CSE 311: Foundations of Computing

Topic 7: Languages

Theoretical Computer Science

Strings

- An *alphabet* Σ is any finite set of characters

- The set Σ^* of *strings* over the alphabet Σ
  - example: {0,1}* is the set of *binary strings*

    0, 1, 00, 01, 10, 11, 000, 001, ... and “”

- Σ^* is defined recursively by
  - **Basis:** $\varepsilon \in \Sigma^*$ (ε is the empty string, i.e., “”)
  - **Recursive:** if $w \in \Sigma^*$, $a \in \Sigma$, then $wa \in \Sigma^*$

Languages: Sets of Strings

- Subsets of strings are called *languages*
- Examples:
  - Σ^* = All strings over alphabet Σ
  - palindromes over Σ
  - binary strings with an equal # of 0's and 1's
  - syntactically correct Java/C/C++ programs
  - valid English sentences
  - correct solutions to coding problems:

$$S = \{x\#y \mid y \text{ is Java code that does what } x \text{ says}\}$$

Foreword on Intro to Theory C.S.

- Look at different ways of defining languages
- See which are more expressive than others
  - i.e., which can define more languages
- Later: connect ways of defining languages to different types of (restricted) computers
  - computers capable of recognizing those languages
    i.e., distinguishing strings in the language from not
- Consequence: computers that recognize more expressive languages are more powerful

Palindromes

Palindromes are strings that are the same when read backwards and forwards

Basis:

ε is a palindrome

any $a \in \Sigma$ is a palindrome

Recursive step:

If p is a palindrome,

then apa is a palindrome for every $a \in \Sigma$

Regular Expressions

Regular expressions over Σ

- **Basis**:

  ε is a regular expression (could also include $\varnothing$)

  a is a regular expression for any $a \in \Sigma$

- **Recursive step**:

  If **A** and **B** are regular expressions, then so are:

  $A \cup B$

  AB

  A*

Each Regular Expression is a "pattern"

ε matches only the **empty string**

a matches only the one-character string *a*

A ∪ **B** matches all strings that either **A** matches or **B** matches (or both)

AB matches all strings that have a first part that **A** matches followed by a second part that **B** matches

A* matches all strings that have any number of strings (even 0) that **A** matches, one after another (ε ∪ **A** ∪ **AA** ∪ **AAA** ∪ **…**)

Definition of the *language* matched by a regular expression

Language of a Regular Expression

The language defined by a regular expression:

$$\mathrm{L}(\varepsilon) = \{\varepsilon\}$$

$$\mathrm{L}(a) = \{a\}$$

$$\mathrm{L}(A \cup B) = L(A) \cup L(B)$$

$$\mathrm{L}(AB) = \{y \bullet z : \ y \in L(A), z \in L(B)\}$$

$$\mathrm{L}(A^*) = \bigcup_{n=0}^{\infty} L(A^n)$$

A^n **defined recursively by**

$$A^0 = \emptyset$$

$$A^{n+1} = A^n A$$

Examples

001*

0*1*

Examples

001*

{00, 001, 0011, 00111, ...}

0*1*

Any number of 0's followed by any number of 1's

Examples

$(0 \cup 1)\,0\,(0 \cup 1)\,0$

$(0^*1^*)^*$

Examples

$(0 \cup 1)\,0\,(0 \cup 1)\,0$

{0000, 0010, 1000, 1010}

$(0^*1^*)^*$

All binary strings

Examples

- All binary strings that contain 0110

$$(0 \cup 1)^* \, 0110 \, (0 \cup 1)^*$$

- All binary strings that begin with a string of doubled characters (00 or 11) followed by 01010 or 10001

$$(00 \cup 11)^* \, (01010 \cup 10001) \, (0 \cup 1)^*$$

Examples

- All binary strings that have an even # of 1's

  e.g., $0^*(10^*10^*)^*$

- All binary strings that *don't* contain 101

  e.g., $0^*(1 \cup 1000^*)^*(\varepsilon \cup 10)$

  at least two 0s between 1s

Finite languages vs Regular Expressions

- All finite languages have a regular expression.

  (a language is finite if its elements can be put into a list)

  Why?

- Given a list of strings s_1, s_2, …, s_n

  Construct the regular expression

$$s_1 \cup s_2 \cup \ldots \cup s_n$$

(Could make this formal by induction on n)

Finite languages vs Regular Expressions

- Every regular expression that does not use * generates a finite language.

  Why?

- Prove by structural induction on the syntax of regular expressions!

Star-free implies finite

Let A be a regular expression that does not use *. Then L(A) is finite.

Proof: We proceed by structural induction on A.

Case ε: $L(\varepsilon) = \{\varepsilon\}$, which is finite

Case a: $L(a) = \{a\}$, which is finite

Case $A \cup B$:

$L(A \cup B) = L(A) \cup L(B)$

By the IH, each is finite, so their union is finite.

Star-free implies finite

Let A be a regular expression that does not use *. Then L(A) is finite.

Proof: We proceed by structural induction on A.

Case AB:

$$\mathrm{L}(\mathrm{AB}) = \{y \bullet z : \ y \in L(A), z \in L(B)\}$$

By the IH, $\mathrm{L}(\mathrm{A})$ and $\mathrm{L}(\mathrm{B})$ are finite.

Every element of $\mathrm{L}(\mathrm{AB})$ is covered by a pair (y, z) where $y \in L(A)$ and $z \in L(B)$, so $\mathrm{L}(\mathrm{AB})$ is finite.

(No case for A*!)

Finite languages vs Regular Expressions

Key takeaways:

- Regular expressions can represent all finite languages
- To prove a language is "regular", just give the regular expression that describes it.
- Regular expressions are more powerful than finite languages (e.g., 0* is an infinite language)
- To prove something about *all* regular expressions, use structural induction on the syntax.

Regular Expressions in Practice

- Used to define the "tokens": e.g., legal variable names, keywords in programming languages and compilers
- Used in **`grep`**, a program that does pattern matching searches in UNIX/LINUX
- Pattern matching using regular expressions is an essential feature of PHP
- We can use regular expressions in programs to process strings!

Regular Expressions in Java

- Pattern p = Pattern.compile("a*b");
- Matcher m = p.matcher("aaaaab");
- boolean b = m.matches();

`[01]`	a 0 or a 1	`^` start of string `$` end of string
`[0-9]`	any single digit	`\.` period `\,` comma `\-` minus
`.`	any single character	
ab	a followed by b	(**AB**)
(a|b)	a or b	(**A** ∪ **B**)
a`?`	zero or one of a	(**A** ∪ ε)
a`*`	zero or more of a	**A***
a`+`	one or more of a	**AA***

- e.g. `^[\-+]?[0-9]*(\.|\,)?[0-9]+$`

  General form of decimal number e.g. 9.12 or -9,8 (Europe)

Limitations of Regular Expressions

- Not all languages can be specified by regular expressions
- Even some easy things like
  - Palindromes
  - Strings with equal number of 0's and 1's
- But also more complicated structures in programming languages
  - Matched parentheses
  - Properly formed arithmetic expressions
  - etc.

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow 0\mathbf{S}0 \mid 1\mathbf{S}1 \mid 0 \mid 1 \mid \varepsilon$

Example Context-Free Grammars

Example: $S \rightarrow 0S0 \mid 1S1 \mid 0 \mid 1 \mid \varepsilon$

How does this grammar generate 0110?

Example Context-Free Grammars

Example: $S \rightarrow 0S0 \mid 1S1 \mid 0 \mid 1 \mid \varepsilon$

How does this grammar generate 0110?

$S \rightarrow 0S0 \rightarrow 01S10 \rightarrow 01\varepsilon10 = 0110$

Example Context-Free Grammars

Example: $S \rightarrow 0S0 \mid 1S1 \mid 0 \mid 1 \mid \varepsilon$

How to describe all strings generated?

The set of all binary palindromes

Two ways to Define Binary Palindromes

Recursively-Defined Set

Basis:

ε is a palindrome

any $a \in \{0, 1\}$ is a palindrome

Recursive step:

If p is a palindrome,

then apa is a palindrome for every $a \in \{0, 1\}$

Grammar $\quad S \rightarrow 0S0 \mid 1S1 \mid 0 \mid 1 \mid \varepsilon$

Example Context-Free Grammars

Example: $S \rightarrow A \mid B$

$A \rightarrow 0A \mid \varepsilon$

$B \rightarrow 1B \mid \varepsilon$

How does this grammar generate 000?

Example Context-Free Grammars

Example: $S \rightarrow A \mid B$

$A \rightarrow 0A \mid \varepsilon$

$B \rightarrow 1B \mid \varepsilon$

How does this grammar generate 000?

$S \rightarrow A \rightarrow 0A \rightarrow 00A \rightarrow 000A \rightarrow 000\varepsilon = 000$

Example Context-Free Grammars

Example: $S \rightarrow A \mid B$

$A \rightarrow 0A \mid \varepsilon$

$B \rightarrow 1B \mid \varepsilon$

How to describe all strings generated?

strings of all 0s or all 1s

(all 0s) ∪ (all 1s)

Context-Free Grammars

- A Context-Free Grammar (CFG) is given by a finite set of substitution rules involving
  - A finite set **V** of *variables* that can be replaced
  - Alphabet Σ of *terminal symbols* that can't be replaced
  - One variable, usually **S**, is called the *start symbol*

- The substitution rules involving a variable **A**, written as

$$\mathbf{A} \rightarrow w_1 \mid w_2 \mid \cdots \mid w_k$$

  where each w_i is a string of variables and terminals
  - that is $w_i \in (\mathbf{V} \cup \Sigma)^*$

How CFGs generate strings

- Begin with start symbol **S**

- If there is some variable **A** in the current string you can replace it by one of the w's in the rules for **A**
  - $\mathbf{A} \rightarrow w_1 \mid w_2 \mid \cdots \mid w_k$
  - Write this as $x\mathbf{A}y \Rightarrow xwy$
  - Repeat until no variables left

- The set of strings the CFG describes are all strings, containing no variables, that can be *generated* in this manner (after a finite number of steps)

Example Context-Free Grammars

Example: $\mathbf{S} \rightarrow 0\mathbf{S} \mid \mathbf{S}1 \mid \varepsilon$

Example Context-Free Grammars

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0*1*

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching 0*1* but with same number of 0's and 1's)

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Grammar for $\{0^n 1^{2n} : n \geq 0\}$

Example Context-Free Grammars

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(i.e., matching 0*1* but with same number of 0's and 1's)

$$S \rightarrow 0S1 \mid \varepsilon$$

Grammar for $\{0^n 1^{2n} : n \geq 0\}$

$$S \rightarrow 0S11 \mid \varepsilon$$

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching 0*1* but with same number of 0's and 1's)

$$S \rightarrow 0S1 \mid \varepsilon$$

Grammar for $\{0^n 1^{n+1} 0 : n \geq 0\}$

Example Context-Free Grammars

Grammar for $\{0^n 1^n : n \geq 0\}$

(i.e., matching 0*1* but with same number of 0's and 1's)

$$S \rightarrow 0S1 \mid \varepsilon$$

Grammar for $\{0^n 1^{n+1} 0 : n \geq 0\}$

$$S \rightarrow A10$$

$$A \rightarrow 0A1 \mid \varepsilon$$

Example Context-Free Grammars

Example: $S \rightarrow (S) \mid SS \mid \varepsilon$

Example Context-Free Grammars

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The set of all strings of matched parentheses

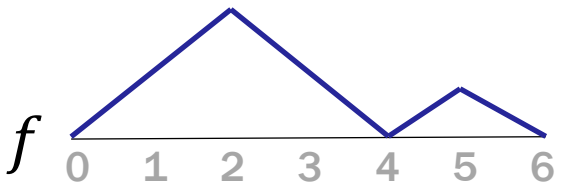
Example Context-Free Grammars

Example: $S \rightarrow (S) \mid SS \mid \varepsilon$

The set of all strings of matched parentheses

Suppose S generates x. Define $f(k)$ to be number of "("s – ")"s in first k characters of x

E.g., for x = (())()

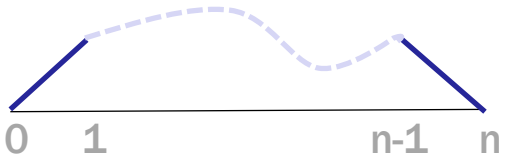


Example Context-Free Grammars

Three possibilities for $f(\mathrm{k})$ for $k \in \{1, \ldots, n-1\}$

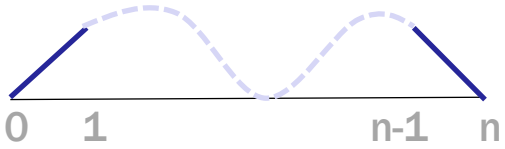
- $f(k) > 0$ **for all such** k

  S $\rightarrow$ **(S)**



- $f(k) = 0$ **for some such** k

  S $\rightarrow$ **SS**



Example Context-Free Grammars

Binary strings with equal numbers of 0s and 1s
(not just 0^n1^n, also 0101, 0110, etc.)

$$S \rightarrow SS \mid 0S1 \mid 1S0 \mid \varepsilon$$

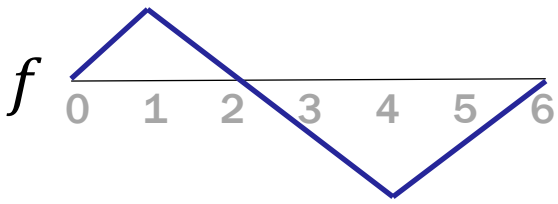
Example Context-Free Grammars

Binary strings with equal numbers of 0s and 1s
(not just 0^n1^n, also 0101, 0110, etc.)

$$S \rightarrow SS \mid 0S1 \mid 1S0 \mid \varepsilon$$

Suppose S generates x. Define $f(k)$ to be #0s – #1s in the first k characters of x.

E.g., for x = 011100



Example Context-Free Grammars

Binary strings with equal numbers of 0s and 1s
(not just 0^n1^n, also 0101, 0110, etc.)

$$S \rightarrow SS \mid 0S1 \mid 1S0 \mid \varepsilon$$

Suppose S generates x. Define $f(k)$ to be #0s – #1s in the first k characters of x.

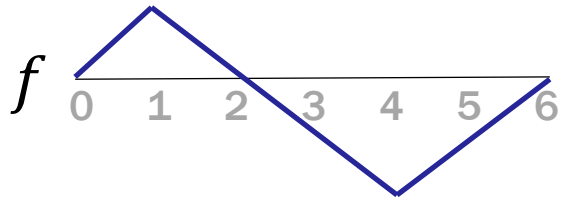
If k-th character is 0, then $f(k) = f(k-1) + 1$

If k-th character is 1, then $f(k) = f(k-1) - 1$

Example Context-Free Grammars

Let $x \in (0 \cup 1)^*$. Define $f_x(k)$ to be the number 0s minus the number of 1s in the k characters of x.

E.g., for x = 011100



$f(k) = 0$ when first k characters have #0s = #1s

– starts out at 0 $\quad f(0) = 0$

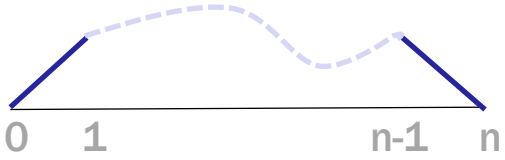
– ends at 0 $\quad f(n) = 0$

Example Context-Free Grammars

Three possibilities for $f(\mathrm{k})$ for $k \in \{1, \ldots, n-1\}$

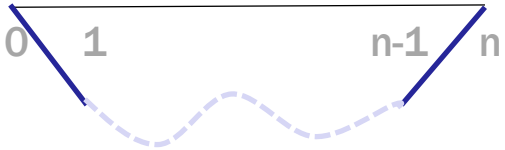
- $f(k) > 0$ for all such k

  $S \rightarrow 0S1$



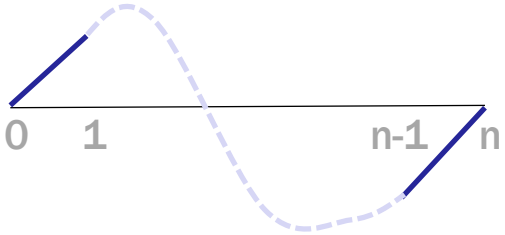
- $f(k) < 0$ for all such k

  $S \rightarrow 1S0$



- $f(k) = 0$ for some such k

  $S \rightarrow SS$



Simple Arithmetic Expressions

$$E \rightarrow E+E \mid E*E \mid (E) \mid x \mid y \mid z \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

Generate (2 + x) * y

Simple Arithmetic Expressions

$$E \rightarrow E+E \mid E*E \mid (E) \mid x \mid y \mid z \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

Generate (2 + x) * y

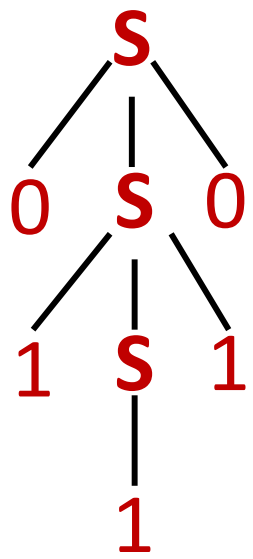
$$E \Rightarrow E*E \Rightarrow (E)*E \Rightarrow (E+E)*E \Rightarrow (2+E)*E \Rightarrow (2+x)*E \Rightarrow (2+x)*y$$

Parse Trees

Suppose that grammar G generates a string x

- A *parse tree* of x for G has
  - Root labeled S (start symbol of G)
  - The children of any node labeled A are labeled by symbols of w left-to-right for some rule $A \rightarrow w$
  - The symbols of x label the leaves ordered left-to-right

$S \rightarrow 0S0 \mid 1S1 \mid 0 \mid 1 \mid \varepsilon$



Parse tree of 01110

Simple Arithmetic Expressions

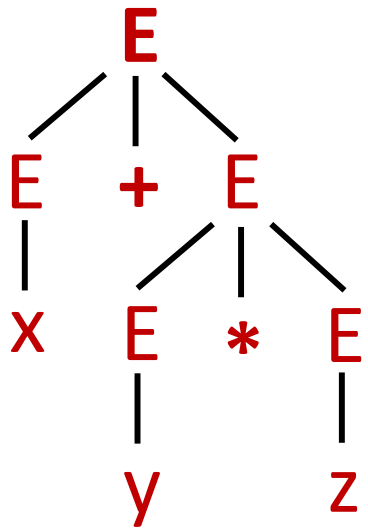
$$E \rightarrow E+E \mid E*E \mid (E) \mid x \mid y \mid z \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

Generate x+y∗z in two ways that give two *different* parse trees

Simple Arithmetic Expressions

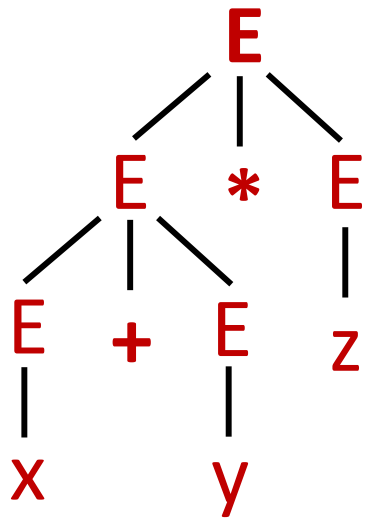
$$E \rightarrow E+E \mid E*E \mid (E) \mid x \mid y \mid z \mid 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

Generate x+y*z in ways that give two *different* parse trees



$E \Rightarrow E+E \Rightarrow x+E \Rightarrow x+E*E \Rightarrow x+y*E \Rightarrow x+y*z$

(multiply y with z and then add to x)



$E \Rightarrow E*E \Rightarrow E+E*E \Rightarrow x+E*E$

$\Rightarrow x+y*E \Rightarrow x+y*z$

(add x to y, then multiply by z)

Induction on Parse Trees

Structural induction is the tool used to prove many more interesting theorems

- General associativity follows from our one rule
  - likewise for generalized De Morgan's laws
- Okay to substitute y for x everywhere in a modular equation when we know that $x \equiv_m y$
- The "Meta Theorem" on set operators

These are proven by induction on parse trees

- parse trees are recursively defined

CFGs and Regular Expressions

Theorem: For any set of strings (language) A described by a regular expression, there is a CFG that recognizes A.

Proof idea:

P(A) is "A is recognized by some CFG"

Structural induction based on the recursive definition of regular expressions...

Regular Expressions over Σ

- **Basis:**
  - ε is a regular expression
  - ***a*** is a regular expression for any $a \in \Sigma$
- **Recursive step:**
  - If **A** and **B** are regular expressions then so are:

    A $\cup$ **B**

    AB

    A*

CFGs are more general than REs

- CFG to match RE **ε**

  $\mathbf{S} \rightarrow \varepsilon$

- CFG to match RE ***a*** (for any $a \in \Sigma$)

  $\mathbf{S} \rightarrow$ a

CFGs are more general than REs

Suppose CFG with start symbol $\mathbf{S_1}$ matches RE **A**
CFG with start symbol $\mathbf{S_2}$ matches RE **B**

- CFG to match RE $\mathbf{A} \cup \mathbf{B}$

  $\mathbf{S} \rightarrow \mathbf{S_1} \mid \mathbf{S_2}$ + rules from original CFGs

- CFG to match RE **AB**

  $\mathbf{S} \rightarrow \mathbf{S_1}\,\mathbf{S_2}$ + rules from original CFGs

CFGs are more general than REs

Suppose CFG with start symbol $\mathbf{S_1}$ matches RE **A**

- CFG to match RE $\mathbf{A}^*$ (= $\varepsilon \cup \mathbf{A} \cup \mathbf{AA} \cup \mathbf{AAA} \cup \ldots$)

  $\mathbf{S} \rightarrow \mathbf{S_1}\,\mathbf{S} \mid \varepsilon$ + rules from CFG with $\mathbf{S_1}$