d_1 + d_0' + L'$$

$$d_2 + d_1' + d_0 + L'$$

$$d_2 + d_1' + d_0' + L'$$

$$d_2' + d_1 + d_0' + L$$

$$d_2' + d_1' + d_0$$

No matter what  $L$  is, we always say it's 1.  
So, we don't need  $L$  in the expression.

# Truth Table to Logic (Part 3)

$d_2d_1d_0$	$L$	$c_0$	$c_1$	$c_2$	$c_3$	
SUN 000	0	0	1	0	0	
SUN 000	1	0	0	0	1	$d_2 + d_1 + d_0 + L'$
MON 001	0	0	1	0	0	
MON 001	1	0	0	0	1	$d_2 + d_1 + d_0' + L'$
TUE 010	0	0	1	0	0	
TUE 010	1	0	0	1	0	$d_2 + d_1' + d_0 + L'$
WED 011	0	0	1	0	0	
WED 011	1	0	0	1	0	$d_2 + d_1' + d_0' + L'$
THU 100	-	0	1	0	0	
FRI 101	0	1	0	0	0	$d_2' + d_1 + d_0' + L$
FRI 101	1	0	1	0	0	
SAT 110	-	1	0	0	0	$d_2' + d_1' + d_0$

For  $c_1$ , let's look at the 0s:

$$d_2 + d_1 + d_0 + L'$$

$$d_2 + d_1 + d_0' + L'$$

$$d_2 + d_1' + d_0 + L'$$

$$d_2 + d_1' + d_0' + L'$$

$$d_2' + d_1 + d_0' + L$$

$$d_2' + d_1' + d_0$$

How do we combine them?

# Truth Table to Logic (Part 3)

$d_2d_1d_0$	$L$	$c_0$	$c_1$	$c_2$	$c_3$	
SUN 000	0	0	1	0	0	
SUN 000	1	0	0	0	1	$d_2 + d_1 + d_0 + L'$
MON 001	0	0	1	0	0	
MON 001	1	0	0	0	1	$d_2 + d_1 + d_0' + L'$
TUE 010	0	0	1	0	0	
TUE 010	1	0	0	1	0	$d_2 + d_1' + d_0 + L'$
WED 011	0	0	1	0	0	
WED 011	1	0	0	1	0	$d_2 + d_1' + d_0' + L'$
THU 100	-	0	1	0	0	
FRI 101	0	1	0	0	0	$d_2' + d_1 + d_0' + L$
FRI 101	1	0	1	0	0	
SAT 110	-	1	0	0	0	$d_2' + d_1' + d_0$

For  $c_1$ , let's look at the 0s:

$$\begin{aligned}
 c_1 = & (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')(d_2 + d_1' + d_0 + L') \\
 & (d_2 + d_1' + d_0' + L')(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)
 \end{aligned}$$

# Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

For  $c_1$ , let's look at the 0s:

Is  $c_1$  still in CNF form?

Yes, but not canonical CNF

$$\begin{aligned}
 c_1 = & (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')(d_2 + d_1' + d_0 + L') \\
 & (d_2 + d_1' + d_0' + L')(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)
 \end{aligned}$$

# Truth Table to Logic (Part 4)

---

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$	
SUN	000	0	0	1	0	0	$c_1 = (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')$ $(d_2 + d_1' + d_0 + L')(d_2 + d_1' + d_0' + L')$ $(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)$
SUN	000	1	0	0	0	1	$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$
MON	001	0	0	1	0	0	$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	
FRI	101	0	1	0	0	0	
FRI	101	1	0	1	0	0	
SAT	110	-	1	0	0	0	

# Truth Table to Logic (Part 4)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$	
SUN	000	0	0	1	0	0	$c_1 = (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')$ $(d_2 + d_1' + d_0 + L')(d_2 + d_1' + d_0' + L')$ $(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)$
SUN	000	1	0	0	0	1	$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$
MON	001	0	0	1	0	0	$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	
FRI	101	0	1	0	0	0	Finally, we do $c_0$ :
FRI	101	1	0	1	0	0	
SAT	110	-	1	0	0	0	

# Truth Table to Logic (Part 4)

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$	
SUN	000	0	0	1	0	0	$c_1 = (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')$ $(d_2 + d_1' + d_0 + L')(d_2 + d_1' + d_0' + L')$ $(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)$
SUN	000	1	0	0	0	1	$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$
MON	001	0	0	1	0	0	$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$
MON	001	1	0	0	0	1	
TUE	010	0	0	1	0	0	
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	
FRI	101	0	1	0	0	0	$d_2 \cdot d_1' \cdot d_0 \cdot L'$
FRI	101	1	0	1	0	0	
SAT	110	-	1	0	0	0	$d_2 \cdot d_1 \cdot d_0'$

Finally, we do  $c_0$ :

$$d_2 \cdot d_1' \cdot d_0 \cdot L'$$

$$d_2 \cdot d_1 \cdot d_0'$$

# Truth Table to Logic (Part 4)

---

	$d_2d_1d_0$	L	$c_0$	$c_1$	$c_2$	$c_3$	
SUN	000	0	0	1	0	0	$c_1 = (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')$ $(d_2 + d_1' + d_0 + L')(d_2 + d_1' + d_0' + L')$ $(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)$
SUN	000	1	0	0	0	1	$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$
MON	001	0	0	1	0	0	$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$
MON	001	1	0	0	0	1	$c_0 = d_2 \cdot d_1' \cdot d_0 \cdot L' + d_2 \cdot d_1 \cdot d_0'$
TUE	010	0	0	1	0	0	
TUE	010	1	0	0	1	0	
WED	011	0	0	1	0	0	
WED	011	1	0	0	1	0	
THU	100	-	0	1	0	0	
FRI	101	0	1	0	0	0	
FRI	101	1	0	1	0	0	
SAT	110	-	1	0	0	0	

# Mapping Truth Tables to Logic Gates

---

**Given a truth table:**

1. Write the output in a table
2. Write the Boolean expression
3. Draw as gates

# Equivalence

# One Application of Equivalence

---

Given a truth table:

1. Write the output in a table
2. Write the Boolean expression
3. Draw as gates

This will give us *some* circuit.  
But is it the best circuit?

# Tautologies!

---

**Terminology:** A compound proposition is a...

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$a$	$b$	...	$f(a,b)$	$g(a,b)$
T	T	...	T	F
F	T	...	T	F
T	F	...	T	F
F	F	...	T	F
...	...	...	...	...

# Tautologies!

---

**Terminology:** A compound proposition is a...

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$$p \vee \neg p$$

$$p \oplus p$$

$$(p \rightarrow r) \wedge p$$

# Tautologies!

---

**Terminology:** A compound proposition is a...

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$$p \vee \neg p$$

This is a tautology. It's called the “law of the excluded middle”.  
If  $p$  is true, then  $p \vee \neg p$  is true. If  $p$  is false, then  $p \vee \neg p$  is true.

$$p \oplus p$$

This is a contradiction. It's always false no matter what truth value  $p$  takes on.

$$(p \rightarrow r) \wedge p$$

This is a contingency. When  $p=T$ ,  $r=T$ ,  $(T \rightarrow T) \wedge T$  is true.  
When  $p=T$ ,  $r=F$ ,  $(T \rightarrow F) \wedge T$  is false.

# Logical Equivalence

---

**A = B** means **A** and **B** are the same thing written twice:

- $p \wedge r = p \wedge r$
- $p \wedge r \neq r \wedge p$

# Logical Equivalence

---

**A = B means A and B are the same thing written twice:**

- $p \wedge r = p \wedge r$

These are equal, because they are character-for-character identical.

- $p \wedge r \neq r \wedge p$

These are NOT equal, because they are different sequences of characters. They “mean” the same thing though.

- $p \wedge q \wedge r = (p \wedge q) \wedge r$

# Logical Equivalence

---

**A = B means A and B are the same thing written twice:**

- $p \wedge r = p \wedge r$

These are equal, because they are character-for-character identical.

- $p \wedge r \neq r \wedge p$

These are NOT equal, because they are different sequences of characters. They “mean” the same thing though.

- $p \wedge q \wedge r = (p \wedge q) \wedge r$

These are equal. The parentheses are *implicit* in the first case.

(Also, things whitespace also should not matter. In full detail, equality is between parse trees not strings. More later on...)

# Logical Equivalence

---

**A = B** means A and B are the same thing written twice:

- $p \wedge r = p \wedge r$

These are equal, because they are character-for-character identical.

- $p \wedge r \neq r \wedge p$

These are NOT equal, because they are different sequences of characters. They “mean” the same thing though.

**A ≡ B** means A and B have identical truth values:

- $p \wedge r \equiv p \wedge r$

- $p \wedge r \equiv r \wedge p$

- $p \wedge r \not\equiv r \vee p$

# Logical Equivalence

---

**A = B** means A and B are the same thing written twice:

- $p \wedge r = p \wedge r$

These are equal, because they are character-for-character identical.

- $p \wedge r \neq r \wedge p$

These are NOT equal, because they are different sequences of characters. They “mean” the same thing though.

**A  $\equiv$  B** means A and B have identical truth values:

- $p \wedge r \equiv p \wedge r$

Two formulas that are equal also are equivalent.

- $p \wedge r \equiv r \wedge p$

These two formulas have the same truth table!

- $p \wedge r \not\equiv r \vee p$

When  $p=T$  and  $r=F$ ,  $p \wedge r$  is false, but  $p \vee r$  is true!

## **$A \leftrightarrow B$ vs. $A \equiv B$**

---

**$A \leftrightarrow B$  is a proposition that may be true or false depending on the truth values of  $A$  and  $B$ .**

Example:  $(p \wedge r) \leftrightarrow (r \vee p)$  has 4 rows in its truth table

**$A \equiv B$  is an assertion over all possible truth values that  $A$  and  $B$  always have the same truth values.**

Example:  $p \wedge r \equiv r \wedge p$  is true

**$A \equiv B$  and  $(A \leftrightarrow B) \equiv T$  have the same meaning as does “ $A \leftrightarrow B$  is a tautology”**

# Logical Equivalence $A \equiv B$

---

$A \equiv B$  is an assertion that *two propositions A and B* always have the same truth values.

$A \equiv B$  and  $(A \leftrightarrow B) \equiv T$  have the same meaning.

$$p \wedge r \equiv r \wedge p$$

$p$	$r$	$p \wedge r$	$r \wedge p$	$(p \wedge r) \leftrightarrow (r \wedge p)$
T	T			
T	F			
F	T			
F	F			

# Logical Equivalence $A \equiv B$

---

$A \equiv B$  is an assertion that *two propositions A and B* always have the same truth values.

$A \equiv B$  and  $(A \leftrightarrow B) \equiv T$  have the same meaning.

$$p \wedge r \equiv r \wedge p$$

$p$	$r$	$p \wedge r$	$r \wedge p$	$(p \wedge r) \leftrightarrow (r \wedge p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

# Familiar Equivalence

---

## Double Negation

$$p \equiv \neg \neg p$$

$p$	$\neg p$	$\neg \neg p$
T	F	T
F	T	F

# De Morgan's Laws

---

$$\neg(p \wedge r) \equiv \neg p \vee \neg r$$

$$\neg(p \vee r) \equiv \neg p \wedge \neg r$$

**Negate the statement:**

“My code compiles or there is a bug.”

**To negate the statement,  
ask “when is the original statement false”.**

It's false when not(my code compiles) AND not(there is a bug).

Translating back into English, we get:  
My code doesn't compile and there is not a bug.

# De Morgan's Laws

---

Example:  $\neg(p \wedge r) \equiv \neg p \vee \neg r$

$p$	$r$	$\neg p$	$\neg r$	$\neg p \vee \neg r$	$p \wedge r$	$\neg(p \wedge r)$
T	T	F	F			
T	F	F	T			
F	T	T	F			
F	F	T	T			

# De Morgan's Laws

---

Example:  $\neg(p \wedge r) \equiv \neg p \vee \neg r$

$p$	$r$	$\neg p$	$\neg r$	$\neg p \vee \neg r$	$p \wedge r$	$\neg(p \wedge r)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

# De Morgan's Laws

---

$$\neg(p \wedge r) \equiv \neg p \vee \neg r$$

$$\neg(p \vee r) \equiv \neg p \wedge \neg r$$

```
if (!(front != null && value > front.data)) {  
    front = new ListNode(value, front);  
} else {  
    ListNode current = front;  
    while (current.next != null && current.next.data < value)  
        current = current.next;  
    current.next = new ListNode(value, current.next);  
}
```

# De Morgan's Laws

---

$$\neg(p \wedge r) \equiv \neg p \vee \neg r$$

$$\neg(p \vee r) \equiv \neg p \wedge \neg r$$

`!(front != null && value > front.data)`

$\equiv$

`front == null || value <= front.data`

# Law of Implication

---

$$p \rightarrow r \equiv \neg p \vee r$$

$p$	$r$	$p \rightarrow r$	$\neg p$	$\neg p \vee r$
T	T			
T	F			
F	T			
F	F			

# Law of Implication

---

$$p \rightarrow r \equiv \neg p \vee r$$

$p$	$r$	$p \rightarrow r$	$\neg p$	$\neg p \vee r$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

## Biconditional: $p \leftrightarrow r$

---

- $p$  if and only if  $r$     ( $p$  iff  $r$ )
- $p$  implies  $r$  and  $r$  implies  $p$
- $p$  is necessary and sufficient for  $r$

$p$	$r$	$p \leftrightarrow r$	$p \rightarrow r$	$r \rightarrow p$	$(p \rightarrow r) \wedge (r \rightarrow p)$
T	T	T	T	T	
T	F	F	F	T	
F	T	F	T	F	
F	F	T	T	T	

## Biconditional: $p \leftrightarrow r$

---

- $p$  if and only if  $r$     ( $p$  iff  $r$ )
- $p$  implies  $r$  and  $r$  implies  $p$
- $p$  is necessary and sufficient for  $r$

$p$	$r$	$p \leftrightarrow r$	$p \rightarrow r$	$r \rightarrow p$	$(p \rightarrow r) \wedge (r \rightarrow p)$
T	T	T	T	T	T
T	F	F	F	T	F
F	T	F	T	F	F
F	F	T	T	T	T

# Some Familiar Properties of Arithmetic

---

- $x + y = y + x$  **(Commutativity)**
- $x \cdot (y + z) = x \cdot y + x \cdot z$  **(Distributivity)**
- $(x + y) + z = x + (y + z)$  **(Associativity)**

# Important Equivalences

---

- **Identity**
  - $p \wedge \text{T} \equiv p$
  - $p \vee \text{F} \equiv p$
- **Domination**
  - $p \vee \text{T} \equiv \text{T}$
  - $p \wedge \text{F} \equiv \text{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \text{T}$
  - $p \wedge \neg p \equiv \text{F}$

# Some Familiar Properties of Arithmetic

---

- $x \cdot 1 = x$  (Identity)
- $x + 0 = x$
- $x \cdot 0 = 0$  (Domination)

# Important Equivalences

---

- **Identity**
  - $p \wedge \text{T} \equiv p$
  - $p \vee \text{F} \equiv p$
- **Domination**
  - $p \vee \text{T} \equiv \text{T}$
  - $p \wedge \text{F} \equiv \text{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \text{T}$
  - $p \wedge \neg p \equiv \text{F}$

# Some Familiar Properties of Arithmetic

---

- Usual properties hold under relabeling:
  - 0, 1 becomes F, T
  - “+” becomes “ $\vee$ ”
  - “.” becomes “ $\wedge$ ”
- But there are some new facts:
  - Distributivity works for both “ $\wedge$ ” and “ $\vee$ ”
  - Domination works with T
- There are some other facts specific to logic...

# Important Equivalences

---

- **Identity**
  - $p \wedge \text{T} \equiv p$
  - $p \vee \text{F} \equiv p$
- **Domination**
  - $p \vee \text{T} \equiv \text{T}$
  - $p \wedge \text{F} \equiv \text{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \text{T}$
  - $p \wedge \neg p \equiv \text{F}$

# Important Equivalences

---

- **Identity**
  - $p \wedge \text{T} \equiv p$
  - $p \vee \text{F} \equiv p$
- **Domination**
  - $p \vee \text{T} \equiv \text{T}$
  - $p \wedge \text{F} \equiv \text{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \text{T}$
  - $p \wedge \neg p \equiv \text{F}$

# Using Equivalences

---

- Note that  $p$ ,  $q$ , and  $r$  can be any propositions (not just atomic propositions)
- Ex:  $(r \rightarrow s) \wedge (\neg t) \equiv (\neg t) \wedge (r \rightarrow s)$ 
  - apply commutativity:  $p \wedge q \equiv q \wedge p$   
with  $p := r \rightarrow s$   
and  $q := \neg t$

# Computing Equivalence

---

**Given two propositions, can we write an algorithm to determine if they are equivalent?**

**What is the runtime of our algorithm?**

# Computing Equivalence

---

**Given two propositions, can we write an algorithm to determine if they are equivalent?**

Yes! Generate the truth tables for both propositions and check if they are the same for every entry.

**What is the runtime of our algorithm?**

Every atomic proposition has two possibilities (T, F). If there are  $n$  atomic propositions, there are  $2^n$  rows in the truth table.

# Another approach: Equivalence Chains

---

## To show A is equivalent to B

- Apply a series of logical equivalences to sub-expressions to convert A to B

## To show A is a tautology

- Apply a series of logical equivalences to sub-expressions to convert A to T

# Another approach: Equivalence Chains

---

**To show A is equivalent to B**

- Apply a series of logical equivalences to sub-expressions to convert A to B

**Example:**

Let A be “ $p \vee (p \wedge p)$ ”, and B be “ $p$ ”.

Our general equivalence proof looks like:

$$\begin{aligned} p \vee (p \wedge p) &\equiv \\ &\equiv p \end{aligned}$$

# Another approach: Logical Equivalences

---

- **Identity**

- $p \wedge T \equiv p$
  - $p \vee F \equiv p$

- **Domination**

- $p \vee T \equiv T$
  - $p \wedge F \equiv F$

- **Idempotent**

- $p \vee p \equiv p$
  - $p \wedge p \equiv p$

- **Commutative**

- $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$

- **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- **Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **Absorption**

- $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$

- **Negation**

- $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$

**De Morgan's Laws**

$$\begin{aligned}\neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q\end{aligned}$$

**Law of Implication**

$$p \rightarrow q \equiv \neg p \vee q$$

**Contrapositive**

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

**Biconditional**

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

**Double Negation**

$$p \equiv \neg \neg p$$

## Example:

Let A be " $p \vee (p \wedge p)$ ", and B be " $p$ ".

Our general equivalence proof looks like:

$$\begin{aligned}p \vee (p \wedge p) &\equiv \\ &\equiv p\end{aligned}$$

# Logical Equivalences

---

- **Identity**

- $p \wedge T \equiv p$
  - $p \vee F \equiv p$

- **Domination**

- $p \vee T \equiv T$
  - $p \wedge F \equiv F$

- **Idempotent**

- $p \vee p \equiv p$
  - $p \wedge p \equiv p$

- **Commutative**

- $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$

- **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- **Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **Absorption**

- $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$

- **Negation**

- $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$

**De Morgan's Laws**

$$\begin{aligned}\neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q\end{aligned}$$

**Law of Implication**

$$p \rightarrow q \equiv \neg p \vee q$$

**Contrapositive**

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

**Biconditional**

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

**Double Negation**

$$p \equiv \neg \neg p$$

## Example:

Let A be " $p \vee (p \wedge p)$ ", and B be " $p$ ".

Our general equivalence proof looks like:

$$\begin{aligned}p \vee (p \wedge p) &\equiv p \vee p \\ &\equiv p\end{aligned}$$

**Idempotent**

**Idempotent**

# Logical Equivalences

---

**To show A is a tautology**

- Apply a series of logical equivalences to sub-expressions to convert A to T

**Example:**

Let A be “ $\neg p \vee (p \vee p)$ ”.

Our general equivalence proof looks like:

$$\begin{aligned}\neg p \vee (p \vee p) &\equiv \\ &\equiv \\ &\equiv T\end{aligned}$$

# Logical Equivalences

---

- **Identity**

- $p \wedge T \equiv p$
  - $p \vee F \equiv p$

- **Domination**

- $p \vee T \equiv T$
  - $p \wedge F \equiv F$

- **Idempotent**

- $p \vee p \equiv p$
  - $p \wedge p \equiv p$

- **Commutative**

- $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$

- **Associative**

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- **Distributive**

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- **Absorption**

- $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$

- **Negation**

- $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$

**De Morgan's Laws**

$$\begin{aligned}\neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q\end{aligned}$$

**Law of Implication**

$$p \rightarrow q \equiv \neg p \vee q$$

**Contrapositive**

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

**Biconditional**

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

**Double Negation**

$$p \equiv \neg \neg p$$

## Example:

Let A be " $\neg p \vee (p \vee p)$ ".

Our general equivalence proof looks like:

$$\begin{aligned}\neg p \vee (p \vee p) &\equiv \\ &\equiv \\ &\equiv T\end{aligned}$$

# Logical Equivalences

---

- **Identity**
  - $p \wedge T \equiv p$
  - $p \vee F \equiv p$
- **Domination**
  - $p \vee T \equiv T$
  - $p \wedge F \equiv F$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$

- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$

## De Morgan's Laws

$$\begin{aligned}\neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q\end{aligned}$$

## Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

## Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

## Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

## Double Negation

$$p \equiv \neg \neg p$$

## Example:

Let A be " $\neg p \vee (p \vee p)$ ".

Our general equivalence proof looks like:

$$\begin{aligned}\neg p \vee (p \vee p) &\equiv \neg p \vee p \\ &\equiv p \vee \neg p \\ &\equiv T\end{aligned}$$

**Idempotent**  
**Commutative**  
**Negation**

# Prove these propositions are equivalent: Option 1

---

Prove:  $p \wedge (p \rightarrow r) \equiv p \wedge r$

Make a Truth Table and show:

$$(p \wedge (p \rightarrow r)) \leftrightarrow (p \wedge r) \equiv \mathbf{T}$$

$p$	$r$	$p \rightarrow r$	$(p \wedge (p \rightarrow r))$	$p \wedge r$	$(p \wedge (p \rightarrow r)) \leftrightarrow (p \wedge r)$
T	T				
T	F				
F	T				
F	F				

# Prove these propositions are equivalent: Option 1

---

Prove:  $p \wedge (p \rightarrow r) \equiv p \wedge r$

Make a Truth Table and show:

$$(p \wedge (p \rightarrow r)) \leftrightarrow (p \wedge r) \equiv \text{T}$$

$p$	$r$	$p \rightarrow r$	$(p \wedge (p \rightarrow r))$	$p \wedge r$	$(p \wedge (p \rightarrow r)) \leftrightarrow (p \wedge r)$
T	T	T	T	T	T
T	F	F	F	F	T
F	T	T	F	F	T
F	F	T	F	F	T

# Prove these propositions are equivalent: Option 2

---

**Prove:**  $p \wedge (p \rightarrow r) \equiv p \wedge r$

$$\begin{aligned} p \wedge (p \rightarrow r) &\equiv \\ &\equiv \\ &\equiv \\ &\equiv \\ &\equiv p \wedge r \end{aligned}$$

- **Identity**
  - $p \wedge T \equiv p$
  - $p \vee F \equiv p$
- **Domination**
  - $p \vee T \equiv T$
  - $p \wedge F \equiv F$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$

- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv T$
  - $p \wedge \neg p \equiv F$

**De Morgan's Laws**

$$\begin{aligned} \neg(p \wedge q) &\equiv \neg p \vee \neg q \\ \neg(p \vee q) &\equiv \neg p \wedge \neg q \end{aligned}$$

**Law of Implication**

$$p \rightarrow q \equiv \neg p \vee q$$

**Contrapositive**

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

**Biconditional**

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

**Double Negation**

$$p \equiv \neg \neg p$$

# Prove these propositions are equivalent: Option 2

---

**Prove:**  $p \wedge (p \rightarrow r) \equiv p \wedge r$

$$\begin{aligned} p \wedge (p \rightarrow r) &\equiv p \wedge (\neg p \vee r) \\ &\equiv (p \wedge \neg p) \vee (p \wedge r) \\ &\equiv \mathbf{F} \vee (p \wedge r) \\ &\equiv (p \wedge r) \vee \mathbf{F} \\ &\equiv p \wedge r \end{aligned}$$

Law of Implication  
Distributive  
Negation  
Commutative  
Identity

• <b>Identity</b> <ul style="list-style-type: none"><li>– <math>p \wedge \mathbf{T} \equiv p</math></li><li>– <math>p \vee \mathbf{F} \equiv p</math></li></ul>	• <b>Associative</b> <ul style="list-style-type: none"><li>– <math>(p \vee q) \vee r \equiv p \vee (q \vee r)</math></li><li>– <math>(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)</math></li></ul>	<b>De Morgan's Laws</b> $\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$
• <b>Domination</b> <ul style="list-style-type: none"><li>– <math>p \vee \mathbf{T} \equiv \mathbf{T}</math></li><li>– <math>p \wedge \mathbf{F} \equiv \mathbf{F}</math></li></ul>	• <b>Distributive</b> <ul style="list-style-type: none"><li>– <math>p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)</math></li><li>– <math>p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)</math></li></ul>	<b>Law of Implication</b> $p \rightarrow q \equiv \neg p \vee q$
• <b>Idempotent</b> <ul style="list-style-type: none"><li>– <math>p \vee p \equiv p</math></li><li>– <math>p \wedge p \equiv p</math></li></ul>	• <b>Absorption</b> <ul style="list-style-type: none"><li>– <math>p \vee (p \wedge q) \equiv p</math></li><li>– <math>p \wedge (p \vee q) \equiv p</math></li></ul>	<b>Contrapositive</b> $p \rightarrow q \equiv \neg q \rightarrow \neg p$
• <b>Commutative</b> <ul style="list-style-type: none"><li>– <math>p \vee q \equiv q \vee p</math></li><li>– <math>p \wedge q \equiv q \wedge p</math></li></ul>	• <b>Negation</b> <ul style="list-style-type: none"><li>– <math>p \vee \neg p \equiv \mathbf{T}</math></li><li>– <math>p \wedge \neg p \equiv \mathbf{F}</math></li></ul>	<b>Biconditional</b> $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
		<b>Double Negation</b> $p \equiv \neg \neg p$

# Prove this is a Tautology: Option 1

---

$$(p \wedge r) \rightarrow (r \vee p)$$

Make a Truth Table and show:

$$(p \wedge r) \rightarrow (r \vee p) \equiv \top$$

$p$	$r$	$p \wedge r$	$r \vee p$	$(p \wedge r) \rightarrow (r \vee p)$
T	T			
T	F			
F	T			
F	F			

# Prove this is a Tautology: Option 1

---

$$(p \wedge r) \rightarrow (r \vee p)$$

Make a Truth Table and show:

$$(p \wedge r) \rightarrow (r \vee p) \equiv \mathbf{T}$$

$p$	$r$	$p \wedge r$	$r \vee p$	$(p \wedge r) \rightarrow (r \vee p)$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

# Prove this is a Tautology: Option 2

$$(p \wedge r) \rightarrow (r \vee p)$$

Use a series of equivalences like so:

$$(p \wedge r) \rightarrow (r \vee p) \equiv$$

≡

≡

≡

≡

≡

≡

≡

≡ T

## Identity

- $p \wedge T \equiv p$
- $p \vee F \equiv p$

## Domination

- $p \vee T \equiv T$
- $p \wedge F \equiv F$

## Idempotent

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

## Commutative

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

## Associative

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

## Distributive

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

## Absorption

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

## Negation

- $p \vee \neg p \equiv T$
- $p \wedge \neg p \equiv F$

# Prove this is a Tautology: Option 2

$$(p \wedge r) \rightarrow (r \vee p)$$

Use a series of equivalences like so:

$$\begin{aligned}(p \wedge r) \rightarrow (r \vee p) &\equiv \neg(p \wedge r) \vee (r \vee p) \\ &\equiv (\neg p \vee \neg r) \vee (r \vee p) \\ &\equiv \neg p \vee (\neg r \vee (r \vee p)) \\ &\equiv \neg p \vee ((\neg r \vee r) \vee p) \\ &\equiv \neg p \vee (p \vee (\neg r \vee r)) \\ &\equiv (\neg p \vee p) \vee (\neg r \vee r) \\ &\equiv (p \vee \neg p) \vee (r \vee \neg r) \\ &\equiv \top \vee \top \\ &\equiv \top\end{aligned}$$

## Identity

- $p \wedge \top \equiv p$
- $p \vee \text{F} \equiv p$

## Domination

- $p \vee \top \equiv \top$
- $p \wedge \text{F} \equiv \text{F}$

## Idempotent

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

## Commutative

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

## Associative

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

## Distributive

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

## Absorption

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

## Negation

- $p \vee \neg p \equiv \top$
- $p \wedge \neg p \equiv \text{F}$

Law of Implication

De Morgan

Associative

Associative

Commutative

Associative

Commutative (twice)

Negation (twice)

Domination/Identity

# Chains of Equivalence/Tautology

---

- Not smaller than truth tables when there are only a few propositional variables...
- ...but usually *much shorter* than truth table proofs when there are many propositional variables
- A big advantage will be that we can extend them to a more in-depth understanding of logic for which truth tables don't apply.

# Recall: Truth Table to Logic

---

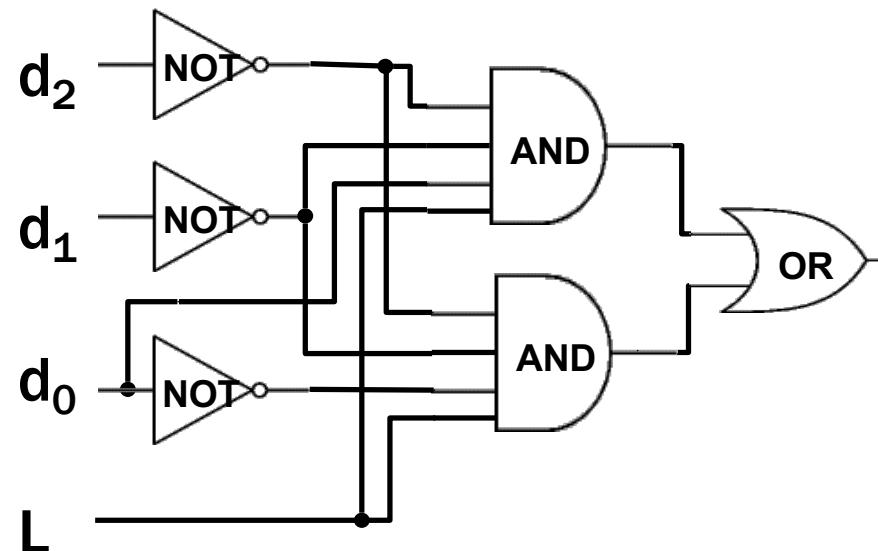
$$c_0 = d_2 \cdot d_1' \cdot d_0 \cdot L' + d_2 \cdot d_1 \cdot d_0' + d_2 \cdot d_1 \cdot d_0$$

$$c_1 = d_2' \cdot d_1' \cdot d_0' \cdot L' + d_2' \cdot d_1' \cdot d_0 \cdot L' + d_2' \cdot d_1 \cdot d_0' \cdot L' + d_2' \cdot d_1 \cdot d_0 \cdot L' + d_2 \cdot d_1' \cdot d_0' + d_2 \cdot d_1' \cdot d_0 \cdot L$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

Here's  $c_3$  as a circuit:



# Simplifying using Boolean Algebra

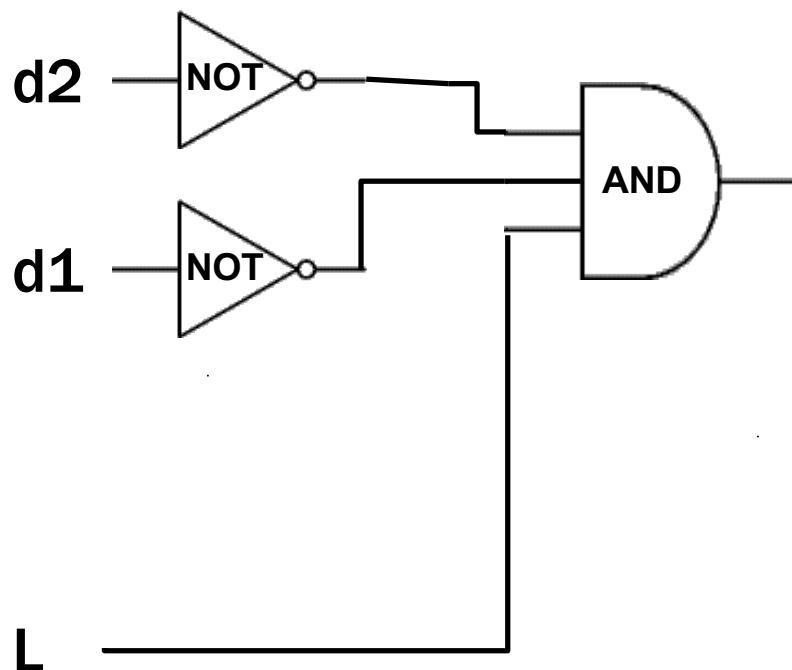
---

$$\begin{aligned}c3 &= d2' \cdot d1' \cdot d0' \cdot L + d2' \cdot d1' \cdot d0 \cdot L \\&= d2' \cdot d1' \cdot (d0' + d0) \cdot L \\&= d2' \cdot d1' \cdot 1 \cdot L \\&= d2' \cdot d1' \cdot L\end{aligned}$$

Distributivity  
Negation  
Identity

In Boolean Algebra,  
we skip Associativity,  
Commutativity, and  
Identity steps

In Boolean Algebra,  
write "=" instead of " $\equiv$ "



# Uses of Equivalence

---

- Working with logical formulas
  - simplification
- Working with circuits
  - hardware verification
- Software applications
  - query optimization and caching
  - artificial intelligence
  - program verification

# **Predicate Logic**

# Predicate Logic

---

- **Propositional Logic**
  - Allows us to analyze complex propositions in terms of their simpler constituent parts (a.k.a. atomic propositions) joined by connectives
- **Predicate Logic**
  - Lets us analyze them at a deeper level by expressing how those propositions depend on the objects they are talking about

“All positive integers  $x$ ,  $y$ , and  $z$  satisfy  $x^3 + y^3 \neq z^3$ .”

# Predicate Logic

---

**Adds two key notions to propositional logic**

- Predicates
- Quantifiers

# Predicates

---

## Predicate

– A function that returns a truth value, e.g.,

$\text{Cat}(x) := \text{"x is a cat"}$

$\text{Prime}(x) := \text{"x is prime"}$

$\text{HasTaken}(x, y) := \text{"student x has taken course y"}$

$\text{LessThan}(x, y) := \text{"x < y"}$

$\text{Sum}(x, y, z) := \text{"x + y = z"}$

$\text{GreaterThan5}(x) := \text{"x > 5"}$

$\text{HasNChars}(s, n) := \text{"string s has length n"}$

**Predicates can have varying numbers of arguments and input types.**

# Domain of Discourse

---

For ease of use, we define one “type”/“domain” that we work over. This non-empty set of objects is called the **“domain of discourse”**.

For each of the following, what might the domain be?

(1) “ $x$  is a cat”, “ $x$  barks”, “ $x$  ruined my couch”

“mammals” or “sentient beings” or “cats and dogs” or ...

(2) “ $x$  is prime”, “ $x = 0$ ”, “ $x > 0$ ”, “ $x$  is a power of two”

“numbers” or “integers” or “non-negative integers” or ...

(3) “student  $x$  has taken course  $y$ ” “ $x$  is a pre-req for  $z$ ”

“students and courses” or “university entities” or ...

# Quantifiers

---

We use ***quantifiers*** to talk about collections of objects.

$\forall x P(x)$

$P(x)$  is true **for every**  $x$  in the domain  
read as “**for all**  $x$ ,  $P$  of  $x$ ”



$\exists x P(x)$

$P(x)$  is true **for some**  $x$  in the domain  
read as “**there exists**  $x$ ,  $P$  of  $x$ ”

# Truth Depends on the Domain of Discourse

---

- Example: "Everybody was Kung Fu fighting"
  - true if the domain is people in the arena
  - false if the domain is the whole world



# Quantifiers

---

We use ***quantifiers*** to talk about collections of objects.

**Universal Quantifier (“for all”):**  $\forall x P(x)$

$P(x)$  is true for **every**  $x$  in the domain

read as “**for all  $x$ ,  $P$  of  $x$** ”

**Examples:** Are these true?

- $\forall x \text{Odd}(x)$
- $\forall x \text{LessThan4}(x)$

# Quantifiers

---

We use ***quantifiers*** to talk about collections of objects.

**Universal Quantifier (“for all”):**  $\forall x P(x)$

$P(x)$  is true for **every**  $x$  in the domain

read as “**for all  $x$ ,  $P$  of  $x$** ”

**Examples:** Are these true? It depends on the domain. For example:

- $\forall x \text{ Odd}(x)$
- $\forall x \text{ LessThan4}(x)$

$\{1, 3, -1, -27\}$	Integers	Odd Integers
True	False	True
True	False	False

# Quantifiers

---

We use ***quantifiers*** to talk about collections of objects.

**Existential Quantifier (“exists”):**  $\exists x P(x)$

**There is** an  $x$  in the domain for which  $P(x)$  is true  
read as “**there exists**  $x$ , **P of x**”

**Examples:** Are these true?

- $\exists x \text{ Odd}(x)$
- $\exists x \text{ LessThan4}(x)$

# Quantifiers

---

We use **quantifiers** to talk about collections of objects.

**Existential Quantifier (“exists”):**  $\exists x P(x)$

**There is** an  $x$  in the domain for which  $P(x)$  is true  
read as “**there exists**  $x$ , **P of x**”

**Examples:** Are these true? It depends on the domain. For example:

- $\exists x \text{ Odd}(x)$
- $\exists x \text{ LessThan4}(x)$

$\{1, 3, -1, -27\}$	Integers	Positive Multiples of 5
True	True	True
True	True	False

# Statements with Quantifiers

Domain of Discourse  
Positive Integers

## Predicate Definitions

Even(x) := "x is even"      Greater(x, y) := "x > y"  
Odd(x) := "x is odd"      Equal(x, y) := "x = y"  
Prime(x) := "x is prime"      Sum(x, y, z) := "x + y = z"

Determine the truth values of each of these statements:

$\exists x$  Even(x)

T      e.g. 2, 4, 6, ...

$\forall x$  Odd(x)

F      e.g. 2, 4, 6, ...

$\forall x$  (Even(x)  $\vee$  Odd(x))

T      every integer is either even or odd

$\exists x$  (Even(x)  $\wedge$  Odd(x))

F      no integer is both even and odd

$\forall x$  Greater(x+1, x)

T      adding 1 makes a bigger number

$\exists x$  (Even(x)  $\wedge$  Prime(x))

T      Even(2) is true and Prime(2) is true

# Syntax of Quantifiers

---

		Precedence
<b>Negation (not)</b>	$\neg p$	highest
<b>For all</b>	$\forall x P(x)$	
<b>Exists</b>	$\exists x P(x)$	
<b>Conjunction (and)</b>	$p \wedge q$	
<b>Disjunction (or)</b>	$p \vee q$	
<b>Exclusive Or</b>	$p \oplus q$	
<b>Implication</b>	$p \rightarrow r$	
<b>Biconditional</b>	$p \leftrightarrow q$	lowest

$\forall x \neg P(x) \wedge Q(y)$  means  $(\forall x \neg P(x)) \wedge Q(y)$

# Syntax of Quantifiers

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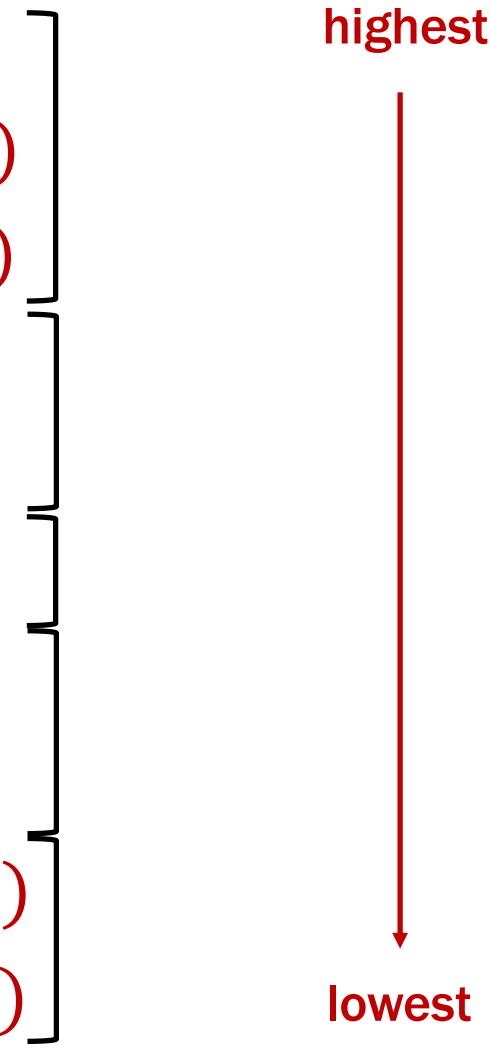
<b>Negation (not)</b>	$\neg p$
<b>For all</b>	$\forall x P(x)$
<b>Exists</b>	$\exists x P(x)$
<b>Conjunction (and)</b>	$p \wedge q$
<b>Disjunction (or)</b>	$p \vee q$
<b>Exclusive Or</b>	$p \oplus q$
<b>Implication</b>	$p \rightarrow r$
<b>Biconditional</b>	$p \leftrightarrow q$

Not everyone uses  
this convention!

We will try to  
accommodate  
both approaches...

# Syntax of Quantifiers (Two Conventions)

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<b>Negation (not)</b>	$\neg p$		<b>highest</b>
<b>For all</b>	$\forall x P(x)$		
<b>Exists</b>	$\exists x P(x)$		
<b>Conjunction (and)</b>	$p \wedge q$		
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<b>Exclusive Or</b>	$p \oplus q$		
<b>Implication</b>	$p \rightarrow r$		
<b>Biconditional</b>	$p \leftrightarrow q$		
<b>For all</b>	$\forall x, P(x)$		
<b>Exists</b>	$\exists x, P(x)$		<b>lowest</b>

# Syntax of Quantifiers (Two Conventions)

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<b>Negation (not)</b>	$\neg p$	$\left. \begin{array}{l} \forall x P(x) \\ \exists x P(x) \\ p \wedge q \\ p \vee q \\ p \oplus q \\ p \rightarrow r \\ p \leftrightarrow q \\ \forall x, P(x) \\ \exists x, P(x) \end{array} \right\}$
<b>For all</b>	$\forall x P(x)$	
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<b>For all</b>	$\forall x, P(x)$	
<b>Exists</b>	$\exists x, P(x)$	

$\forall x, \neg P(x) \wedge Q(y)$

means

$\forall x (\neg P(x) \wedge Q(y))$

# Statements with Quantifiers (Literal Translations)

**Domain of Discourse**  
Positive Integers

## Predicate Definitions

Even(x) := "x is even"	Greater(x, y) := "x > y"
Odd(x) := "x is odd"	Equal(x, y) := "x = y"
Prime(x) := "x is prime"	Sum(x, y, z) := "x + y = z"

Translate the following statements to English

$\forall x \exists y \text{ Greater}(y, x)$

For every positive integer x, there is a positive integer y, such that  $y > x$ .

$\exists y \forall x \text{ Greater}(y, x)$

There is a positive integer y such that, for every pos. int. x, we have  $y > x$ .

$\forall x \exists y (\text{Prime}(y) \wedge \text{Greater}(y, x))$

For every positive integer x, there is a pos. int. y such that y is prime and  $y > x$ .

$\forall x (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

For each positive integer x, if x is prime, then  $x = 2$  or x is odd.

$\exists x \exists y (\text{Prime}(x) \wedge \text{Prime}(y) \wedge \text{Sum}(x, 2, y))$

There exist positive integers x and y such that x and y are prime and  $x + 2 = y$ .

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# Statements with Quantifiers (Natural Translations)

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Translate the following statements to English

$\forall x \exists y \text{ Greater}(y, x)$

For every positive integer, there is some larger positive integer.

$\exists y \forall x \text{ Greater}(y, x)$

There is a positive integer that is larger than every other positive integer.

$\forall x \exists y (\text{Prime}(y) \wedge \text{Greater}(y, x))$

For every positive integer, there is a prime that is larger.

Sound more natural without introducing variable names

# Statements with Quantifiers (Literal Translations)

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There exist positive integers x and y such that x and y are prime and  $x + 2 = y$ .

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For each positive integer x, if x is prime, then  $x = 2$  or x is odd.

Spot the domain restriction patterns

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There exist primes x and y such that  $x + 2 = y$ .

$$\forall x (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$$

Every prime number is either 2 or odd.

Spot the domain restriction patterns

# English to Predicate Logic

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Domain of Discourse  
Mammals

## Predicate Definitions

Cat(x) := “x is a cat”  
Red(x) := “x is red”  
LikesTofu(x) := “x likes tofu”

“All red cats like tofu”

$$\forall x ((\text{Red}(x) \wedge \text{Cat}(x)) \rightarrow \text{LikesTofu}(x))$$

“Some red cats don’t like tofu”

$$\exists y ((\text{Red}(y) \wedge \text{Cat}(y)) \wedge \neg \text{LikesTofu}(y))$$

# English to Predicate Logic

Domain of Discourse  
Mammals

## Predicate Definitions

$\text{Cat}(x) := "x \text{ is a cat}"$

$\text{Red}(x) := "x \text{ is red}"$

$\text{LikesTofu}(x) := "x \text{ likes tofu}"$

When putting two predicates together like this, we use an “and”.

“All Red cats like tofu”

When restricting to a smaller domain in a “for all” we use implication.

“Some red cats don’t like tofu”

When restricting to a smaller domain in an “exists” we use and.

“Some” means “there exists”.

# English to Predicate Logic

---

Domain of Discourse  
Mammals

## Predicate Definitions

$\text{Cat}(x) := "x \text{ is a cat}"$   
 $\text{Red}(x) := "x \text{ is red}"$   
 $\text{LikesTofu}(x) := "x \text{ likes tofu}"$

“All Red cats like tofu”

“Red cats like tofu”



When there's no leading quantification,  
it *usually* means “for all”.

“Some red cats don’t like tofu”

“A red cat doesn’t like tofu”



“A” means “there exists”.

# Statements with Quantifiers (Natural Translations)

---

Translations usually sound more natural if we

## 1. Notice “domain restriction” patterns

$$\forall x (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$$

Every prime number is either 2 or odd.

## 2. Avoid introducing *unnecessary* variable names

$$\forall x \exists y \text{Greater}(y, x)$$

For every positive integer, there is some larger positive integer.

## 3. Can sometimes drop “all” or “there is”

$$\neg \exists x (\text{Even}(x) \wedge \text{Prime}(x) \wedge \text{Greater}(x, 2))$$

No even prime is greater than 2.

# More English Ambiguity

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Implicit quantifiers in English are often **ambiguous**

Three people that are all friends can form a raiding party  $\forall$

Three people that I know were all friends with Paul Allen  $\exists$

Formal logic removes this ambiguity

- quantifiers can always be specified
- unquantified variables that are not known constants (e.g,  $\pi$ )  
are **implicitly**  $\forall$ -quantified (mostly... one special case coming later)

# Quantifiers in Java

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- Implementing quantifiers in Java...

```
boolean forAll(Map<Integer, Boolean> P) {  
    for (Integer x : P.keySet()) {  
        if (!P.get(x)) return false;  
    }  
    return true;  
}  
     $\forall x P(x)$ 
```

(Bound) variable names don't matter:  $\forall x P(x) \equiv \forall a P(a)$

```
boolean exists(Map<Integer, Boolean> P) {  
    for (Integer x : P.keySet()) {  
        if (P.get(x)) return true;  
    }  
    return false;  
}  
     $\exists x P(x)$ 
```

# Scope of Quantifiers

---

Domain of Discourse
{1, 2, 3, ..., 100}

**Example:**  $\exists x \text{ Greater}(x, y) \equiv \exists z \text{ Greater}(z, y)$

truth value:

doesn't depend on x or z “**bound variables**”

does depend on y “**free variable**”

# Scope of Quantifiers

---

Domain of Discourse

{1, 2, 3, ..., 100}

**Example:**  $\exists x \text{ Greater}(x, y) \equiv \exists z \text{ Greater}(z, y)$

truth value:

doesn't depend on x or z “**bound variables**”

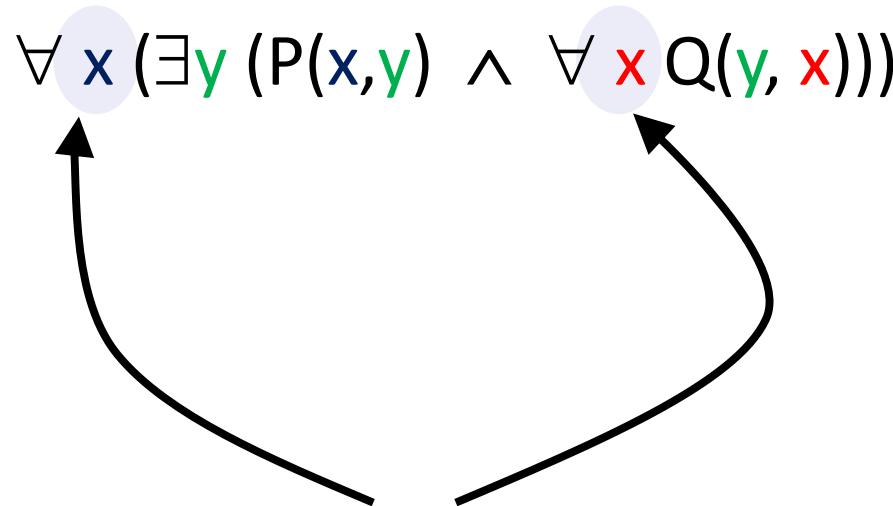
does depend on y “**free variable**”

**quantifiers only act on free variables of the formula**

$$\forall x \exists y (P(x, y) \rightarrow \forall x Q(y, x)))$$

# Quantifier “Style”

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$$\forall x (\exists y (P(x, y) \wedge \forall x Q(y, x)))$$


The diagram shows a logical formula with two quantifiers. The first quantifier is  $\forall x$  (blue  $x$ ) and the second is  $\forall x$  (red  $x$ ). Two arrows point from the blue  $x$  to the red  $x$ , indicating that the same variable is being used in both quantifiers, which is a common source of confusion in logic.

This isn't “wrong”, it's just horrible style.  
Don't confuse your reader by using the same  
variable multiple times...there are a lot of letters...

# Scope of Quantifiers

---

$$\exists x \ (P(x) \wedge Q(x)) \quad \text{vs.} \quad (\exists x \ P(x)) \wedge (\exists x \ Q(x))$$

This one asserts P  
and Q of the *same* x.

This one asserts P and Q  
of potentially different x's.

*Variables with the same name do not  
necessarily refer to the same object.*

# Nested Quantifiers

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- **Bound variable names don't matter**

$$\forall x \exists y P(x, y) \equiv \forall a \exists b P(a, b)$$

- **Positions of quantifiers can sometimes change**

$$\forall x (Q(x) \wedge \exists y P(x, y)) \equiv \forall x \exists y (Q(x) \wedge P(x, y))$$

- **But: order is important...**

# Quantifier Order Can Matter

Domain of Discourse
$\{1, 2, 3, 4\}$

Predicate Definitions
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GreaterEq(x, y) ::= “ $x \geq y$ ”

	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

“Every number has a number greater than or equal to it.”

$\forall y \exists x \text{ GreaterEq}(x, y)$

# Quantifier Order Can Matter

Domain of Discourse  
 $\{1, 2, 3, 4\}$

## Predicate Definitions

“There is a number greater than or equal to all numbers.”

$\exists x \forall y \text{GreaterEq}(x, y)$

**“Every number has a number greater than or equal to it.”**

$\forall y \exists x \text{GreaterEq}(x, y)$

		1	2	3	4
		1	F	F	F
		2	T	T	F
		3	T	T	F
		4	T	T	T

The purple statement requires an entire row to be true.

**The red statement requires one entry in each column to be true.**

**Important:** both include the case  $x = y$

*Different names does not imply different objects!*

# Quantification with Two Variables

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expression	when <b>true</b>	when <b>false</b>
$\forall x \forall y P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y P(x, y)$	We can find a specific $y$ for each $x$ . $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some $x$ doesn't have a corresponding $y$ .
$\exists y \forall x P(x, y)$	We can find ONE $y$ that works no matter what $x$ is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate $y$ , there is an $x$ that it doesn't work for.

# De Morgan's Laws for Quantifiers

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$$\begin{aligned}\neg \forall x P(x) &\equiv \exists x \neg P(x) \\ \neg \exists x P(x) &\equiv \forall x \neg P(x)\end{aligned}$$

There is no unicorn

$$\neg \exists x \text{ Unicorn}(x)$$

Every animal is not a unicorn

$$\forall x \neg \text{ Unicorn}(x)$$

These are **equivalent** but not **equal**

# De Morgan's Laws for Quantifiers

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It is easy to check that

$$\neg \exists x (P(x) \wedge R(x)) \equiv \forall x (P(x) \rightarrow \neg R(x))$$

and similarly that

$$\neg \forall x (P(x) \rightarrow R(x)) \equiv \exists x (P(x) \wedge \neg R(x))$$

De Morgan's Laws respect domain restrictions!  
(It leaves them in place and only negates the other parts.)