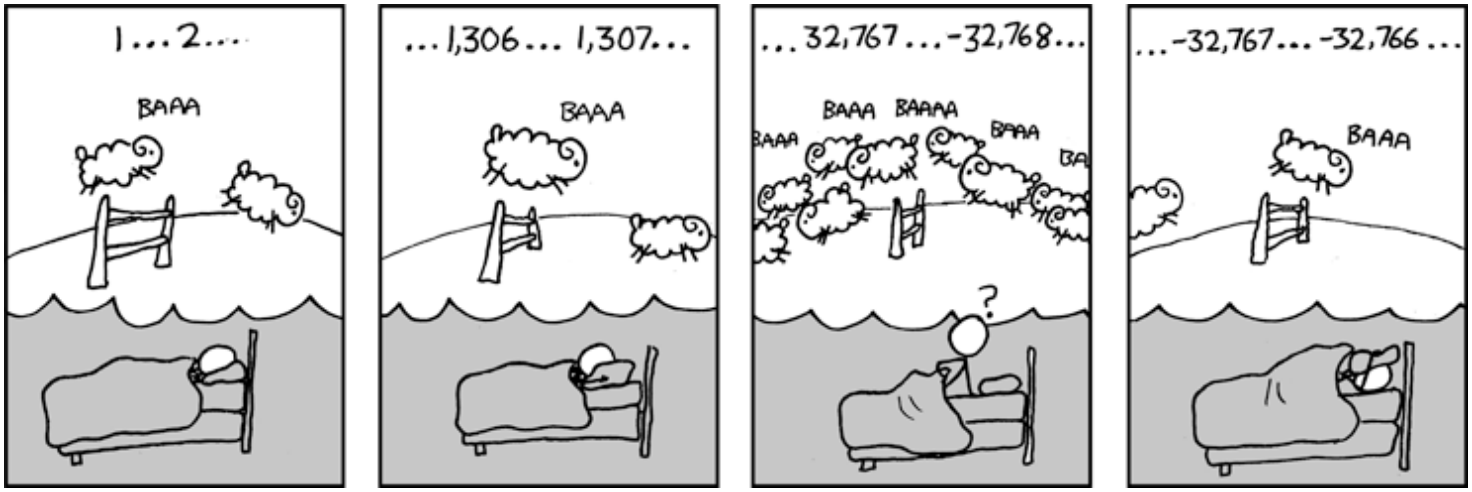
CSE 311: Foundations of Computing

Topic 7: Applications



Applications of Computability

Recall: Undecidability of the Halting Problem

CODE(**P**) means "the code of the program **P**"

The Halting Problem

Given: - CODE(**P**) for any program **P**
- input **x**

Output: **true** if **P** halts on input **x**
false if **P** does not halt on input **x**

Theorem [Turing]: **There is no program that solves the Halting Problem**

Recall: Proving Problems Undecidable

- Can use the fact that the Halting Problem is undecidable to show that other problems are undecidable

General method (a "reduction" from A to B):

Prove that, if there were a program deciding B, then there would be a program deciding the Halting Problem.

"B decidable → A decidable"

Show how to use a solution to B to build a solution to A

Proves that B is *at least* as hard as A

Recall: Proving Problems Undecidable

- Can use the fact that the Halting Problem is undecidable to show that other problems are undecidable

General method (a "reduction" from Halting Problem):

Prove that, if there were a program deciding B, then there would be a program deciding the Halting Problem.

"B decidable → Halting Problem decidable"

Contrapositive:

"Halting Problem undecidable → B undecidable"

Therefore, B is undecidable

More Reductions

- Can use undecidability of these problems to show that other problems are undecidable.

- Mentioned examples about CFG last time

- More examples:

  $\mathrm{EQUIV}(P, Q)$: **True** if $P(x)$ and $Q(x)$ have the same behavior for every input x

  False otherwise

Some Problems About Code Are Decidable

Not *every* problem on programs is undecidable!

Which of these is decidable?

- Input CODE(Q) and x
  Output: true if Q prints "ERROR" on input x after less than 100 steps
  false otherwise

- Input CODE(Q) and x
  Output: true if Q prints "ERROR" on input x after more than 100 steps
  false otherwise

Another undecidable problem

```
public class Q {
  private static String x = "...";

  public static void main(String[] args) {
    PrintStream out = System.out;
    System.setOut(new PrintStream(
        new WriterOutputStream(new StringWriter())));
    System.setIn(new ReaderInputStream(new StringReader(x)));

    for (int i = 0; i < 100; i++);  // waste 100 steps

    P.main(args);

    out.println("ERROR");
  }
}

class P {
  public static void main(String[] args) { ... }
  ...
}
```

Rice's Theorem

Not *every* problem on programs is undecidable...

but almost all of them are!

Rice's Theorem:

Any "non-trivial" property of the input-output behavior of Java programs is undecidable.

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Not *every* problem on programs is undecidable...

but almost all of them are!

Rice's Theorem (a.k.a. "Compilers ARE DIFFICULT

Any "non-trivial" property of the input-output behavior of Java programs is undecidable.

Proof of Rice's Theorem

Rice's Theorem:

Any "non-trivial" property of the **input-output behavior** of Java programs is undecidable.

- "Non-trivial" means there is at least one program with the property and at least one program without it.
  - if all programs have (or don't have) the property, it's easy
  - suppose that program **A** has the property and **B** does not

- Consider the infinite loop program **L**:

```
public static void main(String[] args) {
  while (true);
}
```

Proof of Rice's Theorem

- Suppose that program **A** has the property and **B** does not
- Suppose that **L** doesn't have the property
  - so **A** as the property and **L** doesn't
  - (other case is similar)
- Suppose, for a contradiction, that **T** checks for the property
- Given a program **P**, construct this program **Q**...

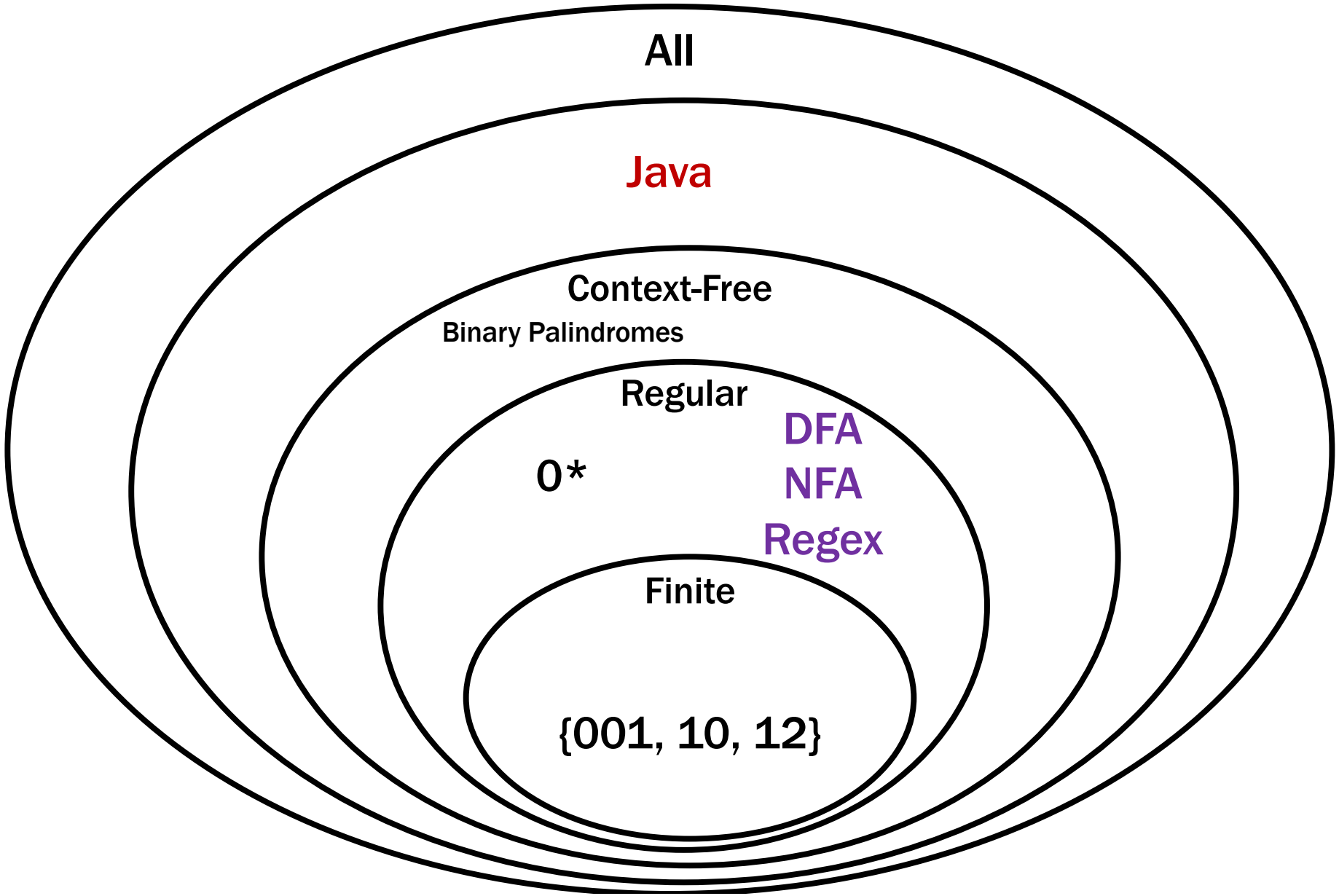
Proof of Rice's Theorem

```
public class Q {
  private static String x = "...";

  public static void main(String[] args) {
    InputStream in = System.in;  // save System.in and System.out
    PrintStream out = System.out;
    System.setOut(new PrintStream(
        new WriterOutputStream(new StringWriter())));
    System.setIn(new ReaderInputStream(new StringReader(x)));

    P.main(args);  // If P doesn't halt, then it acts like L

    System.setOut(out);  // restore System.in and System.out
    System.setIn(in);

    A.main(args);  // If P does halt, then it acts like A
  }
}

class P { … }
```

Proof of Rice's Theorem

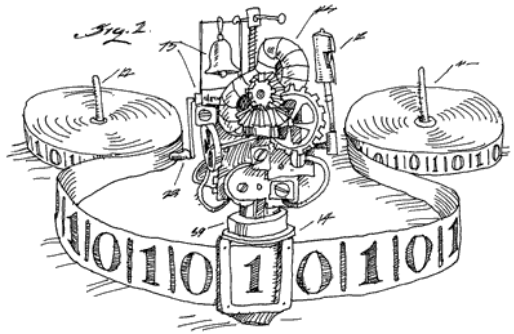
- Suppose that program A has the property and B does not
- Suppose that L has the property (other case is similar).
  - so L has the property and B does not
- Suppose, for a contradiction, that T checks for the property
- Construct this program Q that
  - acts like A (has the property) if P halts on x
  - acts like L (doesn't have property) if P doesn't halt on x
- Ask T if Q has the property
  - if it does, then P halts on x
  - if it doesn't, then P doesn't halt on x

Recall our language picture



Done

- **We proved that there is no computer program that can solve the Halting Problem.**
  - There was nothing special about Java*
    [Church-Turing thesis]



This is essentially a law of physics
(not mathematics)

Before Java

1930's:

How can we formalize what algorithms are possible?

- **Turing machines** (Turing, Post)
  - basis of modern computer hardware
- **Lambda Calculus** (Church, Kleene)
  - basis for functional programming languages

Turing machines

Church-Turing Thesis:

Any reasonable model of computation that includes all possible algorithms is equivalent in power to a Turing machine

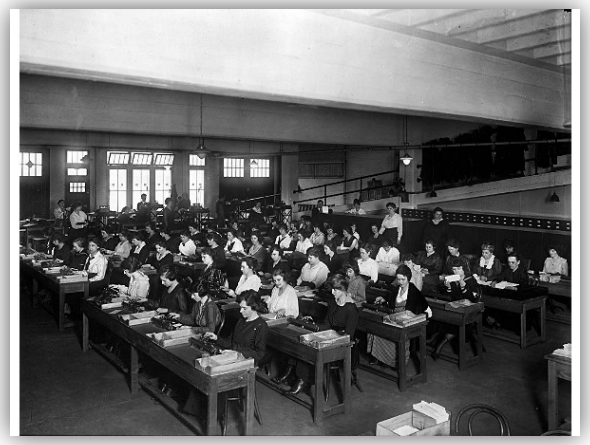
Evidence

- Huge numbers of models based on radically different ideas turned out to be equivalent to TMs
- TM can simulate the physics of any machine that we could build (even quantum computers)
- In physics, this is considered a law of nature

Computers and algorithms

- Does Java (or any programming language) cover all possible computation? Every possible algorithm?

- There was a time when computers were people who did calculations on sheets paper to solve computational problems



- Computers as we known them arose from trying to understand everything these people could do.

Turing machines

- **Finite Control**
  - **Brain/CPU** that has only a **finite # of possible "states of mind"**
- **Recording medium**
  - **An unlimited supply of** blank **"scratch paper"** on which to **write & read symbols,** each chosen from a **finite set of possibilities**
  - **Input** also supplied **on** the **scratch paper**
- **Focus of attention**
  - **Finite control** can only **focus on** a **small portion of the recording medium at once**
  - **Focus of attention** can only **shift a small amount at a time**

Turing machines

- **Recording medium**
  - An **infinite** read/write **“tape”** marked off into **cells**
  - Each **cell** can store **one symbol or** be **“blank”**
  - **Tape** is **initially** all **blank except** a few **cells** of the tape **containing the input string**
  - **Read/write head** can **scan one cell** of the tape - **starts on input**

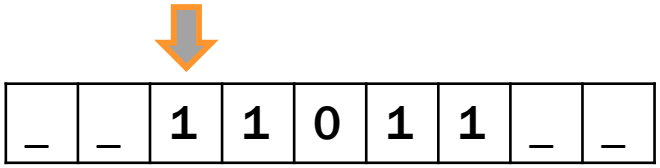
Turing machines

- In each step, a Turing machine
  1. Reads the currently scanned cell
  2. Based on current state and scanned symbol
     i. Overwrites symbol in scanned cell
     ii. Moves read/write head left or right one cell
     iii. Changes to a new state

- Each Turing Machine is specified by its finite set of rules

Turing machines

	_	0	1
s_1	(1, L, s_3)	(1, L, s_4)	(0, R, s_2)
s_2	(0, R, s_1)	(1, R, s_1)	(0, R, s_1)
s_3			
s_4			



UW CSE's Steam-Powered Turing Machine

Original in Sieg Hall stairwell

Turing machines

Ideal Java/C programs:

- Just like the Java/C you're used to programming with, except you never run out of memory
  - no OutOfMemoryError

Equivalent to Turing machines but easier to program:

- Turing machine definition is useful for breaking computation down into simplest steps

  easier to analyze TMs than Java programs
- We only care about high level so we use programs

Turing-Complete Programming Languages

- Programming languages equal in power to Java / C / etc. are called "Turing-complete"

- Some surprising languages are Turing-complete
  - any general-purpose programming language you've heard of
    proof: show that you write a TM simulator in your language
  - Excel macros
  - TypeScript's type system
  - Conway's "Game of Life"
  - Magic: The Gathering

Turing-Complete Programming Languages

- Turing's contemporary, Alonzo Church, proposed a different language called "Lambda calculus"
  - first "high level" programming language
    a simple "functional" programming language
  - widely used in programming languages even today

- Turing's proof of equivalence
  - TM implements LC: world's first interpreter
  - LC implements TM: world's first hardware simulator

Turing's big idea part 1: Machines as data

Original Turing machine definition:

- A different "machine" **M** for each task
- Each machine **M** is defined by a finite set of possible operations on finite set of symbols
- So... **M** has a finite description as a sequence of symbols, its "code", which we denote **<M>**

You already are used to this idea with the notion of the program code, but this was a new idea in Turing's time.

Turing's big idea part 2: A Universal TM

- **A Turing machine interpreter U**
  - On input <M> and its input **x**,
    U outputs the same thing as **M** does on input **x**
  - At each step it decodes which operation **M** would have performed and simulates it.
- **One Turing machine is enough**
  - Basis for modern stored-program computer (& "software")

    Von Neumann studied Turing's UTM design

Applications of Languages

Recall: Context-Free Grammars

- A Context-Free Grammar (CFG) is given by a finite set of substitution rules involving
  - A finite set **V** of *variables* that can be replaced
  - Alphabet Σ of *terminal symbols* that can't be replaced
  - One variable, usually **S**, is called the *start symbol*

- The substitution rules involving a variable **A**, written as

$$\mathbf{A} \rightarrow w_1 \mid w_2 \mid \cdots \mid w_k$$

  where each w_i is a string of variables and terminals
  - that is $w_i \in (\mathbf{V} \cup \Sigma)^*$

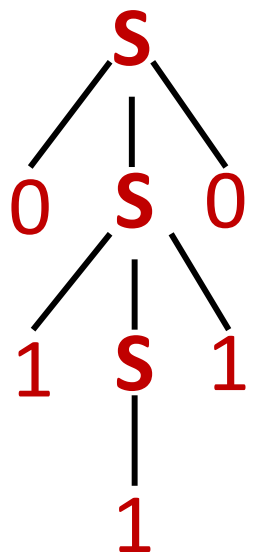
Recall: Parse Trees

Suppose that grammar G generates a string x

- A *parse tree* of x for G has
  - Root labeled S (start symbol of G)
  - The children of any node labeled A are labeled by symbols of w left-to-right for some rule $A \rightarrow w$
  - The symbols of x label the leaves ordered left-to-right

$S \rightarrow 0S0 \mid 1S1 \mid 0 \mid 1 \mid \varepsilon$

Parse tree of 01110

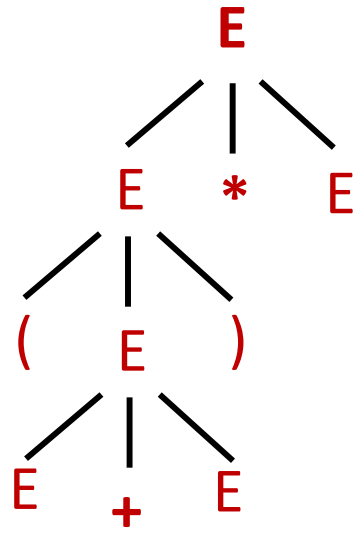


Simple Arithmetic Expressions

$$E \rightarrow E+E \mid E*E \mid (E) \mid x \mid y \mid z \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

Generate (2 + x) * y

$E \rightarrow E*E$
$\rightarrow (E)*E$
$\rightarrow (E+E)*E$



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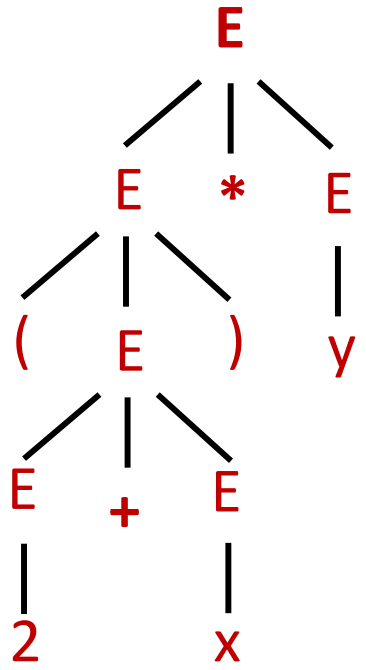
$\rightarrow (2+E)*E$
$\rightarrow (2+x)*E$
$\rightarrow (2+x)*y$



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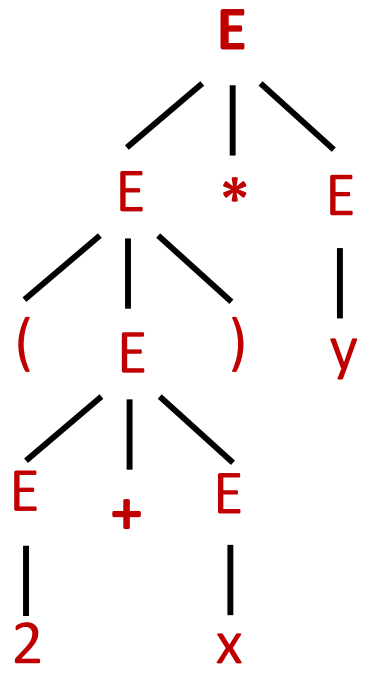
Generate (2 + x) * y

$E \rightarrow E*E$
$\rightarrow (E)*E$
$\rightarrow (E+E)*E$
$\rightarrow (2+E)*E$
$\rightarrow (2+x)*E$
$\rightarrow (2+x)*y$

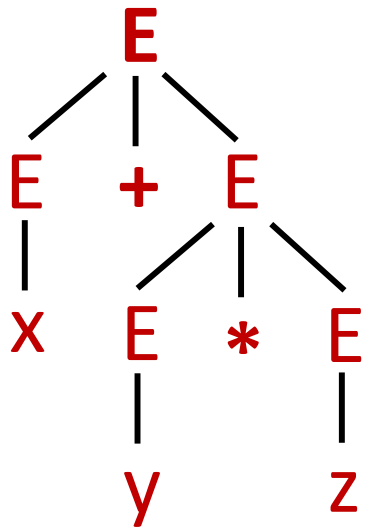
or...
$\rightarrow (E+E)*y$
$\rightarrow (E+x)*y$
$\rightarrow (2+x)*y$



Simple Arithmetic Expressions

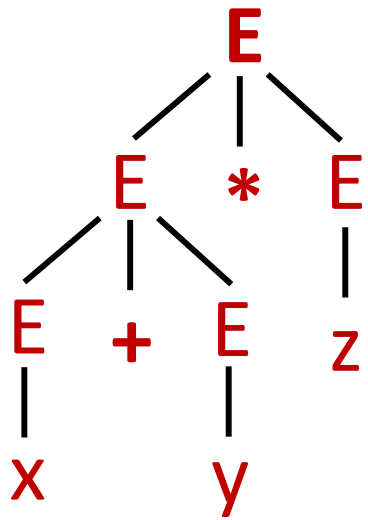
$$E \rightarrow E+E \mid E*E \mid (E) \mid x \mid y \mid z \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

Generate x+y∗z in ways that give two *different* parse trees



$E \Rightarrow E+E \Rightarrow x+E \Rightarrow x+E*E \Rightarrow x+y*E \Rightarrow x+y*z$

(multiply y with z and then add to x)



$E \Rightarrow E*E \Rightarrow E+E*E \Rightarrow x+E*E$

$\Rightarrow x+y*E \Rightarrow x+y*z$

(add x to y, then multiply by z)

building precedence in simple arithmetic expressions

- **E** – expression (start symbol)
- **T** – term **F** – factor **I** – identifier **N** - number

E $\rightarrow$ **T** | **E**+**T**

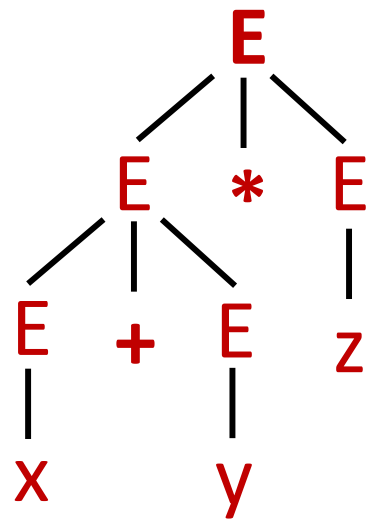
T $\rightarrow$ **F** | **F**∗**T**

F $\rightarrow$ (**E**) | **I** | **N**

I $\rightarrow$ x | y | z

N $\rightarrow$ 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9

No longer allows:



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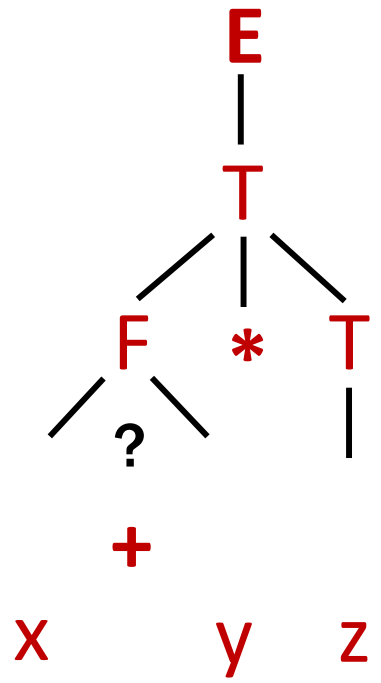

building precedence in simple arithmetic expressions

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E $\rightarrow$ **T** | **E**+**T**

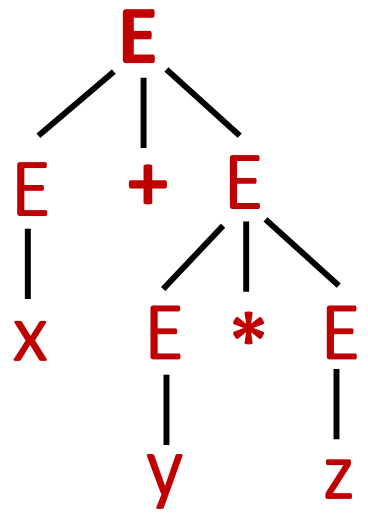
T $\rightarrow$ **F** | **F**∗**T**

F $\rightarrow$ (**E**) | **I** | **N**

I $\rightarrow$ x | y | z

N $\rightarrow$ 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9

Still allows:



building precedence in simple arithmetic expressions

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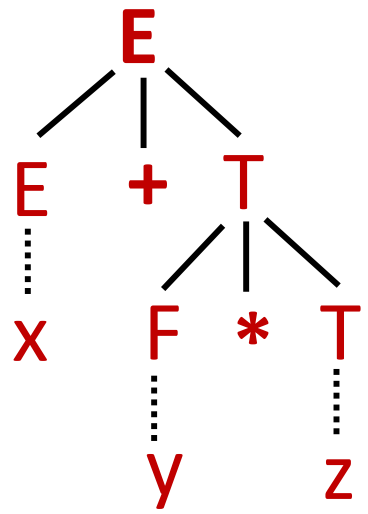
E $\rightarrow$ **T** | **E**+**T**

T $\rightarrow$ **F** | **F**∗**T**

F $\rightarrow$ (**E**) | **I** | **N**

I $\rightarrow$ x | y | z

N $\rightarrow$ 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9



Backus-Naur Form (The same thing...)

BNF (Backus-Naur Form) grammars

- Originally used to define programming languages
- Variables denoted by long names in angle brackets, e.g.

  <identifier>, <if-then-else-statement>, <assignment-statement>, <condition>

  ::= used instead of $\rightarrow$

BNF for C

```
statement:
  ((identifier | "case" constant-expression | "default") ":")*
  (expression? ";" |
   block |
   "if" "(" expression ")" statement |
   "if" "(" expression ")" statement "else" statement |
   "switch" "(" expression ")" statement |
   "while" "(" expression ")" statement |
   "do" statement "while" "(" expression ")" ";" |
   "for" "(" expression? ";" expression? ";" expression? ")" statement |
   "goto" identifier ";" |
   "continue" ";" |
   "break" ";" |
   "return" expression? ";"
  )

block: "{" declaration* statement* "}"

expression:
  assignment-expression%

assignment-expression: (
    unary-expression (
      "=" | "*=" | "/=" | "%=" | "+=" | "-=" | "<<=" | ">>=" | "&=" |
      "^=" | "|="
    )
  )* conditional-expression

conditional-expression:
  logical-OR-expression ( "?" expression ":" conditional-expression )?
```

BNF for (Simple) English

Back to middle school:

<sentence>::=<noun phrase><verb phrase>

<noun phrase>::==<article><adjective><noun>

<verb phrase>::=<verb><adverb>|<verb><object>

<object>::=<noun phrase>

Parse:

The yellow duck squeaked loudly

The red truck hit a parked car

Applications of Set Theory

Recall: Sets

Sets are collections of objects called elements.

Write $a \in B$ to say that a is an element of set B, and $a \notin B$ to say that it is not.

Some simple examples
A = {1}
B = {1, 3, 2}
C = {□, 1}
D = {{17}, 17}
E = {1, 2, 7, cat, dog, ∅, α}

Representing Sets Using Bits

- Suppose universe U is $\{1,2,\ldots,n\}$
- Can represent set $B \subseteq U$ as a vector of bits:

  $b_n \ldots b_2 b_1$ where $b_i = 1$ when $i \in B$

  $b_i = 0$ when $i \notin B$

  – Called the *characteristic vector* of set B

- With $\mathrm{n} = 5$:
  – the characteristic vector for $\{1,2,5\}$ is **10011**
  – what is characteristic vector for $\{2,3,5\}$?
  – characteristic vectors are and **10110**

Representing Sets Using Bits

- Suppose universe U is $\{1,2,\ldots,n\}$
- Can represent set $B \subseteq U$ as a vector of bits:

  $b_1 b_2 \ldots b_n$ where $b_i = 1$ when $i \in B$

  $b_i = 0$ when $i \notin B$

  – Called the *characteristic vector* of set B

- With $\mathrm{n} = 5$, $A = \{1,2,5\}$, and $B = \{2,3,5\}$,
  – characteristic vectors are **10011** and **10110**
  – what is characteristic vector for $A \cup B$? $A \cap B$?
  – characteristic vectors are **10111** and **10010**

Bitwise Operations

01101101	Java:	`z = x | y`
∨ 00110111		
01111111		
00101010	Java:	`z = x & y`
∧ 00001111		
00001010		
01101101	Java:	`z = x ^ y`
⊕ 00110111		
01011010		

A Useful Identity

- If x and y are bits: $(x \oplus y) \oplus y = ?$

x	y	$x \oplus y$	$(x \oplus y) \oplus y$
0	0		
0	1		
1	0		
1	1		

A Useful Identity

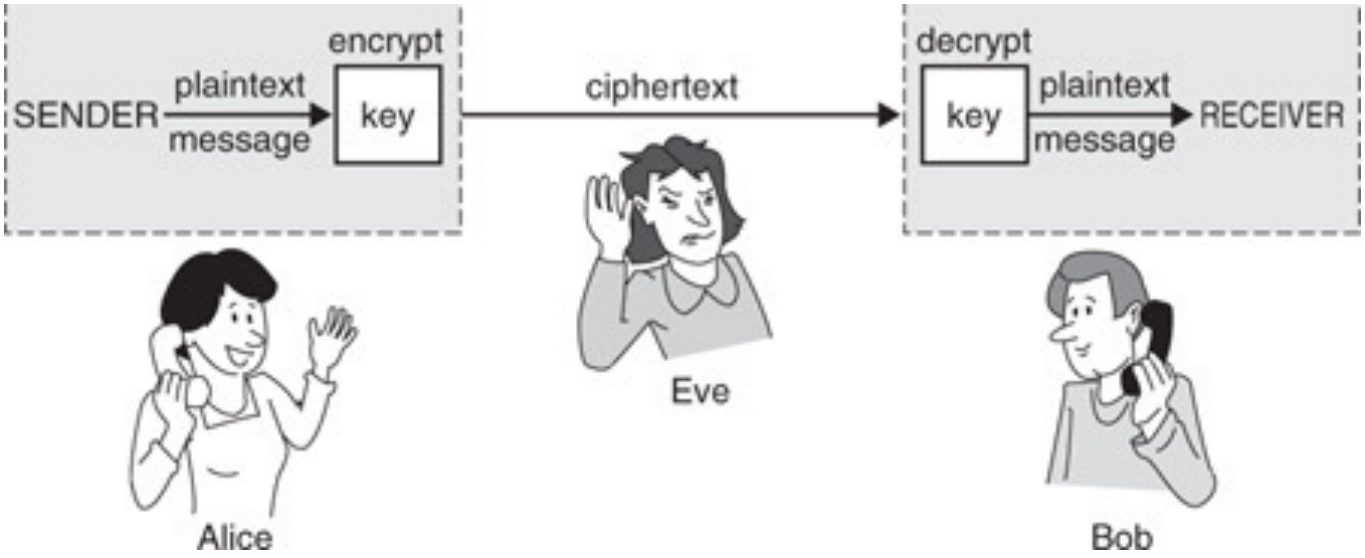
- If x and y are bits: $(x \oplus y) \oplus y = ?$

x	y	$x \oplus y$	$(x \oplus y) \oplus y$
0	0	0	0
0	1	1	0
1	0	1	1
1	1	0	1

- What if x and y are bit-vectors?

Private Key Cryptography

- **Alice** wants to communicate message secretly to **Bob** so that eavesdropper **Eve** who hears their conversation cannot tell what **Alice**'s message is.
- **Alice** and **Bob** can get together and privately share a secret key **K** ahead of time.



Recall: Relations

Let A and B be sets,
A **binary relation from** A **to** B is a subset of $A \times B$

n-ary Relations

Let $\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots, \boldsymbol{An}$ be sets. An **n-ary** relation on these sets is a subset of $\boldsymbol{A_1} \times \boldsymbol{A_2} \times \cdots \times \boldsymbol{A_n}$.

Relational Databases

STUDENT

Student_Name	ID_Number	Office	GPA
Knuth	328012098	022	4.00
Von Neuman	481080220	555	3.78
Russell	238082388	022	3.85
Einstein	238001920	022	2.11
Newton	1727017	333	3.61
Karp	348882811	022	3.98
Bernoulli	2921938	022	3.21

Database Operations: Projection

Find all offices: $\Pi_{\mathbf{Office}}(\mathrm{STUDENT})$

Office
022
555
333

Find offices and GPAs: $\Pi_{\mathbf{Office,GPA}}(\mathrm{STUDENT})$

Office	GPA
022	4.00
555	3.78
022	3.85
022	2.11
333	3.61
022	3.98
022	3.21

Database Operations: Selection

Find students with GPA > 3.9 : $\sigma_{GPA>3.9}(STUDENT)$

Student_Name	ID_Number	Office	GPA
Knuth	328012098	022	4.00
Karp	348882811	022	3.98

Retrieve the name and GPA for students with GPA > 3.9:

$\Pi_{Student\_Name,GPA}(\sigma_{GPA>3.9}(STUDENT))$

Student_Name	GPA
Knuth	4.00
Karp	3.98

Relational Databases

STUDENT

Student_Name	ID_Number	Office	GPA	Course
Knuth	328012098	022	4.00	CSE311
Knuth	328012098	022	4.00	CSE351
Von Neuman	481080220	555	3.78	CSE311
Russell	238082388	022	3.85	CSE312
Russell	238082388	022	3.85	CSE344
Russell	238082388	022	3.85	CSE351
Newton	1727017	333	3.61	CSE312
Karp	348882811	022	3.98	CSE311
Karp	348882811	022	3.98	CSE312
Karp	348882811	022	3.98	CSE344
Karp	348882811	022	3.98	CSE351
Bernoulli	2921938	022	3.21	CSE351

What's not so nice?

Relational Databases

STUDENT

Student_Name	ID_Number	Office	GPA
Knuth	328012098	022	4.00
Von Neuman	481080220	555	3.78
Russell	238082388	022	3.85
Einstein	238001920	022	2.11
Newton	1727017	333	3.61
Karp	348882811	022	3.98
Bernoulli	2921938	022	3.21

TAKES

ID_Number	Course
328012098	CSE311
328012098	CSE351
481080220	CSE311
238082388	CSE312
238082388	CSE344
238082388	CSE351
1727017	CSE312
348882811	CSE311
348882811	CSE312
348882811	CSE344
348882811	CSE351
2921938	CSE351

Better

Database Operations: Natural Join

Student ⋈ Takes

Student_Name	ID_Number	Office	GPA	Course
Knuth	328012098	022	4.00	CSE311
Knuth	328012098	022	4.00	CSE351
Von Neuman	481080220	555	3.78	CSE311
Russell	238082388	022	3.85	CSE312
Russell	238082388	022	3.85	CSE344
Russell	238082388	022	3.85	CSE351
Newton	1727017	333	3.61	CSE312
Karp	348882811	022	3.98	CSE311
Karp	348882811	022	3.98	CSE312
Karp	348882811	022	3.98	CSE344
Karp	348882811	022	3.98	CSE351
Bernoulli	2921938	022	3.21	CSE351

Relational Databases

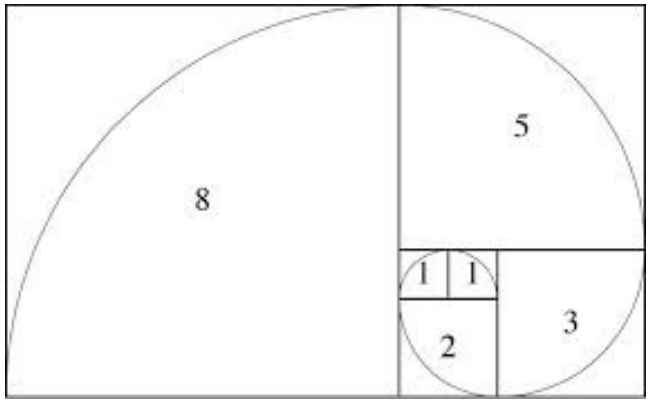
- Store data as a set of relations
- Implements relational operators
  - written in a higher-level language (e.g., SQL)
  - compiled into relational algebra expressions
- Not hard to implement across multiple servers:
  - add a "shuffle" operation that repartitions data
- Not hard to implement more complex data types

Applications of Number Theory

Recall: Fibonacci Numbers

$$f_0 := 0$$
$$f_1 := 1$$
$$f_{n+2} := f_{n+1} + f_n$$

$f(0) := 0$ **and** $f(1) := 1$
$f(n+2) := f(n+1) + f(n)$



Fibonacci Numbers

$f_0 = 0$
$f_1 = 1$
$f_n = f_{n-1} + f_{n-2}$ **for all** $n \geq 2$

Tamás Görbe
@TamasGorbe

A Mathematician's Way* of Converting Miles to Kilometers

$$3 \text{ mi} \approx 5 \text{ km}$$
$$5 \text{ mi} \approx 8 \text{ km}$$
$$8 \text{ mi} \approx 13 \text{ km}$$

f_n mi $\approx$ f_{n+1} km

Recall: Bounding Fibonacci I: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Cases: $f_0 = 0 < 1 = 2^0$ so P(0) is true and
   $f_1 = 1 < 2 = 2^1$ so P(1) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 1$, we have $f_j < 2^j$ for every integer j from 0 to k.
4. Inductive Step: We can calculate that

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} && \text{def of f (since } k+1 \geq 2\text{)} \\ &< 2^k + 2^{k-1} && \text{by IH (since } k-1 \geq 0\text{)} \\ &< 2^k + 2^k \\ &= 2^{k+1} \end{aligned}$$

   so P(k+1) is true.

5. Therefore, by strong induction, $f_n < 2^n$ for all integers $n \geq 0$.

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers n ≥ 2 by strong induction.
2. Base Cases: $f_2 = f_1 + f_0 = 1 \geq 1 = 2^0 = 2^{2/2-1}$ so P(2) holds
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ so P(3) holds
3. Inductive Hypothesis: Assume that for some arbitrary integer k ≥ 3, P(j) is true for every integer j from 2 to k.
4. Inductive Step: We can calculate

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} && \text{def of f (since } k+1 \geq 4\text{)} \\ &\geq 2^{k/2-1} + f_{k-1} && \text{by the IH} \\ &\geq 2^{k/2-1} + 2^{(k-1)/2-1} && \text{by the IH (since } k-1 \geq 2\text{)} \\ &\geq 2 \cdot 2^{(k-1)/2-1} = 2^{(k+1)/2-1} \end{aligned}$$

   so P(k+1) is true.

5. Therefore by strong induction, $f_n \geq 2^{n/2-1}$ for all integers n ≥ 2.

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Why does this help us bound the running time of Euclid's Algorithm?

Running time of Euclid's algorithm

Theorem: Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.

Why does this help us bound the running time of Euclid's Algorithm?

We already proved that $f_n \geq 2^{n/2-1}$ so $f_{n+1} \geq 2^{(n+1)/2}$

Therefore, if Euclid's Algorithm takes n steps
for $\gcd(a, b)$ with $a \geq b > 0$
then $a \geq 2^{(n+1)/2}$

so $\log_2 a \geq (n+1)/2$ or $n \leq 2 \log_2 a - 1$
i.e., # of steps $<$ twice the # of bits in a.

Recall: Euclid's Algorithm

gcd(a, b) = gcd(b, a mod b) gcd(a, 0) = a

```
int gcd(int a, int b) {  /* Assumes: a >= b >= 0 */
    if (b == 0) {
        return a;
    } else {
        return gcd(b, a % b);
    }
}
```

Note: gcd(b, a) = gcd(a, b)

Recall: Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

gcd(660,126)
= gcd(126, 660 mod 126)
= gcd(126, 30)
= gcd(30, 126 mod 30)
= gcd(30, 6)
= gcd(6, 30 mod 6)
= gcd(6, 0)
= 6

Recall: Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

At each step, we calculate "mod" with the division theorem:

gcd(660,126)
= gcd(126, 30)
= gcd(30, 6)
= gcd(6, 0) = 6

660 = 5 · 126 + 30
126 = 4 · 30 + 6
30 = 5 · 6

Recall: Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

At each step, we calculate "mod" with the division theorem:

$660 = 5 \cdot 126 + 30$
$126 = 4 \cdot 30 + 6$
$30 = 5 \cdot 6$

$r_4 = 660$
$r_3 = 126$
$r_2 = 30$
$r_1 = 6$

$r_4 = 5 \cdot r_3 + r_2$
$r_3 = 4 \cdot r_2 + r_1$
$r_2 = 5 \cdot r_1$

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

An informal way to get the idea: Consider an n **step** gcd **calculation starting with** $r_{n+1} = a$ **and** $r_n = b$:

$$r_{n+1} = q_n r_n + r_{n-1}$$
$$r_n = q_{n-1} r_{n-1} + r_{n-2}$$
$$\dots$$
$$r_3 = q_2 r_2 + r_1$$
$$r_2 = q_1 r_1$$

For all $k \geq 2$, $r_{k-1} = r_{k+1} \bmod r_k$

Now $r_1 \geq 1$ **and each** q_k **must be** ≥ 1. **If we replace all the** q_K**'s by** 1 **and replace** r_1 **by** 1 **, we can only reduce the** r_k**'s. After that reduction,** $r_k = f_k$ **for every** k.

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

We go by strong induction on n.

Let P(n) **be** "gcd(a,b) **with** $a \geq b > 0$ **takes** n **steps** $\rightarrow$ $a \geq f_{n+1}$" **for all** $n \geq 1$.

<u>Base Case</u>: n=1 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 1 **step.**

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

We go by strong induction on n.

Let P(n) **be** "gcd(a,b) **with** $a \geq b > 0$ **takes** n **steps** $\rightarrow$ $a \geq f_{n+1}$" **for all** $n \geq 1$.

<u>Base Case</u>: n=1 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 1 **step.**
By assumption, $a \geq b \geq 1 = f_2$ **so** P(1) **holds.**

<u>Base Case</u>: n=2 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 2 **steps.**

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

We go by strong induction on n.

Let P(n) **be** "gcd(a,b) **with** $a \geq b > 0$ **takes** n **steps** $\rightarrow$ $a \geq f_{n+1}$" **for all** $n \geq 1$.

Base Case: n=1 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 1 **step. By assumption,** $a \geq b \geq 1 = f_2$ **so** P(1) **holds.**

Base Case: n=2 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 2 **steps. By assumption,** $a \geq b \geq 2 = f_3$ **so** P(1) **holds.**

Induction Hypothesis: **Suppose that for some integer** $k \geq 2$, P(j) **is true for all integers** j **s.t.** $1 \leq j \leq k$

Running time of Euclid's algorithm

Theorem: **Suppose that Euclid's Algorithm takes n steps for $\gcd(a, b)$ with $a \geq b > 0$. Then, $a \geq f_{n+1}$.**

We go by strong induction on n.

Let P(n) **be** "gcd(a,b) **with** $a \geq b > 0$ **takes** n **steps** $\rightarrow$ $a \geq f_{n+1}$" **for all** $n \geq 1$.

<u>Base Case</u>: n=1 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 1 **step.**
By assumption, $a \geq b \geq 1 = f_2$ **so** P(1) **holds.**

<u>Base Case</u>: n=2 **Suppose Euclid's Algorithm with** $a \geq b > 0$ **takes** 2 **steps.**
By assumption, $a \geq b \geq 2 = f_3$ **so** P(1) **holds.**

<u>Induction Hypothesis</u>: **Suppose that for some integer** $k \geq 2$, P(j) **is true for all integers** j **s.t.** $1 \leq j \leq k$

<u>Inductive Step</u>: **We want to show: if** gcd(a,b) **with** $a \geq b > 0$ **takes** k+1 **steps, then** $a \geq f_{k+2}$.

Running time of Euclid's algorithm

Induction Hypothesis: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

Inductive Step: **Goal: if** $\gcd(a,b)$ **with** $a \geq b > 0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2}$.

Next suppose that $k+1 \geq 3$ **so for the first 3 steps of Euclid's algorithm on** a **and** b **we have**

$$a = q_{k+1}\, b + r_k$$

$$b = q_k\, r_k + r_{k-1}$$

$$r_k = q_{k-1}\, r_{k-1} + r_{k-2}$$

and there are $k-2$ **more steps after this.**

Running time of Euclid's algorithm

Induction Hypothesis: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

Inductive Step: **Goal: if** $\gcd(a,b)$ **with** $a \geq b > 0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2}$.

Next suppose that $k+1 \geq 3$ **so for the first 3 steps of Euclid's algorithm on** a **and** b **we have**

$$a = q_{k+1}\, b + r_k$$
$$b = q_k\, r_k + r_{k-1}$$
$$r_k = q_{k-1}\, r_{k-1} + r_{k-2}$$

and there are $k-2$ **more steps after this. Note that this means that the** $\gcd(b, r_k)$ **takes** k **steps and** $\gcd(r_k, r_{k-1})$ **takes** $k-1$ **steps.**

So since $k \geq 1$ and $k-1 \geq 1$, **by the IH, we have** $b \geq f_{k+1}$ **and** $r_k \geq f_k$.

Running time of Euclid's algorithm

Induction Hypothesis: **Suppose that for some integer** $k \geq 1$, $P(j)$ **is true for all integers** j **s.t.** $1 \leq j \leq k$

Inductive Step: **Goal: if** $\gcd(a,b)$ **with** $a \geq b > 0$ **takes** $k+1$ **steps, then** $a \geq f_{k+2}$.

Next suppose that $k+1 \geq 3$ **so for the first 3 steps of Euclid's algorithm on** a **and** b **we have**

$$a = q_{k+1}\, b + r_k$$
$$b = q_k\, r_k + r_{k-1}$$
$$r_k = q_{k-1}\, r_{k-1} + r_{k-2}$$

and there are $k-2$ **more steps after this. Note that this means that the** $\gcd(b, r_k)$ **takes** k **steps and** $\gcd(r_k, r_{k-1})$ **takes** $k-1$ **steps.**

So since $k \geq 1$ and $k-1 \geq 1$, **by the IH, we have** $b \geq f_{k+1}$ **and** $r_k \geq f_k$. **Also, since** $a \geq b$, **we must have** $q_{k+1} \geq 1$.

So $a = q_{k+1}b + r_k \geq b + r_k \geq f_{k+1} + f_k = f_{k+2}$ **as required.**

Algorithmic Problems

- **Multiplication**
  - Given primes p_1, p_2, ..., p_k, calculate their product $p_1 p_2 \ldots p_k$
- **Factoring**
  - Given an integer n, determine the prime factorization of n

Factoring

Factor the following 232 digit number [RSA768]:

123018668453011775513049495838496272077
285356959533479219732245215172640050726
365751874520219978646938995647494277406
384592519255732630345373154826850791702
612214291346167042921431160222124047927
473779408066535141959745985690214341 3

123018668453011775513049495838496272077285356959533479219732245215172640050726365751874520219978646938995647494277406384592519255732630345373154826850791702612214291346167042921431160222124047927473779408066535141959745985690214341 3

=

33478071698956898786044169848212690817704794983713768568912431388982883793878002287614711652531743087737814467999489

×

3674604366679959042824463379962795263227915816434308764267603228381573966651127923337341714339681027009279873630891 7

Famous Algorithmic Problems

- **Factoring**
  - Given an integer n, determine the prime factorization of n
- **Primality Testing**
  - Given an integer n, determine if n is prime

- **Factoring** is hard
  - (on a classical computer)
- **Primality Testing** is easy

GCD and Factoring

$a = 2^3 \cdot 3 \cdot 5^2 \cdot 7 \cdot 11 = 46{,}200$

$b = 2 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 13 = 204{,}750$

$$\text{GCD}(a, b) = 2^{\min(3,1)} \cdot 3^{\min(1,2)} \cdot 5^{\min(2,3)} \cdot 7^{\min(1,1)} \cdot 11^{\min(1,0)} \cdot 13^{\min(0,1)}$$

Factoring is hard

Yet, we can compute GCD(a,b) without factoring!

Exponentiation

- **Compute** 78365^{81453}

- **Compute** $78365^{81453} \bmod 104729$

- **Output is small**
  - need to keep intermediate results small

Small Multiplications

Since $\boldsymbol{b} = qm + (\boldsymbol{b} \bmod m)$, we have $\boldsymbol{b} \bmod m \equiv_m \boldsymbol{b}$.

And since $\boldsymbol{c} = tm + (\boldsymbol{c} \bmod m)$, we have $\boldsymbol{c} \bmod m \equiv_m \boldsymbol{c}$.

Multiplying these gives $(\boldsymbol{b} \bmod m)(\boldsymbol{c} \bmod m) \equiv_m \boldsymbol{bc}$.

By a Lemma from Topic 3, this tells us that
$\boldsymbol{bc} \bmod m = (\boldsymbol{b} \bmod m)(\boldsymbol{c} \bmod m) \bmod m$.

Okay to mod $\boldsymbol{b}$ and $\boldsymbol{c}$ by m before multiplying if we are planning to mod the result by m

Repeated Squaring – small and fast

Since $b \bmod m \equiv_m b$ and $c \bmod m \equiv_m c$

we have $bc \bmod m = (b \bmod m)(c \bmod m) \bmod m$

So $a^2 \bmod m = (a \bmod m)^2 \bmod m$

and $a^4 \bmod m = (a^2 \bmod m)^2 \bmod m$

and $a^8 \bmod m = (a^4 \bmod m)^2 \bmod m$

and $a^{16} \bmod m = (a^8 \bmod m)^2 \bmod m$

and $a^{32} \bmod m = (a^{16} \bmod m)^2 \bmod m$

Can compute $a^k \bmod m$ for $k = 2^i$ in only i steps

What if k is not a power of 2?

Fast Exponentiation Algorithm

81453 in binary is 1001111000101101

$81453 = 2^{16} + 2^{13} + 2^{12} + 2^{11} + 2^{10} + 2^9 + 2^5 + 2^3 + 2^2 + 2^0$

$a^{81453} = a^{2^{16}} \cdot a^{2^{13}} \cdot a^{2^{12}} \cdot a^{2^{11}} \cdot a^{2^{10}} \cdot a^{2^9} \cdot a^{2^5} \cdot a^{2^3} \cdot a^{2^2} \cdot a^{2^0}$

$a^{81453} \bmod m =$
$(\ldots(((((a^{2^{16}} \bmod m \cdot$
$a^{2^{13}} \bmod m) \bmod m \cdot$
$a^{2^{12}} \bmod m) \bmod m \cdot$
$a^{2^{11}} \bmod m) \bmod m \cdot$
$a^{2^{10}} \bmod m) \bmod m \cdot$
$a^{2^9} \bmod m) \bmod m \cdot$
$a^{2^5} \bmod m) \bmod m \cdot$
$a^{2^3} \bmod m) \bmod m \cdot$
$a^{2^2} \bmod m) \bmod m \cdot$
$a^{2^0} \bmod m) \bmod m$

Uses only 16 + 9 = 25 multiplications

The fast exponentiation algorithm computes $a^k \bmod m$ using $\leq 2\log k$ multiplications $\bmod m$

Fast Exponentiation: $a^k \bmod m$ for all k

Another way....

$$a^{2j} \bmod m = \left(a^j \bmod m\right)^2 \bmod m$$

$$a^{2j+1} \bmod m = \left((a \bmod m) \cdot (a^{2j} \bmod m)\right) \bmod m$$

Fast Exponentiation

```
public static int FastModExp(int a, int k, int modulus) {

        if (k == 0) {
            return 1;

        } else if ((k % 2) == 0) {
            long temp = FastModExp(a,k/2,modulus);
            return (temp * temp) % modulus;

        } else {
            long temp = FastModExp(a,k-1,modulus);
            return (a * temp) % modulus;
        }
}
```

$$a^{2j} \bmod m = \left(a^{j} \bmod m\right)^{2} \bmod m$$

$$a^{2j+1} \bmod m = \left((a \bmod m) \cdot (a^{2j} \bmod m)\right) \bmod m$$

Using Fast Modular Exponentiation

- E-commerce web transactions use SSL, which is based on RSA encryption
- RSA:
  - Vendor chooses random 512-bit or 1024-bit primes p, q and 512/1024-bit exponent e. Computes $m = p \cdot q$
  - Vendor broadcasts (m, e)
  - To send a to vendor, you compute $C = a^e \bmod m$ and send C to the vendor.
  - Using secret p, q the vendor computes d that is the *multiplicative inverse* of e mod $(p-1)(q-1)$.
  - Vendor computes $C^d \bmod m = (a^e)^d \bmod m = a^{ed} \bmod m$
  - Can show that implies $a^{ed} \bmod m = a \bmod m$
  - Assumes we can calculate exponentials quickly!

Basic Applications of mod

- Two's Complement
- Hashing
- Pseudo random number generation

Recall: I'm ALIVE!

```
public class Test {
    final static int SEC_IN_YEAR = 365*24*60*60;
    public static void main(String args[]) {
        System.out.println(
            "I will be alive for at least " +
            SEC_IN_YEAR * 101 + " seconds."
        );
    }
}
```

```
  ----jGRASP exec: java Test
I will be alive for at least -186619904 seconds.

  ----jGRASP: operation complete.
```

n-bit Unsigned Integer Representation

- Represent integer x as sum of powers of 2:

99	= 64 + 32 + 2 + 1	= $2^6 + 2^5 + 2^1 + 2^0$
18	= 16 + 2	= $2^4 + 2^1$

- Binary representation shows which powers are used:

99: 0110 0011
18: 0001 0010

n-bit Unsigned Integer Representation

- Suppose we write numbers with 4 bits:

14	$= 8 + 4 + 2$	$= 2^3 + 2^2 + 2^1$	$= 1110$
11	$= 8 + 2 + 1$	$= 2^3 + 2^1 + 2^0$	$= 1011$

- Largest number we can write in 4 bits is:

15	$= 8 + 4 + 2 + 1$	$= 2^3 + 2^2 + 2^1 + 2^0$	$= 1111$

- Note that $15 = 16 - 1 = 2^4 - 1$
  - we proved this before!

n-bit Unsigned Integer Representation

- Suppose we write numbers with 4 bits (0 .. 15):

14	$= 8 + 4 + 2$	$= 2^3 + 2^2 + 2^1$	$= 1110$
11	$= 8 + 2 + 1$	$= 2^3 + 2^1 + 2^0$	$= 1011$

- Adding these numbers gives us 25 with 5 bits:

25	$= 16 + 8 + 1$	$= 2^4 + 2^3 + 2^0$	$= 11001$

- If we drop the highest bit, we have

9	$= 8 + 1$	$= 2^3 + 2^0$	$= 1001$

n-bit Unsigned Integer Representation

25	$= 16 + 8 + 1$	$= 2^4 + 2^3 + 2^0$	$= 11001$
9	$= 8 + 1$	$= 2^3 + 2^0$	$= 1001$

- Note that $9 \equiv_{16} 25$ since $25 - 9 = 16$
  - dropping 2^4 bit subtracts 16
  - dropping 2^5 bit subtracts $32 = 2 \cdot 16$
  - dropping 2^6 bit subtracts $64 = 4 \cdot 16$

- Throwing away all but 4 bits is arithmetic mod 16
  - easier to implement normal arithmetic!

Sign-Magnitude Integer Representation

n-bit signed integers

Suppose that $-2^{n-1} < x < 2^{n-1}$

First bit as the sign, $n - 1$ bits for the value

99 = 64 + 32 + 2 + 1

18 = 16 + 2

For n = 8:

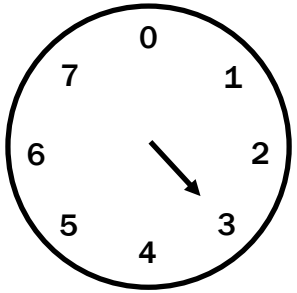
99: 0110 0011

-18: 1001 0010

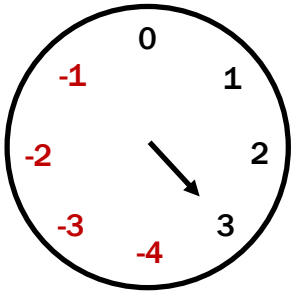
Problem: this has both +0 and -0 (annoying)

Arithmetic on a Clock

3 bits, unsigned



3 bits, signed



Since $-1 \equiv_8 7$, arithmetic is unchanged

Only differences are printing and comparison

Two's Complement Representation

Suppose that $0 \leq x < 2^{n-1}$

x is represented by the binary representation of x

Suppose that $-2^{n-1} \leq x < 0$

x is represented by the binary representation of $x + 2^n$

result is in the range $2^{n-1} \leq x < 2^n$

99 = 64 + 32 + 2 + 1

18 = 16 + 2

For n = 8:

99: 0110 0011

-18: 1110 1110 (-18 + 256 = 238)

Two's Complement Representation

Suppose that $0 \leq x < 2^{n-1}$

x is represented by the binary representation of x

Suppose that $-2^{n-1} \leq x < 0$

x is represented by the binary representation of $x + 2^n$

result is in the range $2^{n-1} \leq x < 2^n$

With 4 bits:

0	1	2	3	4	5	6	7	-8	-7	-6	-5	-4	-3	-2	-1
0000	0001	0010	0011	0100	0101	0110	0111	1000	1001	1010	1011	1100	1101	1110	1111

Key property: First bit is still the sign bit!

Key property: Twos complement representation of any number y is equivalent to $y \bmod 2^n$ so arithmetic works $\bmod 2^n$

$$y + 2^n \equiv_{2^n} y$$

Two's Complement Representation

- For $0 < x \leq 2^{n-1}$, $-x$ is represented by the binary representation of $-x + 2^n$
  - How do we calculate –x from x?
  - E.g., what happens for "`return –x;`" in Java?

$$-x + 2^n = (2^n - 1) - \mathrm{x} + 1$$

- To compute this, flip the bits of x then add 1!

  Flip the bits of x means replace x by $2^n - 1 - x$

  Then add 1 to get $-x + 2^n$

n-bit Unsigned Integer Representation

- Largest representable number is $2^n - 1$

2^n = 100…000 (n+1 bits)

$2^n - 1$ = 11…111 (n bits)

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32 bits
1 = $0.0001
$429,496.7295 max

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Hashing

Scenario:

Map a small number of data values from a large domain $\{0, 1, \ldots, M-1\}$ ...

...into a small set of locations $\{0, 1, \ldots, n-1\}$ so one can quickly check if some value is present

- $\text{hash}(x) = x \bmod p$ for p a prime close to n

  – or $\text{hash}(x) = (ax + c) \bmod p$

- Second avoids clustering nearby inputs
  - $\text{hash}(x)$ and $\text{hash}(x+1)$ can be very far apart

Hashing

- $\text{hash}(x) = (ax + c) \bmod p$ **for prime** p
  - **deterministic function with random-ish behavior**

- **Suppose that** $\text{hash}(x) = \text{hash}(y)$**...**

$$\text{ax} + \text{c} \equiv_p \text{ay} + \text{c}$$

$$\text{ax} \equiv_p \text{ay}$$ add $-\text{c}$ to both sides

$$\text{x} \equiv_p \text{y}$$ multiply both sides by s where $\text{as} \equiv_p 1$

- **Output as evenly spread as** $\text{hash}(x) = x \bmod p$

Hashing

- $\text{hash}(x) = (ax + c) \bmod p$ for prime p
  - deterministic function with random-ish behavior

- Applications
  - map integer to location in array (hash tables)
  - map user ID or IP address to machine

    requests from the same user / IP address go to the same machine

    requests from different users / IP addresses spread randomly

    > (don't want lots of nearby inputs **clustering** into one rack)

Pseudo-Random Number Generation

Linear Congruential method

$$x_{n+1} = (a\, x_n + c) \bmod m$$

Choose random x_0, a, c, m and produce a long sequence of x_n's

Applications of Formal Logic

Satisfiability (SAT)

Problem: Given a Propositional Logic expression, is there a way to set the values of the variables to make the expression evaluate to T?

- if yes, the expression is "satisfiable"
- if not, the expression is "unsatisfiable"

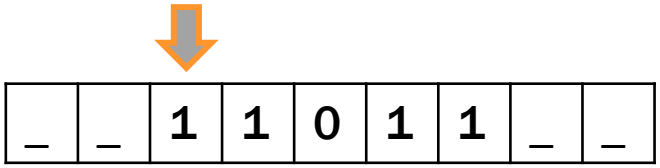
- Many problems can be stated as SAT problems
  - e.g., many "puzzle" type problems

    see 25sp HW1 for an example
  - lots of important & useful problems in this category

    e.g., verifying correctness of hardware

Recall: Turing machines

	_	0	1
s_1	(1, L, s_3)	(1, L, s_4)	(0, R, s_2)
s_2	(0, R, s_1)	(1, R, s_1)	(0, R, s_1)
s_3			
s_4			



P and NP

- P = solvable by TMs in poly time
  - anything outside of this is "too slow" for real use
- NP = … by nondeterministic TMs in poly time
  - any number of options for next state

    e.g., (write a 0, move left, change to state s_1) or
    (write a 1, move right, change to state s_2)
- P =? NP is the most famous open problem in CS
  - DFAs need exponential space to simulate NFAs,
    do TMs need exponential time to simulate NTMs?
  - P ≠ NP is taken as a *law of nature* in physics

NP-Completeness

- **NP = … by nondeterministic TMs in poly time**
  - **any number of options for next state**

    e.g., (write a 0, move left, change to state s_1) or
    (write a 1, move right, change to state s_2)

- **Problems "as hard as any in NP" are "NP-complete"**
  - **prove this by reducing all NP problems to your problem**
  - **Cook reduce all NP problems to SAT in the early 1970s**
  - **Karp then reduced SAT to many other problems**

    enormous list of important problems known to be NP-complete

SAT Solvers

Problem: Given a Propositional Logic expression, is there a way to set the values of the variables to make the expression evaluate to T?

- if yes, the expression is "satisfiable"
- if not, the expression is "unsatisfiable"

- Brute force is doesn't get far with 264 variables...
  - $2^{264} \approx$ # atoms in the observable universe
- Modern SAT solvers handle *millions* of variables
  - would be nice to have access to these!

SAT Solvers

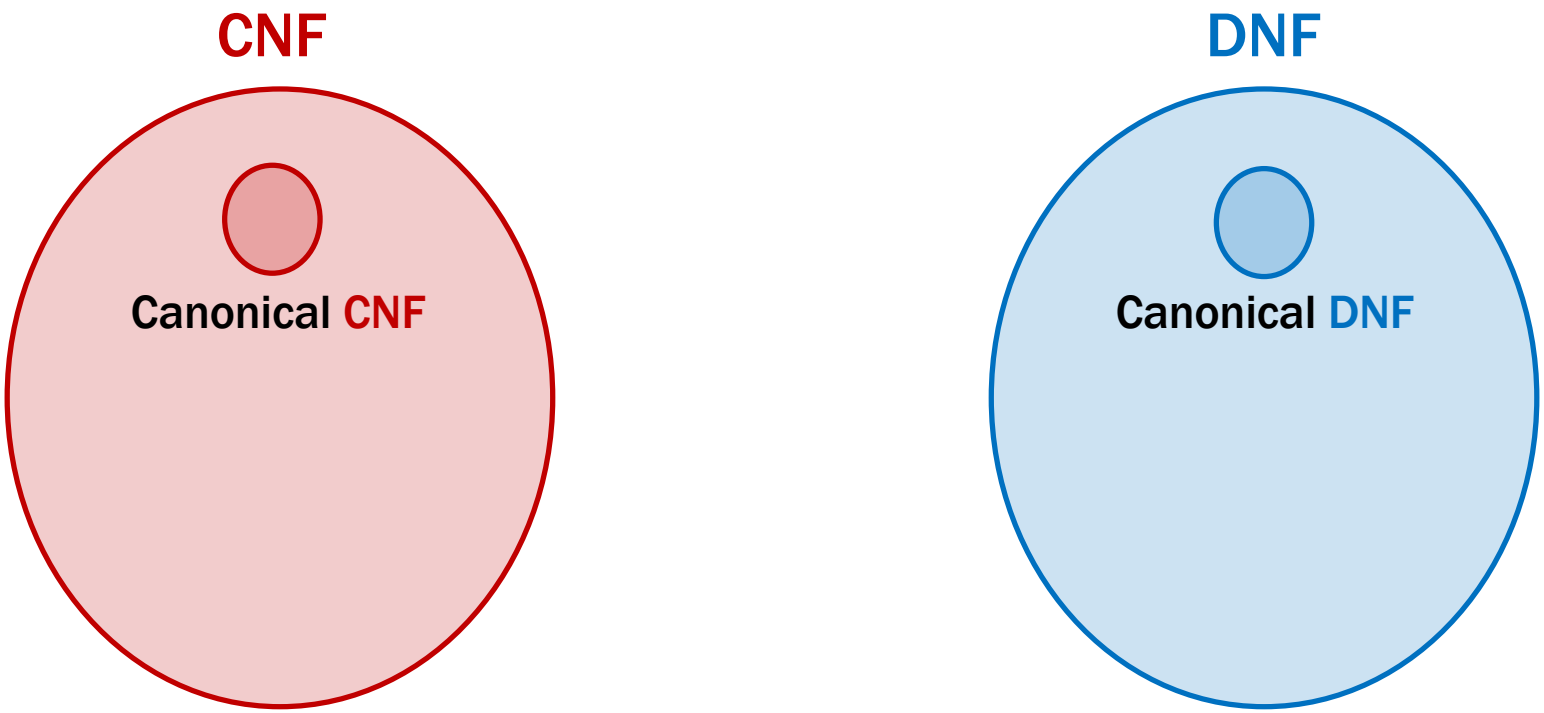
- Usually, do not accept *arbitrary* Logic expressions
  - require the expression to come in a simpler form

- Typically, require the expression in "CNF"
  - CNF not DNF due to how SAT is defined

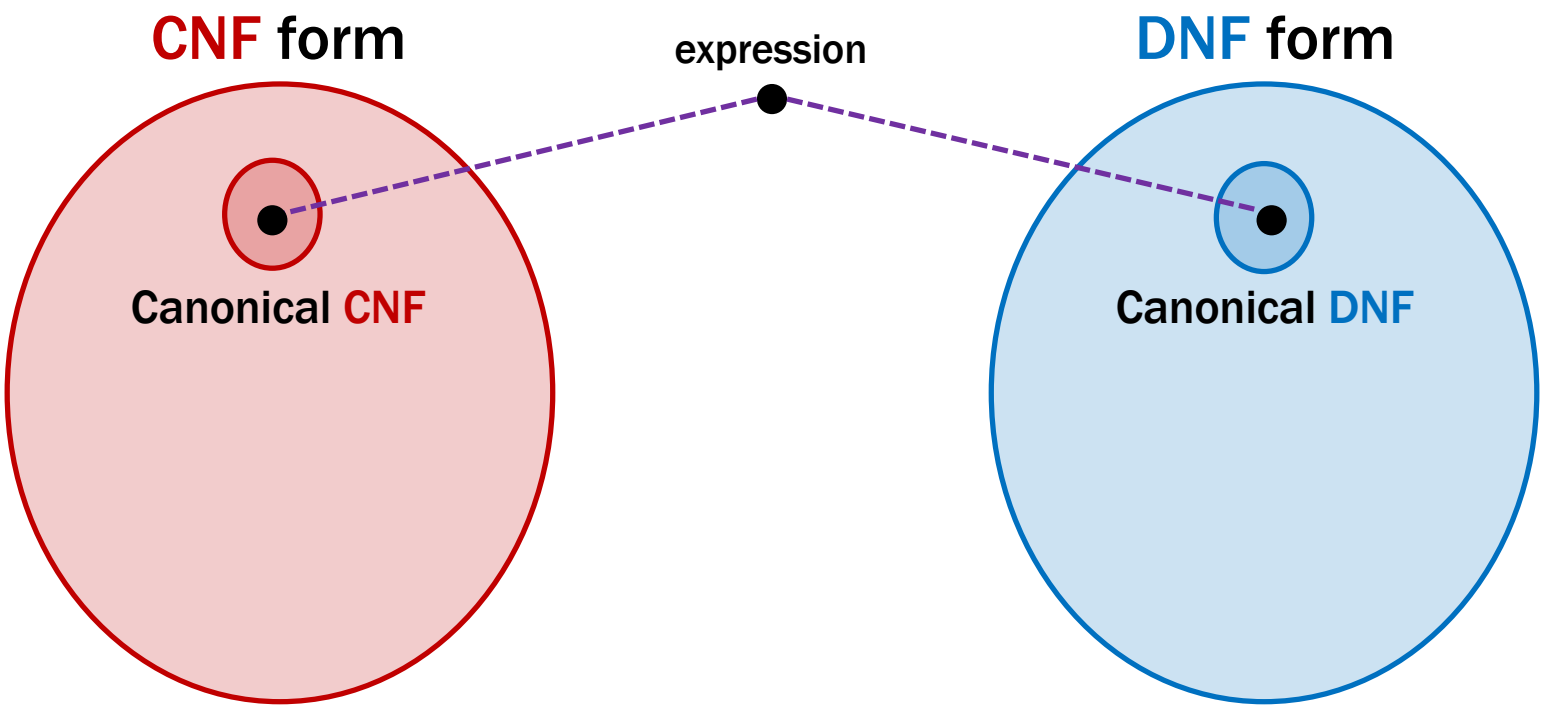
    "tautology" problem would want the input in DNF

- We learned about CNF before…

Recall: CNF & DNF



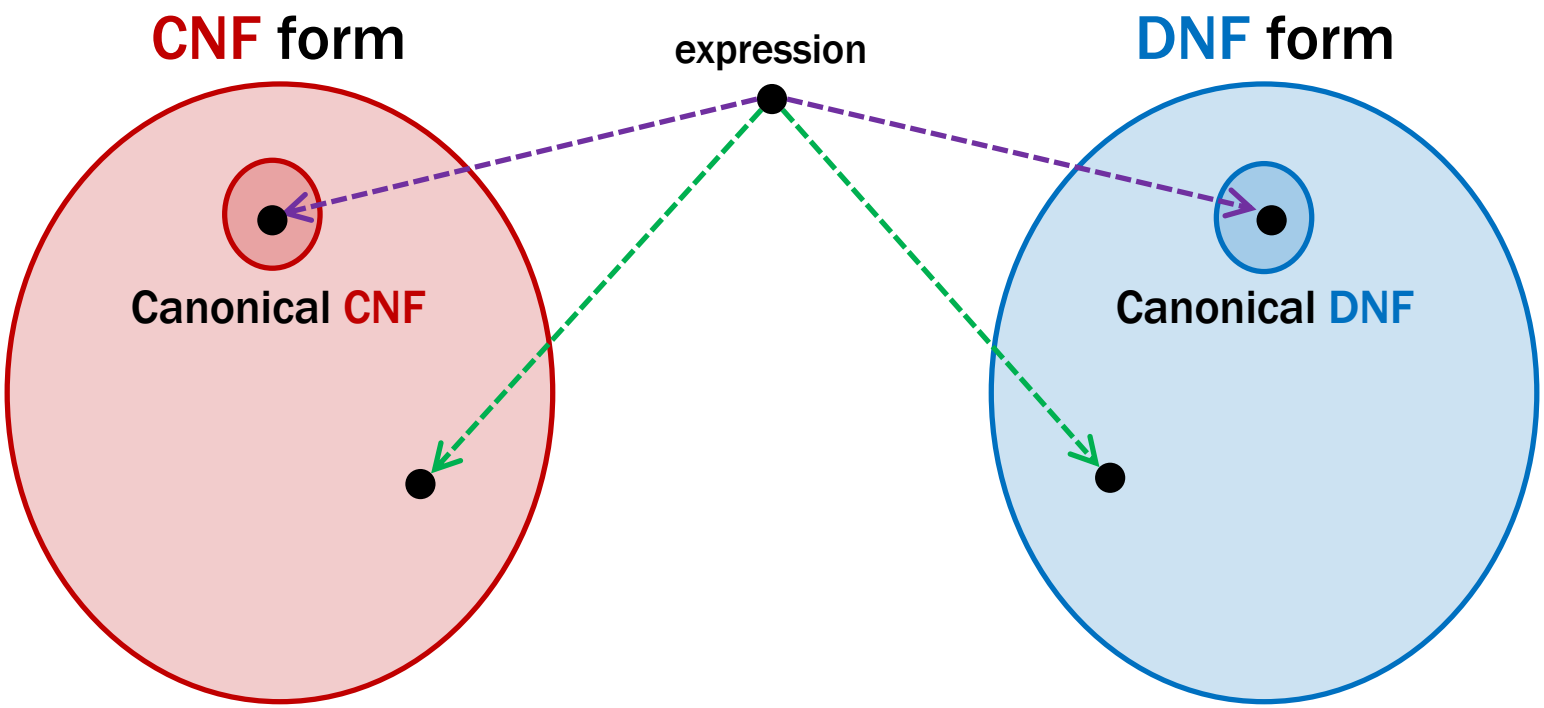
CNF & DNF



equivalent to exactly <u>one</u> in canonical CNF (up to reordering)

if our expressions are in canonical CNF,
then they are equivalent iff they are the same

CNF & DNF



---- equivalent but slow conversion

---- fast conversion but only "equi-satisfiable"
(details are outside the scope of 311)

LLMs + Tools

- **Not great at some problems that require precision**
  - **multiplying numbers with 10+ digits**

    best LLMs routine call **tools** to do this instead
  - **playing chess (legally)**

    until recently, most LLMs could not avoid illegal moves

    reasoning models do this, but are still mediocre (1000–1500 ELO)

- **Best chess engines use neural networks (like LLMs)**
  - **but only for creative parts**
  - **regular code is used to (1) identify legal moves and (2) cleverly search through the space of legal moves**

LLM + Tools

- Do not expect LLMs to solve SAT problems well
  - requires getting millions of details exactly right
- LLM with a SAT solver would be more powerful
  - provide it to the LLM as a tool to use
- Research Area: more (formal) tools for LLMs
  - SAT solvers
  - Integer programming / network flow solvers
  - (this work requires knowledge of CS and formalism)

λ-Calculus

- **Invented by Alonzo Church in the 1930s**
  - **equivalent in power to Turing's machines**
  - **a programming language: software, not hardware**

- **Syntax:**

anonymous functions are called "lambda functions"

S → **V**	variable
| λ **V** . **S**	create a function
| **S S**	call a function
| (**S**)	

V → x | y | z | ...

- **Example:** $(\lambda x . f x) y$

Semantics of λ-Calculus

- If all variables are *bound*, every expression evaluates to a function

E[y/x] means substitute "y" for "x"

- When are two results "the same"?
  - variable renaming: $\lambda x . E \equiv \lambda y . E[y/x]$
  - call unwrapping: $\lambda x . f x \equiv f$

- Any sequence of steps like this is equivalent
  - i.e., we want the reflexive, transitive closure!
  - the result is then an equivalence relation

    rules are already symmetric (either direction works)

Semantics of λ-Calculus

- **Only computational step is a function call:**

$$(\lambda\, x\,.\, E_1)\; E_2 \;\equiv\; E_1[E_2/x] \qquad\qquad \text{e.g., } (\lambda\, x\,.\, f\, x)\; y \;\equiv\; f\, y$$

  - **a function call is just substitution**

- **Much simpler than other languages**
  - **no need to worry about evaluation order**

    in λ-Calculus, any order produces the same result (Church-Rosser)

Infinite Loops

- λ-Calculus is Turing complete…
  - so, it must be possible to write an infinite loop

- You can do so as follows:

$$(\lambda x . x\, x)(\lambda x . x\, x)$$

  - the function call produces what we started with!

Infinite Loops

$(\lambda x . x x)(\lambda x . x x)$ infinite recursion

- Possible to remove this by type checking
  - type system will not allow $\lambda x . x x$
- Result will no longer be Turing-complete!
  - cannot express infinite loops
  - OTOH, no infinite loops!

Typed λ-Calculus

- Let's include...
  - one basic type: the integers ($\mathbb{Z}$)
  - function types

- Syntax of legal types:

$$S \to T \mid T \to S$$

$$T \to \mathbb{Z} \mid (S)$$

- Examples: $\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}$ and $(\mathbb{Z} \to \mathbb{Z}) \to \mathbb{Z}$

Typed λ-Calculus

- How do we assign types to expressions?
  - integer literals will have type "$\mathbb{Z}$"
  - functions are trickier...

- Type of $\mathrm{f}\ \mathrm{x}$ depends on the types of f and x
  - we can write it as follows, where " : " means "has type"

$$\frac{x : A \qquad f : A \to B}{f\ x : B}$$

the typing *judgement* $\mathrm{f}\ \mathrm{x} : \mathrm{B}$ depends on the two *premises* that $\mathrm{x} : \mathrm{A}$ and $\mathrm{f} : \mathrm{A} \to \mathrm{B}$

Typed λ-Calculus

- Type of $\lambda\, \mathrm{x}\, .\, \mathrm{E}$ can also be written this way...

- Trickier because "x" can appear in E
  - E is evaluated in a context where x exists

    it does not have a type without x being defined
  - we will write "$\vdash$" to mean "in context"

    LHS lists the extra variables defined there

$$\frac{x : A \vdash E : B}{\lambda\, x\, .\, E : A \to B}$$

More Types

- Useful programs need more data than this
  - hard to just work with one integer at a time

- Let's try adding a couple more types...
  - pairs of integers

    and pairs of pairs etc.
  - unions of types

    not allowed in Java but normal in set theory

Pairs

- **Syntax and semantics:**
  - **use "(x, y)" to create a pair**
  - **use "$\mathrm{first}(x, y) = x$" to get the left part**
  - **use "second(x, y) = y" to get the right part**

Exercise: add this to the expression grammar

- **A pair of integers has type $\mathbb{Z} \times \mathbb{Z}$**
  - **more generally, pair of A and B has type $A \times B$**

Exercise: add this to the type grammar

Pair Types

- Types of pair creation

$$\frac{x : A \qquad y : B}{(x, y) : A \times B}$$

- Types of pair projection

$$\frac{p : A \times B}{\text{first}(p) : A} \qquad \frac{p : A \times B}{\text{second}(p) : B}$$

Unions

- Common in functional languages
  - Typescript has them; even "C" includes union types
  - must roll your own in Java…

```
class Shape {
  enum Kind { CIRCLE, SQUARE, … }
  Kind tag;
  Shape(Kind tag) { this.tag = tag }
}

class Circle extends Shape {
  double radius;
  Circle(double r) {
    super(Kind.CIRCLE);
    this.radius = r;
  }
}
```

Can use s.tag instead of "`instanceof`" to see what kind of shape it is

Only `Circle`s store a radius

Unions

- At runtime, each instance of $A \cup B$ stores a "tag" to remember whether it is an A or a B

- Syntax and semantics:
  - use "$\text{left}(x)$" to wrap "$x : A$" with a tag for $A \cup B$
  - use "$\text{right}(x)$" to wrap "$x : B$" with a tag for $A \cup B$

- How do we use a tagged union?

Unions

- At runtime, each instance of $A \cup B$ stores a "tag" to remember whether it is an A or a B

- How do we use a tagged union?
  - need to write code for when input is an A and code for when the input is a B
  - use "$\mathrm{case}(x, f, g)$" to switch into f or g

- Examples: $\mathrm{case}(\mathrm{left}(x), f, g) = f\, x$

  $\mathrm{case}(\mathrm{right}(x), f, g) = g\, x$

Union Types

- Types of tagging operations:

$$\frac{x : A}{\text{left}(x) : A \cup B} \qquad \frac{y : B}{\text{right}(y) : A \cup B}$$

- Type of case / switch:

$$\frac{x : A \cup B \qquad f : A \to C \qquad g : B \to C}{\text{case}(x, f, g) : C}$$

Summary of Typing Rules

Pairs

$$\frac{p : A \times B}{\text{first}(p) : A} \qquad \frac{x : A \qquad y : B}{(x, y) : A \times B}$$

Unions

$$\frac{x : A \cup B \quad f : A \to C \quad g : B \to C}{\text{case}(x, f, g) : C} \qquad \frac{x : A}{\text{left}(x) : A \cup B}$$

Functions

$$\frac{x : A \qquad f : A \to B}{f\, x : B} \qquad \frac{x : A \vdash E : B}{\lambda\, x\, .\, E : A \to B}$$

Does this look familiar?

Summary of Typing Rules

Pairs

$$\frac{A \times B}{A} \qquad \frac{A \quad B}{A \times B}$$

Unions

$$\frac{A \cup B \quad A \to C \quad B \to C}{C} \qquad \frac{A}{A \cup B}$$

Functions

$$\frac{A \quad A \to B}{B} \qquad \frac{A \vdash \quad B}{A \to B}$$

How about if I changed these symbols...

Summary of Typing Rules

Pairs

$$\frac{A \wedge B}{A} \qquad \frac{A \quad B}{A \wedge B}$$

Unions

$$\frac{A \vee B \quad A \to C \quad B \to C}{C} \qquad \frac{A}{A \vee B}$$

Functions

$$\frac{A \quad A \to B}{B} \qquad \frac{A \Rightarrow \quad B}{A \to B}$$

These are our inference rules for Propositional Logic

Curry-Howard

- This is not an accident!

- Type systems are logics
  - *any* type system is a logic
  - *any* logic is a type system

- This fact is called the Curry-Howard isomorphism
  - also, proofs are programs and vice versa
  - another deep connection between CS and logic

Logic and LLMs

- Logic and CS are deeply connected
  - both historically and in practice
  - study of programming languages is the study of logic
- Programming languages is exciting right now…
- Research Area: programming languages that work better when LLMs are writing the code
  - what languages catch bugs that LLMs tend to write?
  - may look quite different from languages for humans!