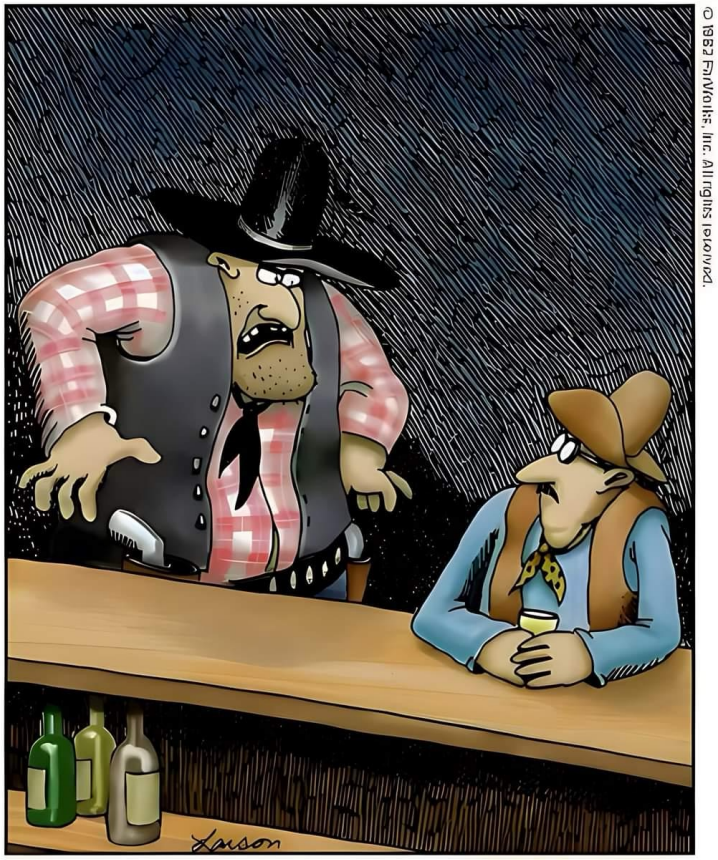
CSE 311: Foundations of Computing

Topic 3: Number Theory



"I *asked* you a question, buddy. ... What's the square root of 5,248?"

Number Theory

- Direct relevance to computing
  - everything in a computer is a number

    colors on the screen are encoded as numbers

- Many significant applications
  - Cryptography & Security
  - Data Structures
  - Distributed Systems

I’m ALIVE!

```
public class Test {
    final static int SEC_IN_YEAR = 365*24*60*60;
    public static void main(String args[]) {
        System.out.println(
            “I will be alive for at least ” +
            SEC_IN_YEAR * 101 + “ seconds.”
        );
    }
}
```

I'm ALIVE!

```
public class Test {
    final static int SEC_IN_YEAR = 365*24*60*60;
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}
```

```
 ----jGRASP exec: java Test
I will be alive for at least -186619904 seconds.

 ----jGRASP: operation complete.
```

Modular Arithmetic

Modular Arithmetic

- Arithmetic over a finite domain
- Almost all computation is over a finite domain

Recall: Elementary School Division

For a, b with $b > 0$, we can divide b into a.

If $b \nmid a$, then we end up with a *remainder* r with $0 < r < b$. Now,

instead of $\frac{a}{b} = q$ we have $\frac{a}{b} = q + \frac{r}{b}$

Multiplying both sides by b gives us $a = qb + r$

Recall: Elementary School Division

For a, b with $b > 0$, we can divide b into a.

If $b \mid a$, then we have $a = qb$ for some q.

If $b \nmid a$, then we have $a = qb + r$ for some q, r with $0 < \mathrm{r} < \mathrm{b}$.

In general, we have $a = qb + r$ for some q, r with $0 \leq r < b$, where $r = 0$ iff $b \mid a$.

Division Theorem

Domain of Discourse
Integers

Division Theorem

For a, b with $b > 0$
there exist *unique* integers *q*, *r* with $0 \leq r < b$
such that $a = qb + r$.

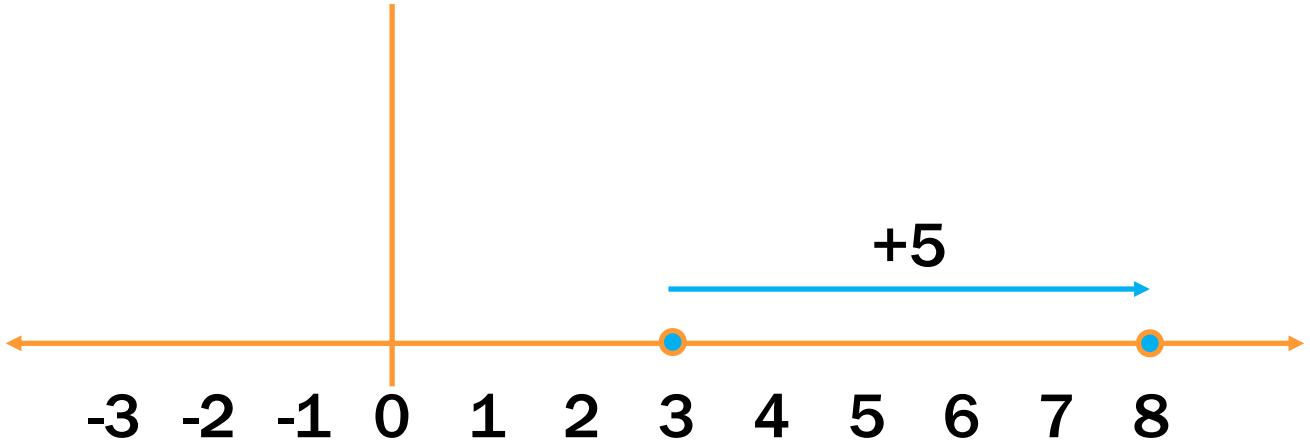
To put it another way, if we divide *b* into *a*, we get a unique quotient $q = a \textbf{ div } b$ **and non-negative remainder** $r = a \textbf{ mod } b$

$$a = (a \textbf{ div } b)\, b + (a \textbf{ mod } b)$$

$$\forall a \, \forall b \, \Big((b > 0) \rightarrow \big(a = (a \textbf{ div } b) b + (a \textbf{ mod } b) \big) \Big)$$

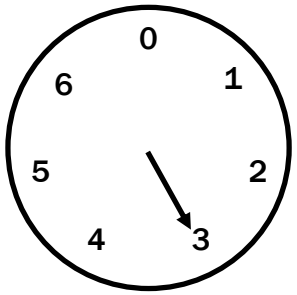
Ordinary arithmetic

$$3 + 5 = 8$$



Arithmetic on a Clock

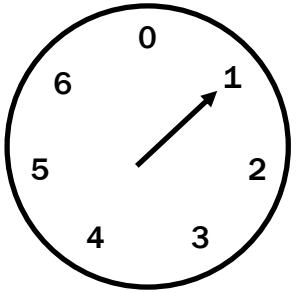
$3 + 5 = 8$



$8 = 7 \cdot 1 + 1$

$15 = 7 \cdot 2 + 1$

$22 = 7 \cdot 3 + 1$

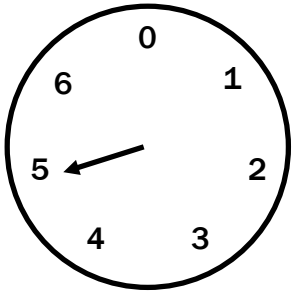


If $a = 7q + r$, then $r \ \ (= a \bmod b)$ is where you <u>stop</u> after taking a steps on the clock

Arithmetic, mod 7

$(a + b) \bmod 7$

$(a \times b) \bmod 7$



+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	0
2	2	3	4	5	6	0	1
3	3	4	5	6	0	1	2
4	4	5	6	0	1	2	3
5	5	6	0	1	2	3	4
6	6	0	1	2	3	4	5

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

Modular Arithmetic

Domain of Discourse
Integers

Definition: "a is congruent to b modulo m"

For a, b, m with $m > 0$

$$a \equiv_m b \;\; := \;\; m \mid (a - b)$$

New notion of "sameness" that will help us understand modular arithmetic

Modular Arithmetic

Domain of Discourse
Integers

Definition: "a is congruent to b modulo m"

For a, b, m with $m > 0$

$$a \equiv_m b \;:=\; m \mid (a - b)$$

The standard math notation is

$$a \equiv b \pmod{m}$$

A chain of equivalences is written

$$a \equiv b \equiv c \equiv d \pmod{m}$$

Many students find this confusing, so we will use $\equiv_m$ instead.

Modular Arithmetic

Domain of Discourse
Integers

Definition: "a is congruent to b modulo m"

For a, b, m with $m > 0$

$$a \equiv_m b \;:=\; m \mid (a - b)$$

Check Your Understanding. What do each of these mean? When are they true?

$-1 \equiv_5 19$

This statement is true. 19 - (-1) = 20 which is divisible by 5

$x \equiv_2 0$

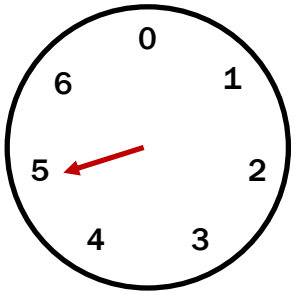
This statement is the same as saying "x is even"; so, any x that is even (including negative even numbers) will work.

$y \equiv_7 2$

This statement is true for y in { ..., -12, -5, 2, 9, 16, ...}. In other words, all y of the form 2+7k for k an integer.

The $\mathrm{mod}\ m$ function vs the $\equiv_m$ predicate

- **The $\mathrm{mod}\ m$ function maps any integer a to a remainder $a \bmod m \in \{0,1,..,m-1\}$.**
  Tells you where it lands on the clock.



- **Imagine grouping together all integers that have the same value of the $\mathrm{mod}\ m$ function.**
  They must differ by a multiple of m ($q_1 m + r$ vs $q_2 m + r$)

- **The $\equiv_m$ predicate compares integers a, b to see if if they differ by a multiple of m.**
  If they differ by a multiple of m, then walking from one to the other leaves you at the same spot on the clock.

Modular Arithmetic: A Property

Let a, b, m be integers with $m > 0$.
Then, $a \equiv_m b$ if and only if $a \bmod m = b \bmod m$.

This claim is an $\leftrightarrow$ ("iff")

Proof Plan:

1. $(a \bmod m = b \bmod m) \rightarrow (a \equiv_m b)$??
2. $(a \equiv_m b) \rightarrow (a \bmod m = b \bmod m)$??
3. $(a \bmod m = b \bmod m) \rightarrow (a \equiv_m b) \wedge$
   $(a \equiv_m b) \rightarrow (a \bmod m = b \bmod m)$ Intro $\wedge$: 1, 2
4. $(a \equiv_m b) \leftrightarrow (a \bmod m = b \bmod m)$ Biconditional: 3

Modular Arithmetic: A Property

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1. $(a \bmod m = b \bmod m) \rightarrow (a \equiv_m b)$??

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Let a, b, m be integers with $m > 0$.
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1.1. $a \bmod m = b \bmod m$ **Assumption**

1.? $a \equiv_m b$ **??**

1. $(a \bmod m = b \bmod m) \rightarrow (a \equiv_m b)$ **Direct Proof**

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1.? $m \mid a - b$ — ??

1.? $a \equiv_m b$ — Def of ≡

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1.? $\exists q\,(a - b = qm)$ — ??
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1.1. $a \bmod m = b \bmod m$	Assumption
1.2. $a = (a \operatorname{div} m)\, m + (a \bmod m)$	Apply Division
1.3. $b = (b \operatorname{div} m)\, m + (b \bmod m)$	Apply Division

1.? $\exists q\, (a - b = qm)$	??
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1.4. $a - b = \big((a \operatorname{div} m) - (b \operatorname{div} m)\big)\, m$	Algebra
1.5. $\exists q\, (a - b = qm)$	Intro ∃
1.6. $m \mid a - b$	Def of |
1.7. $a \equiv_m b$	Def of ≡
1. $(a \bmod m = b \bmod m) \rightarrow (a \equiv_m b)$	Direct Proof

Modular Arithmetic: A Property

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ be integers with $\boldsymbol{m} > \boldsymbol{0}$.
Then, $\boldsymbol{a} \equiv_{\boldsymbol{m}} \boldsymbol{b}$ if and only if $\boldsymbol{a} \bmod \boldsymbol{m} = \boldsymbol{b} \bmod \boldsymbol{m}$.

Suppose that $a \bmod m = b \bmod m$. — Assumption

Apply Division
Apply Division

Algebra

Intro ∃
Def of |
Def of ≡
Direct Proof

Therefore, $a \equiv_m b$.

Modular Arithmetic: A Property

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Suppose that $a \bmod m = b \bmod m$. — Assumption

By the Division Theorem, we can write
$a = (a \operatorname{div} m)\, m + (a \bmod m)$ and — Apply Division
$b = (b \operatorname{div} m)\, m + (b \bmod m)$. — Apply Division

Algebra

Therefore, $a \equiv_m b$. — Intro ∃, Def of |, Def of ≡, Direct Proof

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$a = (a \operatorname{div} m)\, m + (a \bmod m)$ and — Apply Division
$b = (b \operatorname{div} m)\, m + (b \bmod m)$.

Subtracting these we can see that

$$\begin{aligned} a - b &= \big((a \operatorname{div} m) - (b \operatorname{div} m)\big)m + \\ &\quad ((a \bmod m) - (b \bmod m)) \\ &= \big((a \operatorname{div} m) - (b \operatorname{div} m)\big)\, m \end{aligned}$$

— Algebra

since $(a \bmod m) - (b \bmod m) = 0$. — Intro ∃

… — Def of |

Therefore, $a \equiv_m b$. — Def of ≡ — Direct Proof

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— Algebra

since $(a \bmod m) - (b \bmod m) = 0$.

Therefore, by definition of divides, $m \mid (a - b)$ — Intro ∃, Def of |
and so $a \equiv_m b$, by definition of congruent. — Def of ≡, Direct Proof

Modular Arithmetic: A Property

Let a, b, m be integers with $m > 0$.
Then, $a \equiv_m b$ if and only if $a \bmod m = b \bmod m$.

2. $(a \equiv_m b) \rightarrow (a \bmod m = b \bmod m)$??

Modular Arithmetic: A Property

Let a, b, m be integers with $m > 0$.
Then, $a \equiv_m b$ if and only if $a \bmod m = b \bmod m$.

2.1. $a \equiv_m b$ **Assumption**

2.? $a \bmod m = b \bmod m$ **??**

2. $(a \equiv_m b) \rightarrow (a \bmod m = b \bmod m)$ **Direct Proof**

Modular Arithmetic: A Property

Let a, b, m be integers with $m > 0$.
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2.1. $a \equiv_m b$ — Assumption
2.2. $m \mid a - b$ — Def of ≡

2.? $a \bmod m = b \bmod m$ — ??
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2.1. $a \equiv_m b$	Assumption
2.2. $m \mid a - b$	Def of ≡
2.3. $\exists q\,(a - b = qm)$	Def of |
2.4. $a - b = km$	Elim ∃

2.? $a \bmod m = b \bmod m$	??
2. $(a \equiv_m b) \rightarrow (a \bmod m = b \bmod m)$	Direct Proof

Modular Arithmetic: A Property

Let a, b, m be integers with $m > 0$.
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2.5. $a = (a \operatorname{div} m)\, m + (a \bmod m)$	Apply Division

2.? $a \bmod m = b \bmod m$	??
2. $(a \equiv_m b) \rightarrow (a \bmod m = b \bmod m)$	Direct Proof

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2.6. $b = (a \operatorname{div} m - k)\, m + (a \bmod m)$	Algebra

2.? $a \bmod m = b \bmod m$	??
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Modular Arithmetic: A Property

Let a, b, m be integers with $m > 0$.
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2.6. $b = (a \operatorname{div} m - k)\, m + (a \bmod m)$	Algebra
2.7. $b \operatorname{div} m = (a \operatorname{div} m - k) \wedge$ $b \bmod m = a \bmod m$	Apply DivUnique
2.? $a \bmod m = b \bmod m$	??
2. $(a \equiv_m b) \rightarrow (a \bmod m = b \bmod m)$	Direct Proof

Modular Arithmetic: A Property

Let a, b, m be integers with $m > 0$.
Then, $a \equiv_m b$ if and only if $a \bmod m = b \bmod m$.

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2.7. $b \operatorname{div} m = (a \operatorname{div} m - k) \wedge$ $b \bmod m = a \bmod m$	Apply DivUnique
2.8. $a \bmod m = b \bmod m$	Elim ∧
2. $(a \equiv_m b) \rightarrow (a \bmod m = b \bmod m)$	Direct Proof

Modular Arithmetic: A Property

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ be integers with $\boldsymbol{m} > \boldsymbol{0}$.
Then, $\boldsymbol{a} \equiv_{\boldsymbol{m}} \boldsymbol{b}$ if and only if $\boldsymbol{a} \bmod \boldsymbol{m} = \boldsymbol{b} \bmod \boldsymbol{m}$.

Suppose that $a \equiv_m b$. — Assumption

Def of ≡
Def of |
Elim ∃

Apply Division

Algebra

Apply DivUnique
Elim ∃

Therefore, $a \bmod m = b \bmod m$.

Direct Proof

Modular Arithmetic: A Property

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Suppose that $a \equiv_m b$. — Assumption

Then, $m \mid (a - b)$ by the definition of congruence. — Def of ≡
So, $a - b = km$ for some integer k by the definition of divides. Equivalently, $a = b + km$. — Def of | ; Elim ∃

Apply Division

Algebra

Therefore, $a \bmod m = b \bmod m$. — Apply DivUnique ; Elim ∃

Direct Proof

Modular Arithmetic: A Property

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By the Division Theorem, we have $a = (a \operatorname{div} m)\, m + (a \bmod m)$, with $0 \leq (a \bmod m) < m$. — Apply Division

Algebra

Apply DivUnique ; Elim ∃

Therefore, $a \bmod m = b \bmod m$.

Direct Proof

Modular Arithmetic: A Property

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{m}$ be integers with $\boldsymbol{m} > \boldsymbol{0}$.
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By the Division Theorem, we have $a = (a \operatorname{div} m)\, m + (a \bmod m)$, with $0 \leq (a \bmod m) < m$. — Apply Division

Combining these, we have $(a \operatorname{div} m)m + (a \bmod m) = a = b + km$. Solving for b gives $\mathrm{b} = (a \operatorname{div} m)\, m + (a \bmod m) - km = \big((a \operatorname{div} m) - k\big)m + (a \bmod m)$. — Algebra

Apply DivUnique / Elim ∃

Therefore, $a \bmod m = b \bmod m$. — Direct Proof

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By the Division Theorem, we have $a = (a \operatorname{div} m)\, m + (a \bmod m)$, with $0 \leq (a \bmod m) < m$. — Apply Division

Combining these, we have $(a \operatorname{div} m)m + (a \bmod m) = a = b + km$. Solving for b gives $\mathrm{b} = (a \operatorname{div} m)\, m + (a \bmod m) - km = \big((a \operatorname{div} m) - k\big)m + (a \bmod m)$. — Algebra

By the uniqueness property in the Division Theorem, we must have $b \bmod m = a \bmod m$. — Apply DivUnique / Elim ∃

Direct Proof

Recall: Familiar Properties of "="

- If $a = b$ and $b = c$, then $a = c$.
  - i.e., if $a = b = c$, then $a = c$

- If $a = b$ and $c = d$, then $a + c = b + d$.
  - since $c = c$ is true, we can "$+ c$" to both sides

- If $a = b$ and $c = d$, then $ac = bd$.
  - since $c = c$ is true, we can "$\times c$" to both sides

These facts allow us to use algebra to solve problems

The Algebra Rule

$$\text{Algebra}\ \frac{x_1 = y_1 \quad \ldots \quad x_n = y_n}{\therefore x = y}$$

- **Algebra rule applies these properties:**
  - **adding equations**
  - **multiplying equations by a *constant***

Note: no division
(since domain is integers)

- **But also uses knowledge of**
  - **arithmetic with constants**
  - **commutativity of multiplication (e.g.,** $yx = xy$**)**
  - **distributivity (e.g.,** $a(b+c) = ab + bc$**)**

Recall: Familiar Properties of "="

- If $a = b$ and $b = c$, then $a = c$.
  - i.e., if $a = b = c$, then $a = c$
- If $a = b$ and $c = d$, then $a + c = b + d$.
  - since $c = c$ is true, we can "$+ c$" to both sides
- If $a = b$ and $c = d$, then $ac = bd$.
  - since $c = c$ is true, we can "$\times c$" to both sides

Same facts apply to "≤" with non-negative numbers

What about "$\equiv_m$"?

Modular Arithmetic: Basic Property

Let a, b, c and m be integers with $m > 0$.
If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.

1. $(a \equiv_m b \wedge b \equiv_m c) \rightarrow (a \equiv_m c)$??

Modular Arithmetic: Basic Property

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{m}$ be integers with $\boldsymbol{m} > \boldsymbol{0}$.
If $\boldsymbol{a} \equiv_{\boldsymbol{m}} \boldsymbol{b}$ and $\boldsymbol{b} \equiv_{\boldsymbol{m}} \boldsymbol{c}$, then $\boldsymbol{a} \equiv_{\boldsymbol{m}} \boldsymbol{c}$.

Suppose that $a \equiv_m b$ and $b \equiv_m c$.

Assumption

??

Direct Proof

Therefore, $a \equiv_m c$.

Modular Arithmetic: Basic Property

Let a, b, c and m be integers with $m > 0$.
If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.

Suppose that $a \equiv_m b$ and $b \equiv_m c$.

Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (b - c)$.

Elim $\wedge$
Def of $\equiv$

Therefore, $a \equiv_m c$.

??
Direct Proof

Modular Arithmetic: Basic Property

Let a, b, c and m be integers with $m > 0$.
If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.

Suppose that $a \equiv_m b$ and $b \equiv_m c$. — Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (b - c)$. By the definition of divides, we know that $a - b = km$ and $b - c = jm$ for some integers k and j. — Elim $\wedge$ / Def of $\equiv$ / Def of $\mid$ / Elim $\exists$

Therefore, $a \equiv_m c$. — ?? / Direct Proof

Modular Arithmetic: Basic Property

Let a, b, c and m be integers with $m > 0$.
If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.

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By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (b - c)$. By the definition of divides, we know that $a - b = km$ and $b - c = jm$ for some integers k and j. — Elim $\wedge$ / Def of $\equiv$ / Def of $\mid$ / Elim $\exists$

Therefore, we have $m \mid (a - c)$, so $a \equiv_m c$ by the definition of congruence. — ?? / Def of $\equiv$ / Direct Proof

Modular Arithmetic: Basic Property

Let a, b, c and m be integers with $m > 0$.
If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.

Suppose that $a \equiv_m b$ and $b \equiv_m c$. — Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (b - c)$. By the definition of divides, we know that $a - b = km$ and $b - c = jm$ for some integers k and j. — Elim ∧ / Def of ≡ / Def of | / Elim ∃

… ∃q (a – c = mq) … — ??

Therefore, by the definition of divides, we have shown that $m \mid (a - c)$, and then, $a \equiv_m c$ by the definition of congruence. — Def of | / Def of ≡ / Direct Proof

Modular Arithmetic: Basic Property

Let a, b, c and m be integers with $m > 0$.
If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.

Suppose that $a \equiv_m b$ and $b \equiv_m c$. — Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (b - c)$. By the definition of divides, we know that $a - b = km$ and $b - c = jm$ for some integers k and j. — Elim $\wedge$ / Def of $\equiv$ / Def of $\mid$ / Elim $\exists$

Adding these, gives $a - c = km + jm = (k + j)m$. — Algebra

Therefore, by the definition of divides, we have shown that $m \mid (a - c)$, and then, $a \equiv_m c$ by the definition of congruence. — Intro $\exists$ / Def of $\mid$ / Def of $\equiv$ / Direct Proof

Modular Arithmetic: Addition Property

Let m be a positive integer. If $a \equiv_m b$ and $c \equiv_m d$, then $a + c \equiv_m b + d$.

1. $(a \equiv_m b \wedge c \equiv_m d) \rightarrow (a + c \equiv_m b + d)$??

Modular Arithmetic: Addition Property

Let $\boldsymbol{m}$ be a positive integer. If $\boldsymbol{a} \equiv_{\boldsymbol{m}} \boldsymbol{b}$ and $\boldsymbol{c} \equiv_{\boldsymbol{m}} \boldsymbol{d}$, then $\boldsymbol{a} + \boldsymbol{c} \equiv_{\boldsymbol{m}} \boldsymbol{b} + \boldsymbol{d}$.

Suppose that $a \equiv_m b$ and $c \equiv_m d$. Assumption

Therefore, $a + c \equiv_m b + d$. ?? Direct Proof

Modular Arithmetic: Addition Property

Let $\boldsymbol{m}$ be a positive integer. If $\boldsymbol{a \equiv_m b}$ and $\boldsymbol{c \equiv_m d}$, then $\boldsymbol{a + c \equiv_m b + d}$.

Suppose that $a \equiv_m b$ and $c \equiv_m d$. — Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (c - d)$. — Elim $\wedge$, Def of $\equiv$

Therefore, $a + c \equiv_m b + d$. — ??, Direct Proof

Modular Arithmetic: Addition Property

Let $\boldsymbol{m}$ be a positive integer. If $\boldsymbol{a \equiv_m b}$ and $\boldsymbol{c \equiv_m d}$, then $\boldsymbol{a + c \equiv_m b + d}$.

Suppose that $a \equiv_m b$ and $c \equiv_m d$. — Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (c - d)$. By the definition of divides, we know that $a - b = km$ and $c - d = jm$ for some integers k and j. — Elim $\wedge$ / Def of $\equiv$ / Def of $\mid$ / Elim $\exists$

Therefore, $a + c \equiv_m b + d$. — ?? / Direct Proof

Modular Arithmetic: Addition Property

Let m be a positive integer. If $a \equiv_m b$ and $c \equiv_m d$, then $a + c \equiv_m b + d$.

Suppose that $a \equiv_m b$ and $c \equiv_m d$. — Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (c - d)$. By the definition of divides, we know that $a - b = km$ and $c - d = jm$ for some integers k and j. — Elim $\wedge$ / Def of $\equiv$ / Def of $\mid$ / Elim $\exists$

Therefore, we have $m \mid (a + c) - (b + d)$, so we can see that $a + c \equiv_m b + d$ by the definition of congruence. — ?? / Def of $\equiv$ / Direct Proof

Modular Arithmetic: Addition Property

Let m be a positive integer. If $a \equiv_m b$ and $c \equiv_m d$, then $a + c \equiv_m b + d$.

Suppose that $a \equiv_m b$ and $c \equiv_m d$. — Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (c - d)$. By the definition of divides, we know that $a - b = km$ and $c - d = jm$ for some integers k and j. — Elim ∧ / Def of ≡ / Def of | / Elim ∃

… ∃q ((a+c) – (b+d) = mq) … — ??

Therefore, by the definition of divides, we have shown $m \mid (a + c) - (b + d)$, and then, we have $a + c \equiv_m b + d$ by the definition of congruence. — Def of | / Def of ≡ / Direct Proof

Modular Arithmetic: Addition Property

Let $\boldsymbol{m}$ be a positive integer. If $\boldsymbol{a \equiv_m b}$ and $\boldsymbol{c \equiv_m d}$, then $\boldsymbol{a + c \equiv_m b + d}$.

Suppose that $a \equiv_m b$ and $c \equiv_m d$. — Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (c - d)$. By the definition of divides, we know that $a - b = km$ and $c - d = jm$ for some integers k and j. — Elim $\wedge$, Def of $\equiv$, Def of $\mid$, Elim $\exists$

Adding these, gives $(a + c) - (b + d) = (a - b) + (c - d) = km + jm = (k + j)m$. — Algebra

Therefore, by the definition of divides, we have shown $m \mid (a + c) - (b + d)$, and then, we have $a + c \equiv_m b + d$ by the definition of congruence. — Intro $\exists$, Def of $\mid$, Def of $\equiv$

Direct Proof

Modular Arithmetic: Multiplication Property

Let m be a positive integer. If $a \equiv_m b$ and $c \equiv_m d$, then $ac \equiv_m bd$.

Suppose that $a \equiv_m b$ and $c \equiv_m d$. — Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (c - d)$. By the definition of divides, we know that $a - b = jm$ and $c - d = km$ for some integers j and k. — Def of ≡ / Def of | / Elim ∃

Algebra

… ∃q (ac – bd = mq) … — Intro ∃

Therefore, $m \mid ac - bd$ by the definition of divides, so $ac \equiv_m bd$ by the definition of congruence. — Def of | / Def of ≡ / Direct Proof

Modular Arithmetic: Multiplication Property

Let m be a positive integer. If $a \equiv_m b$ and $c \equiv_m d$, then $ac \equiv_m bd$.

Suppose that $a \equiv_m b$ and $c \equiv_m d$. — Assumption

By the definition of congruence, we know that $m \mid (a - b)$ and $m \mid (c - d)$. By the definition of divides, we know that $a - b = jm$ and $c - d = km$ for some integers j and k. — Def of ≡, Def of |, Elim ∃

Equivalently, $a = b + jm$ and $c = d + km$. — Algebra

Multiplying these gives $ac = (b + jm)(d + km) = bd + bkm + djm + jkm = bd + (bk + dj + jk)m$, so $ac - bd = (bk + dj + jkm)m$. — Intro ∃, Def of |

Therefore, $m \mid ac - bd$ by the definition of divides, so $ac \equiv_m bd$ by the definition of congruence. — Def of ≡, Direct Proof

Modular Arithmetic: Properties

If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.

If $a \equiv_m b$ and $c \equiv_m d$, then $a + c \equiv_m b + d$.

Corollary: If $a \equiv_m b$, then $a + c \equiv_m b + c$.

If $a \equiv_m b$ and $c \equiv_m d$, then $ac \equiv_m bd$.

Corollary: If $a \equiv_m b$, then $ac \equiv_m bc$.

These properties are sufficient to allow us to do algebra with congruences

Another Property of "=" Used in Algebra

Can "plug in" (a.k.a. substitute)
the known value of a variable

Example: **given** $2y + 3x = 25$ **and** $x = 7y$,
follows that $2y + 21y = 25$.

We can call this "Algebra",
but it's a more primitive rule.

(See the reference sheet.)

Another Property of "=" Used in Algebra

Can "plug in" (a.k.a. substitute)
the known value of a variable

Example: **given** $2y + 3x = 25$ **and** $x = 7y$,
follows that $2y + 21y = 25$.

This is also true of *congruences*!

(But we don't have the tools to prove it yet....)

Example: **given** $2y + 3x \equiv_{\mathrm{m}} 25$ **and** $x \equiv_{\mathrm{m}} 7y$,
follows that $2y + 21y \equiv_{\mathrm{m}} 25$.

GCD

First GCD Fact

Domain of Discourse
Non-negative Integers

For every positive integer *a*, if *a* > 0, then gcd(*a, 0*) = *a.*

$$\forall a\ ((\mathrm{a} > 0) \rightarrow (a = \gcd(a, 0))$$

We proved this in Topic 2….

Let *a* be arbitrary. Suppose *a* > 0. Show that a satisfies the definition of gcd(*a*, 0).

Useful GCD Fact

Let a and b be positive integers.
We have gcd(a, b) = gcd(b, a mod b)

Proof Idea:
We will show that every number dividing a and b also divides b and $a \bmod b$.
I.e., $d|a$ and $d|b$ iff $d|b$ and $d|(a \bmod b)$.

Hence, their set of common divisors are the same,
which means that their greatest common divisor is the same.

Useful GCD Fact

Let *a* and *b* be positive integers.
We have gcd(*a*, *b*) = gcd(*b*, *a* mod *b*)

Proof (of $d|a$ and $d|b$ iff $d|b$ and $d|(a \bmod b)$)**:**
By the Division Theorem, $a = qb + (a \bmod b)$ for some integer $q = a \operatorname{div} b$.

Suppose $d \mid b$ and $d \mid (a \bmod b)$.
Then $b = md$ and $(a \bmod b) = nd$ for some integers m and n.
Therefore $a = qb + (a \bmod b) = qmd + nd = (qm + n)d$.
So $d \mid a$ by the definition of divides.

Suppose $d \mid a$ and $d \mid b$.
Then $a = kd$ and $b = jd$ for some integers k and j.
Therefore $(a \bmod b) = a - qb = kd - qjd = (k - qj)d$.
So, $d \mid (a \bmod b)$ by the definition of divides.

Since they have the same common divisors, $\gcd(a, b) = \gcd(b, a \bmod b)$. ■

Euclid's Algorithm

gcd(a, b) = gcd(b, a mod b) gcd(a, 0) = a

```
int gcd(int a, int b) {  /* Assumes: a >= b >= 0 */
    if (b == 0) {
        return a;
    } else {
        return gcd(b, a % b);
    }
}
```

Note: gcd(b, a) = gcd(a, b)

Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

gcd(660,126) =

Euclid's Algorithm

Repeatedly use $\gcd(a, b) = \gcd(b, a \bmod b)$ to reduce numbers until you get $\gcd(g, 0) = g$.

gcd(660,126) = gcd(126, 660 mod 126) = gcd(126, 30)
= gcd(30, 126 mod 30) = gcd(30, 6)
= gcd(6, 30 mod 6) = gcd(6, 0)
= 6

Bézout's theorem

If *a* and *b* are positive integers, then there exist integers ***s*** and ***t*** such that

$$\gcd(a,b) = sa + tb.$$

$$(a > 0 \wedge b > 0) \rightarrow \exists s\, \exists t\, (\gcd(a,b) = sa + tb)$$

$$\forall a\, \forall b\, ((a > 0 \wedge b > 0) \rightarrow \exists s\, \exists t\, (\gcd(a,b) = sa + tb))$$

Extended Euclidean algorithm

- **Can use Euclid's Algorithm to find s, t such that**

$$\gcd(a, b) = sa + tb$$

Extended Euclidean algorithm

- **Can use Euclid's Algorithm to find s, t such that**

$$\gcd(a, b) = sa + tb$$

Step 1 (Compute GCD & Keep Tableau Information):

a b b a mod b = r b r

$$\gcd(35, 27) = \gcd(27, 35 \bmod 27) = \gcd(27, 8)$$

a = q * b + r

$$35 = 1 * 27 + 8$$

Extended Euclidean algorithm

- **Can use Euclid's Algorithm to find s, t such that**

$$\gcd(a, b) = sa + tb$$

Step 1 (Compute GCD & Keep Tableau Information):

a b	b a mod b = r	b r	a = q * b + r
$\gcd(35, 27)$	$= \gcd(27, 35 \bmod 27)$	$= \gcd(27, 8)$	$35 = 1 * 27 + 8$
	$= \gcd(8, 27 \bmod 8)$	$= \gcd(8, 3)$	$27 = 3 * 8 + 3$
	$= \gcd(3, 8 \bmod 3)$	$= \gcd(3, 2)$	$8 = 2 * 3 + 2$
	$= \gcd(2, 3 \bmod 2)$	$= \gcd(2, 1)$	$3 = 1 * 2 + 1$
	$= \gcd(1, 2 \bmod 1)$	$= \gcd(1, 0)$	

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 2 (Solve the equations for r):

a = q * b + r
$35 = 1 * 27 + 8$
$27 = 3 * 8 + 3$
$8 = 2 * 3 + 2$
$3 = 1 * 2 + 1$

r = a – q * b

$8 = 35 - 1 * 27$

Extended Euclidean algorithm

- **Can use Euclid's Algorithm to find s, t such that**

$$\gcd(a, b) = sa + tb$$

Step 2 (Solve the equations for r):

a = q * b + r	r = a – q * b
$35 = 1 * 27 + 8$	$8 = 35 - 1 * 27$
$27 = 3 * 8 + 3$	$3 = 27 - 3 * 8$
$8 = 2 * 3 + 2$	$2 = 8 - 2 * 3$
$3 = 1 * 2 + 1$	$1 = 3 - 1 * 2$

Extended Euclidean algorithm

- **Can use Euclid's Algorithm to find s, t such that**

$$\gcd(a, b) = sa + tb$$

Step 3 (Backward Substitute Equations):

$$8 = 35 - 1 * 27$$

$$3 = 27 - 3 * 8$$

$$2 = 8 - 2 * 3$$

$$1 = 3 - 1 * 2$$

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 3 (Backward Substitute Equations):

$8 = 35 - 1 * 27$

$3 = 27 - 3 * 8$

$2 = 8 - 2 * 3$

$1 = 3 - 1 * 2$

Plug in the def of 2

$1 = 3 - 1 * (8 - 2 * 3)$

$= 3 - 8 + 2 * 3$

$= (-1) * 8 + 3 * 3$

Re-arrange into 3's and 8's

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 3 (Backward Substitute Equations):

$8 = 35 - 1 * 27$

$3 = 27 - 3 * 8$

$2 = 8 - 2 * 3$

$1 = 3 - 1 * 2$

Plug in the def of 2

$1 = 3 - 1 * (8 - 2 * 3)$

$= 3 - 8 + 2 * 3$

Re-arrange into 3's and 8's

$= (-1) * 8 + 3 * 3$

Plug in the def of 3

$= (-1) * 8 + 3 * (27 - 3 * 8)$

$= (-1) * 8 + 3 * 27 + (-9) * 8$

$= 3 * 27 + (-10) * 8$

Re-arrange into 8's and 27's

Extended Euclidean algorithm

- Can use Euclid's Algorithm to find s, t such that

$$\gcd(a, b) = sa + tb$$

Step 3 (Backward Substitute Equations):

$8 = 35 - 1 * 27$

$3 = 27 - 3 * 8$

$2 = 8 - 2 * 3$

$1 = 3 - 1 * 2$

Plug in the def of 2

$1 = 3 - 1 * (8 - 2 * 3)$

$= 3 - 8 + 2 * 3$

Re-arrange into 3's and 8's

$= (-1) * 8 + 3 * 3$

Plug in the def of 3

$= (-1) * 8 + 3 * (27 - 3 * 8)$

$= (-1) * 8 + 3 * 27 + (-9) * 8$

$= 3 * 27 + (-10) * 8$

Re-arrange into 8's and 27's

$= 3 * 27 + (-10) * (35 - 1 * 27)$

$= 3 * 27 + (-10) * 35 + 10 * 27$

Re-arrange into 27's and 35's

$= 13 * 27 + (-10) * 35$

Multiplicative inverse $\mod m$

Let $0 \leq a, b < m$. Then, b is the *multiplicative inverse of a (modulo m)* iff $ab \equiv_m 1$.

x	0	1	2	3	4	5	6
0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6
2	0	2	4	6	1	3	5
3	0	3	6	2	5	1	4
4	0	4	1	5	2	6	3
5	0	5	3	1	6	4	2
6	0	6	5	4	3	2	1

mod 7

x	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	0	2	4	6	8
3	0	3	6	9	2	5	8	1	4	7
4	0	4	8	2	6	0	4	8	2	6
5	0	5	0	5	0	5	0	5	0	5
6	0	6	2	8	4	0	6	2	8	4
7	0	7	4	1	8	5	2	9	6	3
8	0	8	6	4	2	0	8	6	4	2
9	0	9	8	7	6	5	4	3	2	1

mod 10

Multiplicative inverse $\mod m$

Suppose $\gcd(a, m) = 1$

By Bézout's Theorem, there exist integers s and t such that $sa + tm = 1$.

s is the multiplicative inverse of a (modulo m):

$1 \equiv_m sa$ since $m \mid 1 - sa$ (since $1 - sa = tm$)

So... we can compute multiplicative inverses with the extended Euclidean algorithm

These inverses let us solve modular equations...

Recall: Multiplicative inverse $\mathrm{mod}\ m$

Suppose $\gcd(a, m) = 1$

By Bézout's Theorem, there exist integers s and t such that $sa + tm = 1$.

s is the multiplicative inverse of a (modulo m):

$1 \equiv_m sa$ since $m \mid 1 - sa$ (since $1 - sa = tm$)

We can compute multiplicative inverses with the Extended Euclidean algorithm

These inverses let us solve modular equations...

Example: Solve a Modular Equation

Solve: $7x \equiv_{26} 3$

Suppose we can show that 15 is the multiplicative inverse of 7 modulo 26, i.e., that $15 \cdot 7 \equiv_{26} 1$

Then, we can multiply on both sides by 15 to see that

$$x \equiv_{26} 1x \equiv_{26} 15 \cdot 7x \equiv_{26} 15 \cdot 3 \equiv_{26} 45 \equiv_{26} 19$$

So, if we are given that $7x \equiv_{26} 3$, then we have shown that $x \equiv_{26} 19$.

Example: Solve a Modular Equation

Solve: $7x \equiv_{26} 3$ Find multiplicative inverse of 7 modulo 26

Example: Solve a Modular Equation

Solve: $7x \equiv_{26} 3$ Find multiplicative inverse of 7 modulo 26

$$\gcd(26, 7) = \gcd(7, 5) = \gcd(5, 2) = \gcd(2, 1) = 1$$

$$26 = 3 * 7 + 5$$
$$7 = 1 * 5 + 2$$
$$5 = 2 * 2 + 1$$

Example: Solve a Modular Equation

Solve: $7x \equiv_{26} 3$ Find multiplicative inverse of 7 modulo 26

$$\gcd(26, 7) = \gcd(7, 5) = \gcd(5, 2) = \gcd(2, 1) = 1$$

$$26 = 3 * 7 + 5 \qquad 5 = 26 - 3 * 7$$
$$7 = 1 * 5 + 2 \qquad 2 = 7 - 1 * 5$$
$$5 = 2 * 2 + 1 \qquad 1 = 5 - 2 * 2$$

Example: Solve a Modular Equation

Solve: $7x \equiv_{26} 3$ Find multiplicative inverse of 7 modulo 26

$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = 1$$

$$26 = 3 * 7 + 5 \qquad 5 = 26 - 3 * 7$$
$$7 = 1 * 5 + 2 \qquad 2 = 7 - 1 * 5$$
$$5 = 2 * 2 + 1 \qquad 1 = 5 - 2 * 2$$

$$\begin{aligned} 1 &= 5 - 2 * (7 - 1 * 5) \\ &= (-2) * 7 + 3 * 5 \\ &= (-2) * 7 + 3 * (26 - 3 * 7) \\ &= (-11) * 7 + 3 * 26 \end{aligned}$$

Example: Solve a Modular Equation

Solve: $7x \equiv_{26} 3$ Find multiplicative inverse of 7 modulo 26

$$\gcd(26,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,1) = 1$$

$$\begin{aligned} 26 &= 3*7 + 5 & 5 &= 26 - 3*7 \\ 7 &= 1*5 + 2 & 2 &= 7 - 1*5 \\ 5 &= 2*2 + 1 & 1 &= 5 - 2*2 \end{aligned}$$

$$\begin{aligned} 1 &= 5 - 2*(7 - 1*5) \\ &= (-2)*7 + 3*5 \\ &= (-2)*7 + 3*(26 - 3*7) \\ &= (-11)*7 + 3*26 \end{aligned}$$

Now $(-11) \bmod 26 = 15.$

"the" multiplicative inverse

(-11 is also "a" multiplicative inverse)

Example: Solve a Modular Equation

Solve: $7x \equiv_{26} 3$

We saw before that... if we are given that $7x \equiv_{26} 3$, then we have shown that $x \equiv_{26} 19$.

$$7x \equiv_{26} 3 \quad \Rightarrow \quad x \equiv_{26} 19$$

But these steps are *all* reversible...

Recall: Properties of Modular Arithmetic

If $a \equiv_m b$ and $b \equiv_m c$, then $a \equiv_m c$.

If $a \equiv_m b$, then $a + c \equiv_m b + c$.

If $a \equiv_m b$, then $ac \equiv_m bc$.

These are the only properties we used...

Multiplicative Inverses and Algebra

Adding to both sides easily reversible:

$$x \equiv_m y$$

$-c$ ↓ $+c$

$$x + c \equiv_m y + c$$

The same is not true of multiplication...

unless we have a multiplicative inverse $cd \equiv_m 1$

$$x \equiv_m y$$

$\times d$ ↓ $\times c$

$$cx \equiv_m cy$$

Example: Solve a Modular Equation

$$7x \equiv_{26} 3 \quad \Rightarrow \quad 15 \cdot 7x \equiv_{26} 15 \cdot 3$$

multiply both sides by 15

$$\Rightarrow \quad x \equiv_{26} 19$$

since $15 \cdot 7 \equiv_{26} 1$ and $15 \cdot 3 \equiv_{26} 19$

$$x \equiv_{26} 19 \quad \Rightarrow \quad 7x \equiv_{26} 7 \cdot 19$$

multiply both sides by 7

$$\Rightarrow \quad 7x \equiv_{26} 3$$

since $7 \cdot 19 \equiv_{26} 3$

Example: Solve a Modular Equation

Solve: $7x \equiv_{26} 3$

We saw before that... if we are given that $7x \equiv_{26} 3$, then we have shown that $x \equiv_{26} 19$.

$$7x \equiv_{26} 3 \quad \Rightarrow \quad x \equiv_{26} 19$$

But all of these steps are reversible...

$$x \equiv_{26} 19 \quad \Rightarrow \quad 7x \equiv_{26} 7 \cdot 19$$

So $7x \equiv_{26} 3$ iff $x \equiv_{26} 19$

Hence, the solutions are all numbers of the form $19 + 26k$ for some integer

Solving Modular Equations in "Standard Form"

Solve: $7x \equiv_{26} 3$ (of the form $Ax \equiv_m B$ for some A and B)

Step 1. Find multiplicative inverse of 7 modulo 26

$$1 \quad = \ \ldots \ = \quad (-11) * 7 \quad + \ 3 * 26$$

Since $(-11) \bmod 26 \ = 15$, the inverse of 7 is 15.

Step 2. Multiply both sides and simplify

Multiplying by 15, we get $x \equiv_{26} 15 \cdot 7x \equiv_{26} 15 \cdot 3 \equiv_{26} 19.$

Step 3. State the full set of solutions

So, the solutions are $19 + 26k$ for any integer k

(must be of the form $a + mk$ with $0 \leq a < m$)

Beware the "Backward Proof"

- **Many classes teach doing proofs backward:**

$15x + 5 = 5(-3x - 1)$	start with we want to *prove*
$(15x + 5)^2 = [5(-3x - 1)]^2$	do some manipulations
… …	
$0 = 0$	end with an obvious truth

- **This is proof of nothing: "$P \rightarrow \mathrm{T}$"is a tautology**
  - **it is true regardless of whether P is true**

- **See notes on the Resources page**

Example: Not in "Standard Form"

Solve: $7(x-3) \equiv_{26} 8 + 2x$

What about equation not in standard form?

Example: Not in "Standard Form"

Solve: $7(x-3) \equiv_{26} 8 + 2x$

Rewrite it in standard form:

$$7x - 21 \equiv_{26} \; 7(x-3) \equiv_{26} 8 + 2\text{x}$$

move 2x to the other side

$$5x - 21 \equiv_{26} 8$$

move -21 to the other side

$$5x \equiv_{26} 29 \equiv_{26} 3$$

These steps are all reversible, so the solutions are the same.

Induction

Mathematical Induction

Method for proving claims about non-negative integers

- **A new logical inference rule!**
  - It only applies over the non-negative numbers
  - The idea is to **use** the special structure of these numbers to prove things more easily

Prove $\forall k\ ((a \equiv_m b) \rightarrow (a^k \equiv_m b^k))$

Let k be an arbitrary *non-negative* integer.

Suppose that $a \equiv_m b$.

We know $((a \equiv_m b) \wedge (a \equiv_m b)) \rightarrow (a^2 \equiv_m b^2)$ by multiplying congruences. So, applying this repeatedly, we have:

$$((a \equiv_m b) \wedge (a \equiv_m b)) \rightarrow (a^2 \equiv_m b^2)$$
$$\big((a^2 \equiv_m b^2) \wedge (a \equiv_m b)\big) \rightarrow (a^3 \equiv_m b^3)$$

…

$$\big((a^{k-1} \equiv_m b^{k-1}) \wedge (a \equiv_m b)\big) \rightarrow (a^k \equiv_m b^k)$$

The "…"s is a problem! We don't have a proof rule that allows us to say "do this over and over".

But there is such a rule for non-negative numbers!

Domain: Non-Negative Numbers

Induction

$$\frac{P(0) \quad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n \;\; P(n)}$$

Induction Is A Rule of Inference

Domain: Non-Negative Numbers

Induction

$$\frac{P(0) \qquad \forall k\ (P(k) \longrightarrow P(k+1))}{\therefore\ \forall n\ \ P(n)}$$

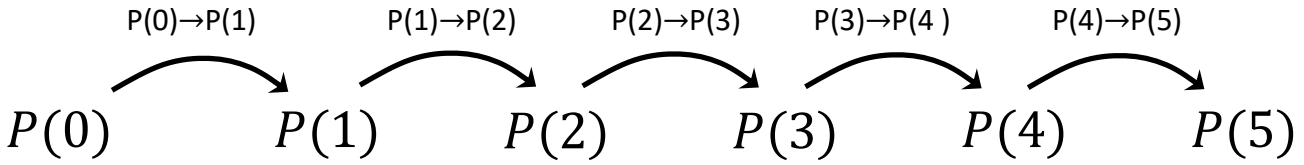
How do the givens prove P(3)?

Induction Is A Rule of Inference

Domain: Non-Negative Numbers

$$\frac{P(0) \qquad \forall k\,(P(k) \rightarrow P(k+1))}{\therefore \forall n\ P(n)}\ \text{Induction}$$

How do the givens prove P(5)?



First, we have P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Using The Induction Rule In A Formal Proof

Induction

$$\frac{P(0) \quad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n \;\; P(n)}$$

Using The Induction Rule In A Formal Proof

Induction

$$\frac{P(0) \quad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n \;\; P(n)}$$

1. P(0)

2. $\forall$k (P(k) $\rightarrow$ P(k+1)) ??
3. $\forall$n P(n) Induction: 1, 2

Using The Induction Rule In A Formal Proof

Induction

$$\frac{P(0) \quad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n \;\; P(n)}$$

1. P(0)

   Let k be an arbitrary integer ≥ 0

   2.1 P(k) $\rightarrow$ P(k+1) ??

2. $\forall$k (P(k) $\rightarrow$ P(k+1)) Intro $\forall$
3. $\forall$n P(n) Induction: 1, 2

Using The Induction Rule In A Formal Proof

Induction

$$\frac{P(0) \quad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n \;\; P(n)}$$

1. P(0)

   Let k be an arbitrary integer ≥ 0

   2.1.1. P(k) — Assumption

   2.1.2. …

   2.1.3. P(k+1)

   2.1 P(k) → P(k+1) — Direct Proof

2. ∀k (P(k) → P(k+1)) — Intro ∀
3. ∀n P(n) — Induction: 1, 2

Translating to an English Proof

Induction

$$\frac{P(0) \quad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n \;\; P(n)}$$

1. Prove P(0) — **Base Case**

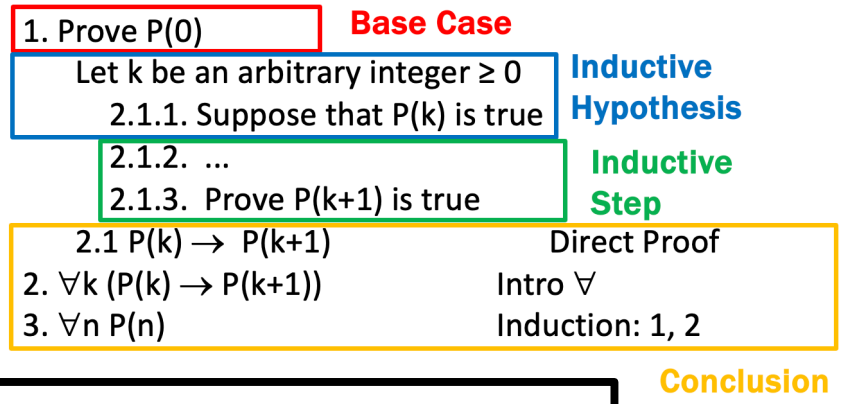
Let k be an arbitrary integer ≥ 0
2.1.1. Suppose that P(k) is true — **Inductive Hypothesis**

2.1.2. …
2.1.3. Prove P(k+1) is true — **Inductive Step**

2.1 P(k) → P(k+1) Direct Proof
2. ∀k (P(k) → P(k+1)) Intro ∀
3. ∀n P(n) Induction: 1, 2

Conclusion

Translating to an English Proof



Induction English Proof Template

[…Define P(n)…]

We will show that $P(n)$ is true for every $n \geq 0$ by induction.

Base Case: *[…proof of $P(0)$ here…]*

Induction Hypothesis:

Suppose that $P(k)$ is true for an arbitrary $k \geq 0$.

Induction Step:

[…proof of $P(k+1)$ here…]

The proof of $P(k+1)$ ***must*** *invoke the IH somewhere.*

So, the claim is true by induction.

Inductive Proofs In 5 Easy Steps

Basic induction template

Proof:

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Suppose $P(k)$ is true for an arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true.

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

What is $1 + 2 + 4 + \ldots + 2^n$?

- $1 = 1$
- $1 + 2 = 3$
- $1 + 2 + 4 = 7$
- $1 + 2 + 4 + 8 = 15$
- $1 + 2 + 4 + 8 + 16 = 31$

It sure looks like this sum is $2^{n+1} - 1$

How can we prove it?

We could prove it for $n = 1, n = 2, n = 3, \ldots$ but that would literally take forever.

Good that we have induction!

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

---

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all non-negative numbers by induction.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

---

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all non-negative numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all non-negative numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$**, i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all non-negative numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. **Induction Step:**

   Goal: Show P(k+1), **i.e. show** $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. Let P(n) be "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". We will show P(n) is true for all non-negative numbers by induction.
2. Base Case (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so P(0) is true.
3. Induction Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 0$, i.e., that $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. Induction Step:

   $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$ by IH

   Adding 2^{k+1} to both sides, we get:

   $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+1} + 2^{k+1} - 1$

   Note that $2^{k+1} + 2^{k+1} = 2(2^{k+1}) = 2^{k+2}$.

   So, we have $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$, which is exactly P(k+1).

Calculation Block

We can do the same with equality:

$2^0 + 2^1 + \ldots + 2^k + 2^{k+1}$

$= (2^0+2^1+ \ldots + 2^k) + 2^{k+1}$

$= (2^{k+1} - 1) + 2^{k+1}$ since $2^0+2^1+ \ldots + 2^k = 2^{k+1} - 1$

$= 2(2^{k+1}) - 1$ (better: "by the IH")

$= 2^{k+2} - 1$

Explanations appear on in right column

- "since" means we substituted LHS for RHS
- ordinary algebra (on integers) does not need explanation
- "def of" will be used to apply the definition of a function

  e.g., replacing f(x) by y when we have f defined as f(x) := y

Calculation Block

We can do the same with equality:

$$
\begin{aligned}
& 2^0 + 2^1 + \ldots + 2^k + 2^{k+1} \\
&\quad = (2^0 + 2^1 + \ldots + 2^k) + 2^{k+1} \\
&\quad = (2^{k+1} - 1) + 2^{k+1} && \text{by the IH} \\
&\quad = 2(2^{k+1}) - 1 \\
&\quad = 2^{k+2} - 1
\end{aligned}
$$

Entire block shows $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

- this is the transitivity property of "="

Can also do calculation with "<" and "≤"

- don't mix directions: ">" and "<" in one block is ><

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. **Let** P(n) **be** "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". **We will show** P(n) **is true for all non-negative numbers by induction.**
2. **Base Case** (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ **so** P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 0$, **i.e., that** $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. **Induction Step:**

   We can calculate

   Show $2^0 + 2^1 + \ldots + 2^k + 2^{k+1} = 2^{k+2} - 1$

$$
\begin{aligned}
2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0 + 2^1 + \ldots + 2^k) + 2^{k+1} \\
&= (2^{k+1} - 1) + 2^{k+1} && \text{by the IH} \\
&= 2(2^{k+1}) - 1 \\
&= 2^{k+2} - 1,
\end{aligned}
$$

   which is exactly P(k+1).

The entire inductive step is one calculation!

We will rely heavily on calculation going forward…

Prove $1 + 2 + 4 + \ldots + 2^n = 2^{n+1} - 1$

1. Let P(n) be "$2^0 + 2^1 + \ldots + 2^n = 2^{n+1} - 1$". We will show P(n) is true for all non-negative numbers by induction.
2. Base Case (n=0): $2^0 = 1 = 2 - 1 = 2^{0+1} - 1$ so P(0) is true.
3. Induction Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 0$, i.e., that $2^0 + 2^1 + \ldots + 2^k = 2^{k+1} - 1$.
4. Induction Step:

   We can calculate

$$\begin{aligned} 2^0 + 2^1 + \ldots + 2^k + 2^{k+1} &= (2^0 + 2^1 + \ldots + 2^k) + 2^{k+1} \\ &= (2^{k+1} - 1) + 2^{k+1} && \text{by the IH} \\ &= 2(2^{k+1}) - 1 \\ &= 2^{k+2} - 1, \end{aligned}$$

   which is exactly P(k+1).
5. Thus P(n) is true for all $n \geq 0$, by induction.

Recall: Inductive Proofs In 5 Easy Steps

Basic induction template

Proof:

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for every $n \geq 0$ by Induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis:

   Suppose $P(k)$ is true for an arbitrary integer $k \geq 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true.

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: Result follows by induction"

Prove $1 + 2 + 3 + \ldots + n = n(n+1)/2$

Prove that $\sum_{i=0}^{n} i = n(n+1)/2$

Summation Notation

$$\sum_{i=0}^{n} i = 0 + 1 + 2 + 3 + \ldots + n$$

Prove $\sum_{i=0}^{n} i = n(n+1)/2$

1. **Let** P(n) **be** "$\sum_{i=0}^{n} i = n(n+1)/2$". **We will show** P(n) **is true for all non-negative numbers by induction.**

Prove $\sum_{i=0}^{n} i = n(n+1)/2$

1. **Let** P(n) **be** "$\sum_{i=0}^{n} i = n(n+1)/2$". **We will show** P(n) **is true for all non-negative numbers by induction.**
2. **Base Case** (n=0): $\sum_{i=0}^{0} i = 0 = 0(0+1)/2$, so P(0) **is true.**

Prove $\sum_{i=0}^{n} i = n(n+1)/2$

1. **Let** P(n) **be** "$\sum_{i=0}^{n} i = n(n+1)/2$". **We will show** P(n) **is true for all non-negative numbers by induction.**
2. **Base Case** (n=0): $\sum_{i=0}^{0} i = 0 = 0(0+1)/2$, so P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**. I.e., suppose** $\sum_{i=0}^{k} i = k(k+1)/2$

"some" or "an"
not <u>any</u>!

Prove $\sum_{i=0}^{n} i = n(n+1)/2$

1. **Let** P(n) **be** "$\sum_{i=0}^{n} i = n(n+1)/2$". **We will show** P(n) **is true for all non-negative numbers by induction.**
2. **Base Case** (n=0): $\sum_{i=0}^{0} i = 0 = 0(0+1)/2$, so P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**. I.e., suppose** $\sum_{i=0}^{k} i = k(k+1)/2$
4. **Induction Step:**

Goal: Show P(k+1)**, i.e.,** $\sum_{i=0}^{k+1} i = (k+1)(k+2)/2$

Prove $\sum_{i=0}^{n} i = n(n+1)/2$

1. Let P(n) be "$\sum_{i=0}^{n} i = n(n+1)/2$". **We will show** P(n) **is true for all non-negative numbers by induction.**
2. **Base Case** (n=0): $\sum_{i=0}^{0} i = 0 = 0(0+1)/2$, so P(0) **is true.**
3. **Induction Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 0**. I.e., suppose** $\sum_{i=0}^{k} i = k(k+1)/2$
4. **Induction Step: We can see that**

$$\begin{aligned}\sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^{k} i\right) + (k+1) \\ &= \mathrm{k}(\mathrm{k}+1)/2 + (k+1) \qquad \text{by the IH} \\ &= (\mathrm{k}+1)(\mathrm{k}/2+1) \\ &= (\mathrm{k}+1)(\mathrm{k}+2)/2\end{aligned}$$

which is exactly P(k+1)**.**

Prove $\sum_{i=0}^{n} i = n(n+1)/2$

1. Let P(n) be "$\sum_{i=0}^{n} i = n(n+1)/2$". We will show P(n) is true for all non-negative numbers by induction.
2. Base Case (n=0): $\sum_{i=0}^{0} i = 0 = 0(0+1)/2$, so P(0) is true.
3. Induction Hypothesis: Suppose that P(k) is true for some arbitrary integer k ≥ 0. I.e., suppose $\sum_{i=0}^{k} i = k(k+1)/2$
4. Induction Step: We can see that

$$\begin{aligned} \sum_{i=0}^{k+1} i &= \left(\sum_{i=0}^{k} i\right) + (k+1) \\ &= \mathrm{k}(\mathrm{k}+1)/2 + (k+1) \qquad \text{by the IH} \\ &= (\mathrm{k}+1)(\mathrm{k}/2+1) \\ &= (\mathrm{k}+1)(\mathrm{k}+2)/2 \end{aligned}$$

   which is exactly P(k+1).

5. Thus P(n) is true for all n ≥ 0, by induction.

Induction: Changing the starting point

- What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b?
- Define predicate $Q(k) = P(k + b)$ for all k.
  - Then $\forall n\ Q(n) \equiv \forall n \geq b\ \ P(n)$
- Ordinary induction for Q:
  - Prove $Q(0) \equiv P(b)$
  - Prove

$$\forall k \left(Q(k) \longrightarrow Q(k+1)\right) \equiv \forall k \geq b \left(P(k) \longrightarrow P(k+1)\right)$$

Inductive Proofs In 5 Easy Steps

Template for induction from a different base case

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq \boldsymbol{b}$ by induction."
2. "Base Case:" Prove $P(\boldsymbol{b})$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $k \geq \boldsymbol{b}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** n ≥ 2 **by induction.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** n ≥ 2 **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let P(n) be "$3^n \geq n^2+3$". We will show P(n) is true for all integers n ≥ 2 by induction.
2. Base Case (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ so P(2) is true.
3. Inductive Hypothesis: Suppose that P(k) is true for some arbitrary integer k ≥ 2. I.e., suppose $3^k \geq k^2+3$.

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. **Let** P(n) **be** "$3^n \geq n^2+3$". **We will show** P(n) **is true for all integers** n ≥ 2 **by induction.**
2. **Base Case** (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ **so** P(2) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** k ≥ 2**. I.e., suppose** $3^k \geq k^2+3$**.**
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $3^{k+1} \geq (k+1)^2+3=k^2+2k+4$

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let P(n) be "$3^n \geq n^2+3$". We will show P(n) is true for all integers $n \geq 2$ by induction.
2. Base Case (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ so P(2) is true.
3. Inductive Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 2$. I.e., suppose $3^k \geq k^2+3$.
4. Inductive Step: We can see that

$$
\begin{aligned}
3^{k+1} &= 3(3^k) \\
&\geq 3(k^2+3) && \text{by the IH} \\
&= k^2+2k^2+9 \\
&\geq k^2+2k+9 && \text{since } k^2 \geq k \\
&\geq k^2+2k+4 && \text{since } 9 \geq 4 \\
&= (k+1)^2+3
\end{aligned}
$$

Therefore P(k+1) is true.

Prove $3^n \geq n^2 + 3$ for all $n \geq 2$

1. Let P(n) be "$3^n \geq n^2+3$". We will show P(n) is true for all integers $n \geq 2$ by induction.
2. Base Case (n=2): $3^2 = 9 \geq 7 = 4+3 = 2^2+3$ so P(2) is true.
3. Inductive Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 2$. I.e., suppose $3^k \geq k^2+3$.
4. Inductive Step: We can see that

$$\begin{aligned} 3^{k+1} = 3(3^k) &\geq 3(k^2+3) && \text{by the IH} \\ &= k^2+2k^2+9 \\ &\geq k^2+2k+9 && \text{since } k^2 \geq k \\ &\geq k^2+2k+4 && \text{since } 9 \geq 4 \\ &= (k+1)^2+3 \end{aligned}$$

   Therefore P(k+1) is true.

5. Thus P(n) is true for all integers $n \geq 2$, by induction.

Induction: Adding Base Cases

- **What if we want to prove that $P(n)$ is true for all integers $n \geq b$ for some integer b but the inductive step only works for $n \geq c$?**

- **Add proofs of $P(b)$, $P(b+1)$, ..., $P(c-1)$**
  - **will call these extra "base cases"**

- **Formally, we are using the fact that**

$$P(b) \wedge \cdots \wedge P(c-1) \wedge \forall n\,((c \leq n) \rightarrow P(n))$$
$$\equiv \forall n\,((b \leq n) \rightarrow P(n))$$

Inductive Proofs In 5 Easy Steps

Template for induction with multiple base cases

1. "Let $P(n)$ be… . We will show that $P(n)$ is true for all integers $n \geq \boldsymbol{b}$ by induction."
2. "Base Case:" Prove $P(\boldsymbol{b})$, …, $P(\boldsymbol{c})$
3. "Inductive Hypothesis:

   Assume $P(k)$ is true for an arbitrary integer $k \geq \boldsymbol{c}$"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

   Use the goal to figure out what you need.

   Make sure you are using I.H. and point out where you are using it. (Don't assume $P(k+1)$!!)
5. "Conclusion: $P(n)$ is true for all integers $n \geq \boldsymbol{b}$"

Recursive Definitions of Functions

Familiar Recursive Definitions

Suppose that $h : \mathbb{N} \to \mathbb{R}$.

Then we have familiar summation notation:

$\sum_{i=0}^{0} h(i) \coloneqq h(0)$

$\sum_{i=0}^{n+1} h(i) \coloneqq \left(\sum_{i=0}^{n} h(i)\right) + h(n+1)$ **for** $n \geq 0$

There is also product notation:

$\prod_{i=0}^{0} h(i) \coloneqq h(0)$

$\prod_{i=0}^{n+1} h(i) \coloneqq \left(\prod_{i=0}^{n} h(i)\right) \cdot h(n+1)$ **for** $n \geq 0$

Recursive definitions of functions

- $0! \coloneqq 1;\ \ (n+1)! \coloneqq (n+1) \cdot n!$ **for all** $n \geq 0$**.**
- $F(0) \coloneqq 0;\ \ F(n+1) \coloneqq F(n) + 1$ **for all** $n \geq 0$**.**
- $G(0) \coloneqq 1;\ \ G(n+1) \coloneqq 2 \cdot G(n)$ **for all** $n \geq 0$**.**
- $H(0) \coloneqq 1;\ \ H(n+1) \coloneqq 2^{H(n)}$ **for all** $n \geq 0$**.**

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "n! ≤ n^n". **We will show that** P(n) **is true for all integers** n ≥ 1 **by induction.**

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "n! ≤ n^n". **We will show that** P(n) **is true for all integers** n ≥ 1 **by induction.**
2. **Base Case** (n=1): 1!=1·0!=1·1=1=1^1 **so** P(1) **is true.**

Prove $n! \leq n^n$ for all $n \geq 1$

1. **Let** P(n) **be** "$n! \leq n^n$". **We will show that** P(n) **is true for all integers** $n \geq 1$ **by induction.**
2. **Base Case** (n=1): $1!=1\cdot 0!=1\cdot 1=1=1^1$ **so** P(1) **is true.**
3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 1$**. I.e., suppose** $k! \leq k^k$.

Prove $n! \leq n^n$ for all $n \geq 1$

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3. **Inductive Hypothesis: Suppose that** P(k) **is true for some arbitrary integer** $k \geq 1$**. I.e., suppose** $k! \leq k^k$.
4. **Inductive Step:**

Goal: Show P(k+1), **i.e. show** $(k+1)! \leq (k+1)^{k+1}$

Prove $n! \leq n^n$ for all $n \geq 1$

1. Let P(n) be "$n! \leq n^n$". We will show that P(n) is true for all integers $n \geq 1$ by induction.
2. Base Case (n=1): $1! = 1 \cdot 0! = 1 \cdot 1 = 1 = 1^1$ so P(1) is true.
3. Inductive Hypothesis: Suppose that P(k) is true for some arbitrary integer $k \geq 1$. I.e., suppose $k! \leq k^k$.
4. Inductive Step:

   We can calculate:

$$
\begin{aligned}
(k+1)! &= (k+1) \cdot k! && \text{by definition of !} \\
&\leq (k+1) \cdot k^k && \text{by the IH} \\
&\leq (k+1) \cdot (k+1)^k && \text{since } k \geq 0 \\
&= (k+1)^{k+1}
\end{aligned}
$$

   Therefore P(k+1) is true.

Prove $n! \leq n^n$ for all $n \geq 1$

1. Let P(n) be "$n! \leq n^n$". We will show that P(n) is true for all integers $n \geq 1$ by induction.
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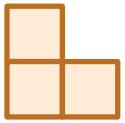
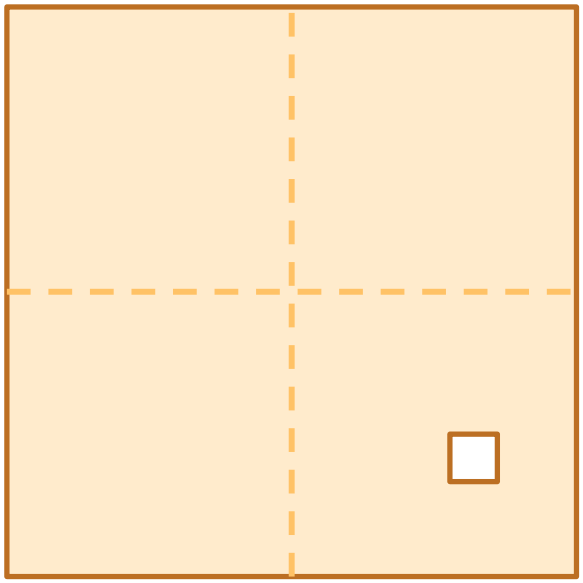
   We can calculate:

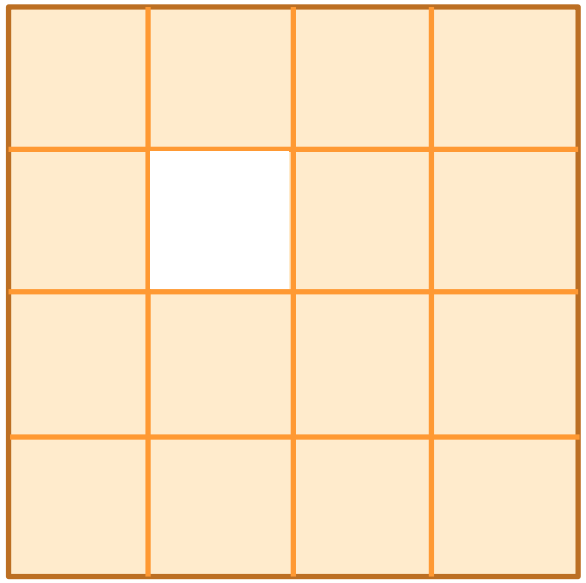
   $$\begin{aligned} (k+1)! &= (k+1) \cdot k! && \text{by definition of !} \\ &\leq (k+1) \cdot k^k && \text{by the IH} \\ &\leq (k+1) \cdot (k+1)^k && \text{since } k \geq 0 \\ &= (k+1)^{k+1} \end{aligned}$$

   Therefore P(k+1) is true.
5. Thus P(n) is true for all $n \geq 1$, by induction.

Checkerboard Tiling

- Prove that a $2^n \times 2^n$ checkerboard with one square removed can be tiled with: 





Checkerboard Tiling

1. **Let** $P(n)$ **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove $P(n)$ **for all** $n \geq 1$ **by induction on** n.

Checkerboard Tiling

1. **Let** $P(n)$ **be any** $2^n \times 2^n$ **checkerboard with one square removed can be tiled with**  **.**
   We prove $P(n)$ **for all** $n \geq 1$ **by induction on** n.
2. **Base Case:** $n=1$ 

Checkerboard Tiling

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   We prove $P(n)$ **for all** $n \geq 1$ **by induction on** n.
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3. **Inductive Hypothesis: Assume** $P(k)$ **for some arbitrary integer** $k \geq 1$

Checkerboard Tiling

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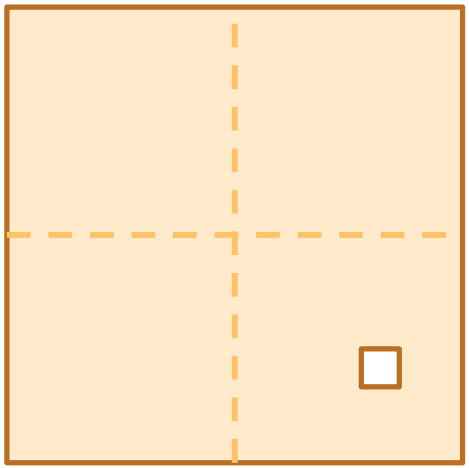
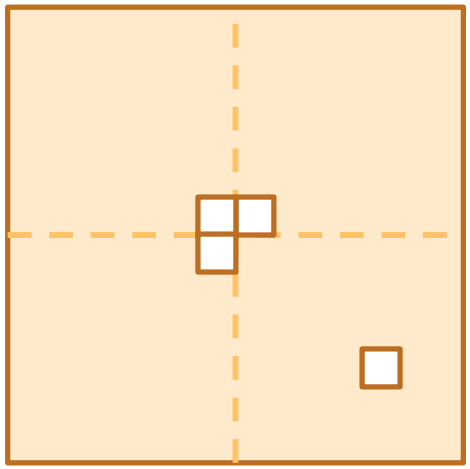
2. **Base Case:** $n=1$ 
3. **Inductive Hypothesis: Assume** $P(k)$ **for some arbitrary integer** $k \geq 1$
4. **Inductive Step: Prove** $P(k+1)$





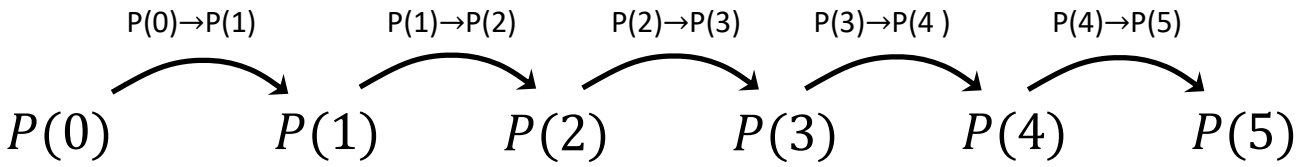
Apply IH to each quadrant then fill with extra tile.

Induction Is A Rule of Inference

Domain: Non-Negative Numbers

$$\frac{P(0) \qquad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n\ P(n)}\ \text{Induction}$$

How do the givens prove P(5)?



Induction Is A Rule of Inference

Domain: Non-Negative Numbers

Induction

$$\frac{P(0) \qquad \forall k\,(P(k) \rightarrow P(k+1))}{\therefore \forall n\ P(n)}$$

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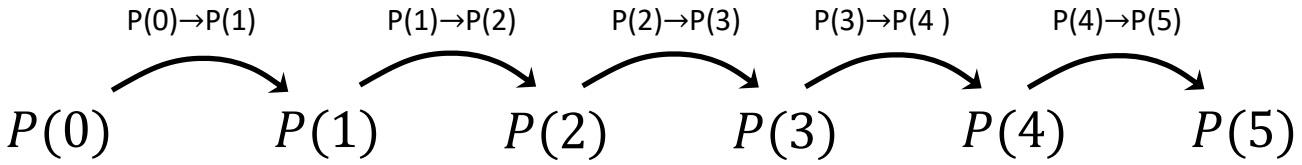


First, we have P(0).
Since P(n) → P(n+1) for all n, we have P(0) → P(1).
Since P(0) is true and P(0) → P(1), by Modus Ponens, P(1) is true.
Since P(n) → P(n+1) for all n, we have P(1) → P(2).
Since P(1) is true and P(1) → P(2), by Modus Ponens, P(2) is true.

Strong Induction

$$\frac{P(0) \qquad \forall k\,(P(k) \longrightarrow P(k+1))}{\therefore \forall n\, P(n)}$$

Induction

$$\frac{P(0) \qquad \forall k\,\Big(\forall j\,\big((0 \le j \le k) \to P(j)\big) \to P(k+1)\Big)}{\therefore \forall n\, P(n)}$$

Strong Induction

Strong Induction

$$\frac{P(0) \qquad \forall k \left(\forall j \left(0 \leq j \leq k \rightarrow P(j)\right) \rightarrow P(k+1)\right)}{\therefore \forall n \, P(n)}$$

Strong induction for P follows from ordinary induction for Q where

$$Q(k) \ ::= \ \forall j \left(0 \leq j \leq k \rightarrow P(j)\right)$$

Note that $Q(0) = P(0)$ and $Q(k+1) \equiv Q(k) \ \wedge P(k+1)$ and $\forall n \, Q(n) \equiv \forall n \, P(n)$

Strong Inductive Proofs In 5 Easy Steps

1. "Let $P(n)$ be... . We will show that $P(n)$ is true for all integers $n \geq b$ by *strong* induction."
2. "Base Case:" Prove $P(b)$
3. "Inductive Hypothesis:

   Assume that for some arbitrary integer $k \geq b$,

   $P(j)$ *is true for every integer* j *from* b to k"
4. "Inductive Step:" Prove that $P(k+1)$ is true:

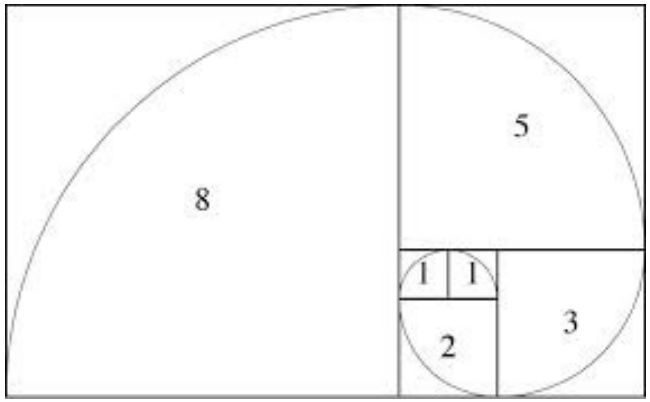
   Use the goal to figure out what you need.

   Make sure you are using I.H. (that $P(b), \ldots, P(k)$ *are true) and point out where you are using it.*
   (Don't assume $P(k+1)$ *!!)*
5. "Conclusion: $P(n)$ is true for all integers $n \geq b$"

Fibonacci Numbers

$$f_0 := 0$$
$$f_1 := 1$$
$$f_{n+2} := f_{n+1} + f_n$$

Will need facts about f_{n-2} to reason about f_n



Bounding Fibonacci: $f_n < 2^n$ for all $n \geq 0$

1. **Let** P(n) **be "**$f_n < 2^n$**". We prove that** P(n) **is true for all integers** n ≥ 0 **by strong induction.**

$$f_0 = 0 \quad f_1 = 1$$
$$f_{n+2} = f_{n+1} + f_n$$

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1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Case: $f_0 = 0 < 1 = 2^0$ so P(0) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 0$, we have $f_j < 2^j$ for every integer j from 0 to k.

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4. Inductive Step:

$f_{k+1} = f_k + f_{k-1}$ def of f

Oops! This is only true if $k + 1 \geq 2$!

Goal: Show P(k+1); that is, $f_{k+1} < 2^{k+1}$

$f_0 = 0 \quad f_1 = 1$
$f_{n+2} = f_{n+1} + f_n$

Bounding Fibonacci: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers n ≥ 0 by strong induction.
2. Base Cases: $f_0 = 0 < 1 = 2^0$ so P(0) is true and
   $f_1 = 1 < 2 = 2^1$ so P(1) is true.

$$f_0 = 0 \quad f_1 = 1$$
$$f_{n+2} = f_{n+1} + f_n$$

Bounding Fibonacci: $f_n < 2^n$ for all $n \geq 0$

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Goal: Show P(k+1); that is, $f_{k+1} < 2^{k+1}$

$$f_0 = 0 \quad f_1 = 1$$
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   $f_1 = 1 < 2 = 2^1$ so P(1) is true.
3. Inductive Hypothesis: Assume that for some arbitrary integer $k \geq 1$, we have $f_j < 2^j$ for every integer j from 0 to k.
4. Inductive Step: We can calculate that

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} && \text{def of f (since } k+1 \geq 2) \\ &< 2^k + f_{k-1} && \text{by IH} \\ &< 2^k + 2^{k-1} && \text{by IH (since } k-1 \geq 0) \\ &< 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

so P(k+1) is true.

$$f_0 = 0 \quad f_1 = 1$$
$$f_{n+2} = f_{n+1} + f_n$$

Bounding Fibonacci: $f_n < 2^n$ for all $n \geq 0$

1. Let P(n) be "$f_n < 2^n$". We prove that P(n) is true for all integers $n \geq 0$ by strong induction.
2. Base Cases: $f_0 = 0 < 1 = 2^0$ so P(0) is true and
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$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} && \text{def of f (since } k+1 \geq 2) \\ &< 2^k + f_{k-1} && \text{by IH} \\ &< 2^k + 2^{k-1} && \text{by IH (since } k-1 \geq 0) \\ &< 2^k + 2^k \\ &= 2^{k+1} \end{aligned}$$

   so P(k+1) is true.

5. Therefore, by strong induction, $f_n < 2^n$ for all integers $n \geq 0$.

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers n ≥ 2 by strong induction.

$$f_0 = 0 \quad f_1 = 1$$
$$f_{n+2} = f_{n+1} + f_n$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers n ≥ 2 by strong induction.
2. Base Case: $f_2 = f_1 + f_0 = 1 \geq 1 = 2^0 = 2^{2/2-1}$ so P(2) holds

$$f_0 = 0 \quad f_1 = 1$$
$$f_{n+2} = f_{n+1} + f_n$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

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2. Base Case: $f_2 = f_1 + f_0 = 1 \geq 1 = 2^0 = 2^{2/2-1}$ so P(2) holds
3. Inductive Hypothesis: Assume that for some arbitrary integer k ≥ 2, P(j) is true for every integer j from 2 to k.

$$f_0 = 0 \quad f_1 = 1$$
$$f_{n+2} = f_{n+1} + f_n$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

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3. Inductive Hypothesis: Assume that for some arbitrary integer k ≥ 2, P(j) is true for every integer j from 2 to k.
4. Inductive Step:

Goal: Show P(k+1); that is, $f_{k+1} \geq 2^{(k+1)/2-1}$

$$f_0 = 0 \quad f_1 = 1$$
$$f_{n+2} = f_{n+1} + f_n$$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

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2. Base Case: $f_2 = f_1 + f_0 = 1 \geq 1 = 2^0 = 2^{2/2-1}$ so P(2) holds
3. Inductive Hypothesis: Assume that for some arbitrary integer k ≥ 2, P(j) is true for every integer j from 2 to k.
4. Inductive Step:

$f_{k+1} = f_k + f_{k-1}$ — def of f (since k+1 ≥ 2)

$\geq 2^{k/2-1} + f_{k-1}$ — by the IH

$\geq 2^{k/2-1} + 2^{(k-1)/2-1}$ — by the IH

Oops! This is only true if $k - 1 \geq 2$!

Goal: Show P(k+1); that is, $f_{k+1} \geq 2^{(k+1)/2-1}$

$f_0 = 0 \quad f_1 = 1$

$f_{n+2} = f_{n+1} + f_n$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers n ≥ 2 by strong induction.
2. Base Cases: $f_2 = f_1 + f_0 = 1 \geq 1 = 2^0 = 2^{2/2-1}$ so P(2) holds
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ so P(3) holds
3. Inductive Hypothesis: Assume that for some arbitrary integer k ≥ 3, P(j) is true for every integer j from 2 to k.
4. Inductive Step:

Goal: Show P(k+1); that is, $f_{k+1} \geq 2^{(k+1)/2-1}$

$f_0 = 0 \quad f_1 = 1$
$f_{n+2} = f_{n+1} + f_n$

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

1. Let P(n) be "$f_n \geq 2^{n/2-1}$". We prove that P(n) is true for all integers n ≥ 2 by strong induction.
2. Base Cases: $f_2 = f_1 + f_0 = 1 \geq 1 = 2^0 = 2^{2/2-1}$ so P(2) holds
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ so P(3) holds
3. Inductive Hypothesis: Assume that for some arbitrary integer k ≥ 3, P(j) is true for every integer j from 2 to k.
4. Inductive Step: We can calculate

$$
\begin{aligned}
f_{k+1} &= f_k + f_{k-1} && \text{def of f (since } k+1 \geq 4\text{)} \\
&\geq 2^{k/2-1} + f_{k-1} && \text{by the IH} \\
&\geq 2^{k/2-1} + 2^{(k-1)/2-1} && \text{by the IH (since } k-1 \geq 2\text{)} \\
&\geq 2 \cdot 2^{(k-1)/2-1} \\
&= 2^{(k+1)/2-1}
\end{aligned}
$$

so P(k+1) is true.

Bounding Fibonacci II: $f_n \geq 2^{n/2-1}$ for all $n \geq 2$

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2. Base Cases: $f_2 = f_1 + f_0 = 1 \geq 1 = 2^0 = 2^{2/2-1}$ so P(2) holds
   $f_3 = f_2 + f_1 = 2 \geq 2^{1/2} = 2^{3/2-1}$ so P(3) holds
3. Inductive Hypothesis: Assume that for some arbitrary integer k ≥ 3, P(j) is true for every integer j from 2 to k.
4. Inductive Step: We can calculate

$$\begin{aligned} f_{k+1} &= f_k + f_{k-1} && \text{def of f (since } k+1 \geq 4\text{)} \\ &\geq 2^{k/2-1} + f_{k-1} && \text{by the IH} \\ &\geq 2^{k/2-1} + 2^{(k-1)/2-1} && \text{by the IH (since } k-1 \geq 2\text{)} \\ &\geq 2 \cdot 2^{(k-1)/2-1} = 2^{(k+1)/2-1} \end{aligned}$$

   so P(k+1) is true.

5. Therefore by strong induction, $f_n \geq 2^{n/2-1}$ for all integers n ≥ 2.