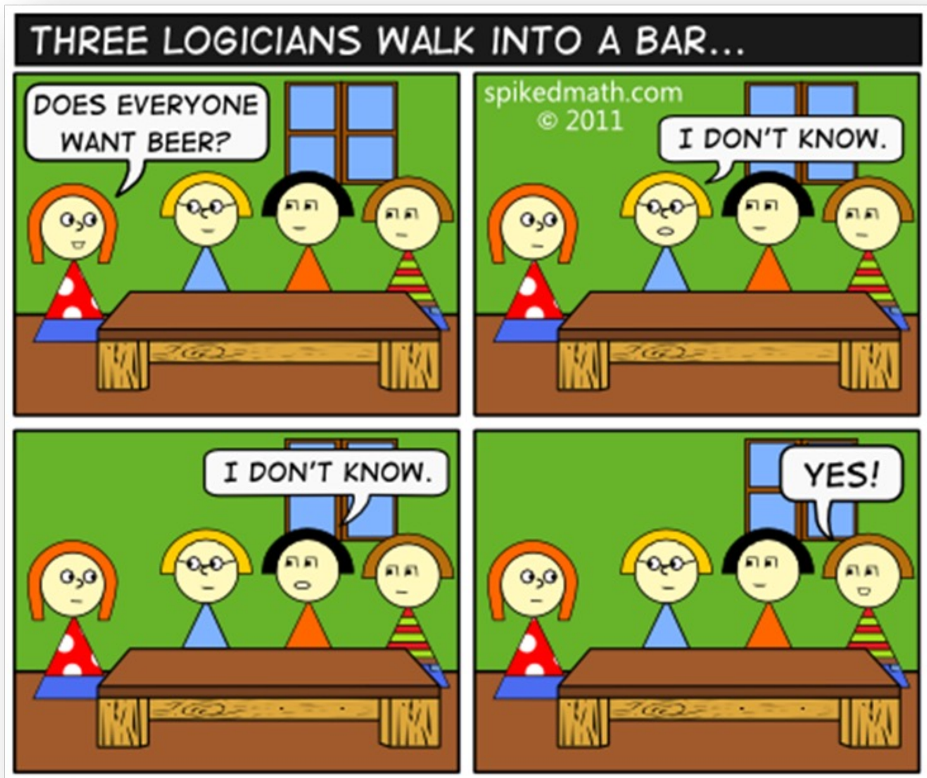
CSE 311: Foundations of Computing

Topic 2: More Logic



More Logic

- This week we will see
  - new applications of Propositional Logic
  - new tools to use with Propositional Logic
  - a new type of Logic (Predicate Logic)

Circuits

Application: Digital Circuits

Computing With Logic

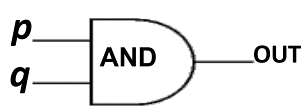
- T corresponds to 1 or "high" voltage
- F corresponds to 0 or "low" voltage

Gates

- Take inputs and produce outputs (functions)
- Several kinds of gates
- Correspond to propositional connectives (most of them)

AND, OR, NOT Gates

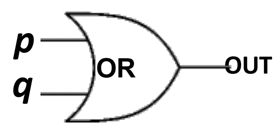
AND Gate



p	q	OUT
1	1	1
1	0	0
0	1	0
0	0	0

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

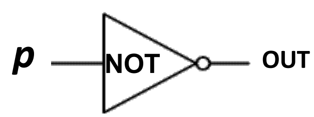
OR Gate



p	q	OUT
1	1	1
1	0	1
0	1	1
0	0	0

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

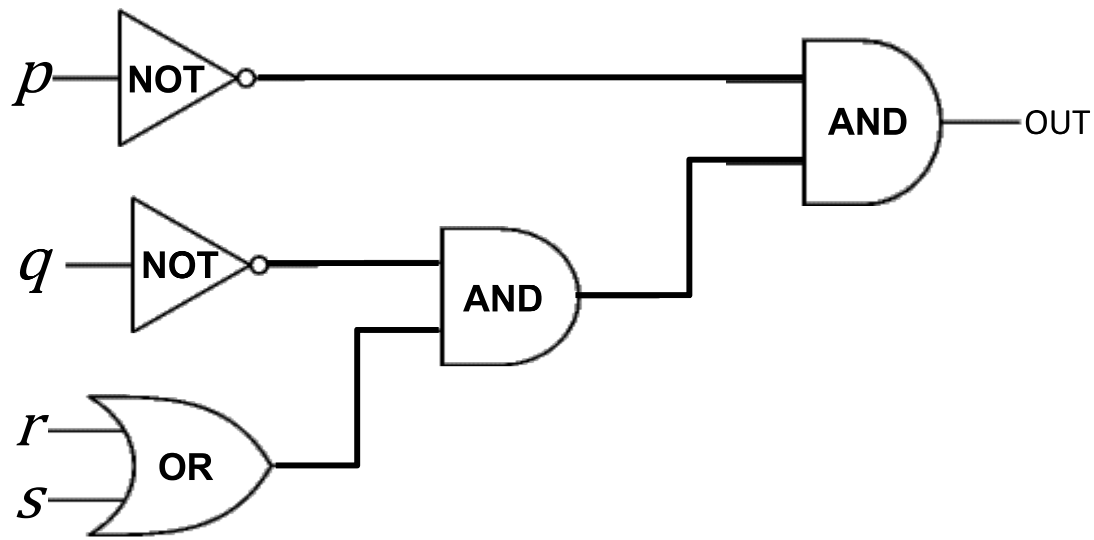
NOT Gate



p	OUT
1	0
0	1

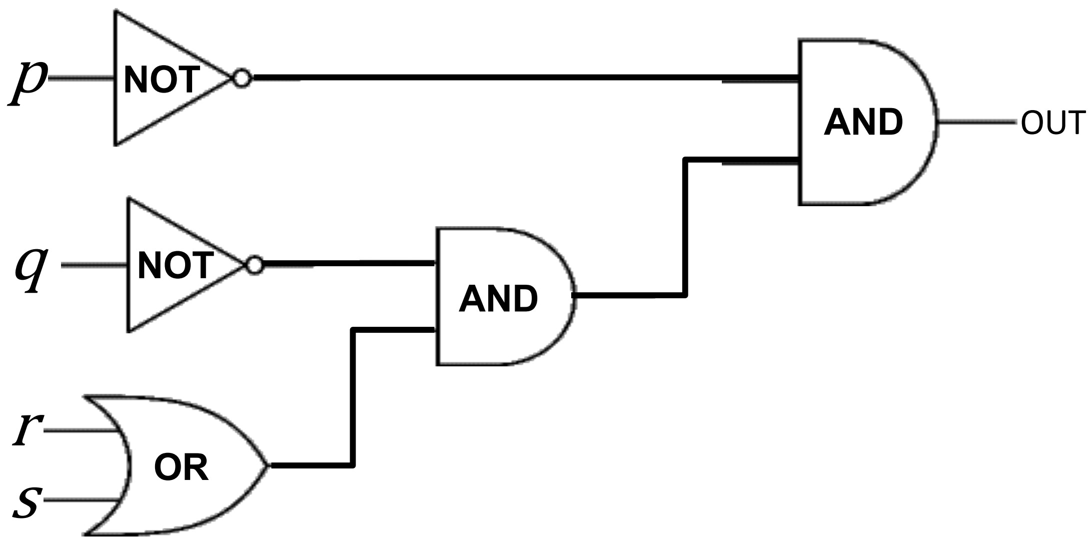
p	$\neg p$
T	F
F	T

Combinational Logic Circuits



Values get sent along wires connecting gates

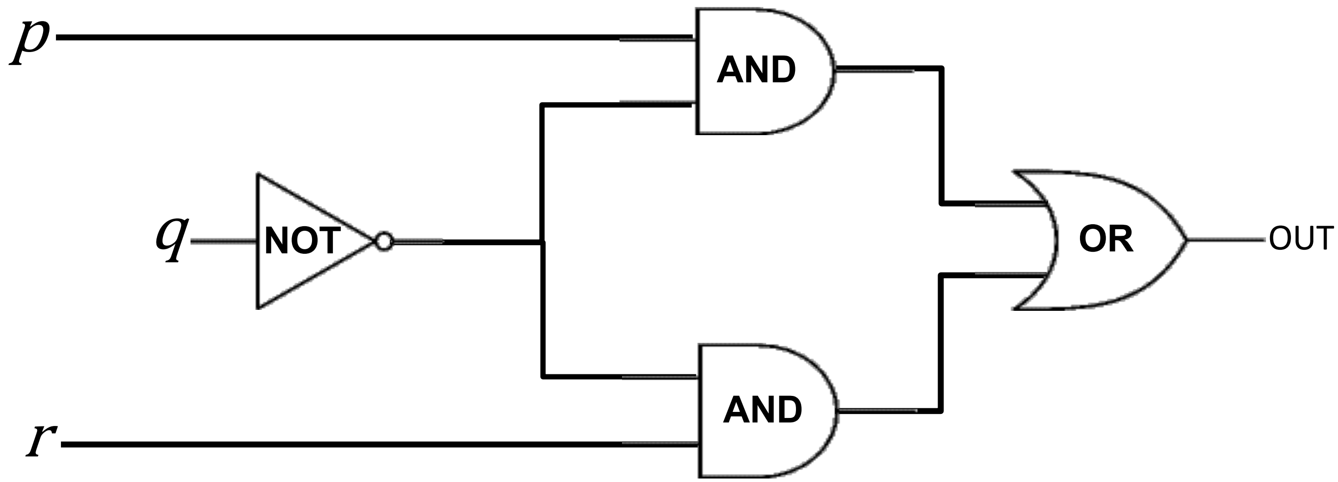
Combinational Logic Circuits



Values get sent along wires connecting gates

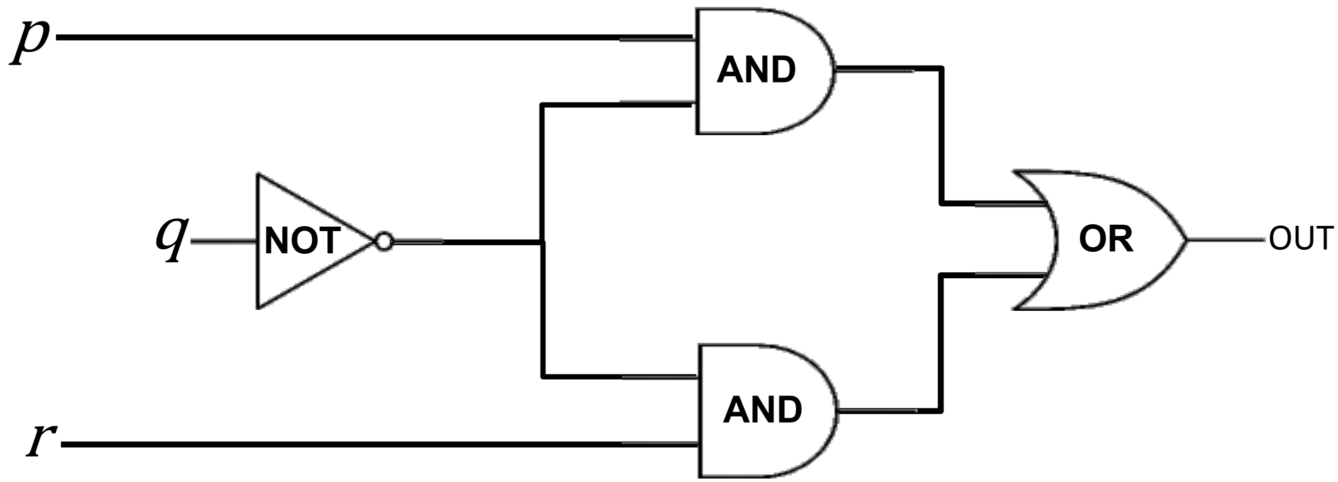
$$\neg p \wedge (\neg q \wedge (r \vee s))$$

Combinational Logic Circuits



Wires can send one value to multiple gates!

Combinational Logic Circuits



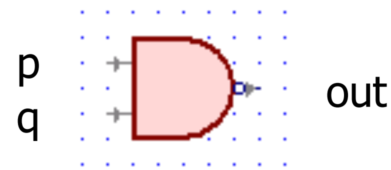
Wires can send one value to multiple gates!

$$(p \wedge \neg q) \vee (\neg q \wedge r)$$

Other Useful Gates

NAND

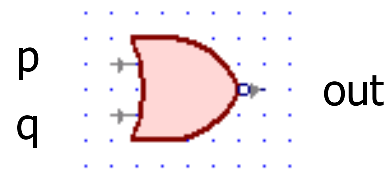
$\neg(p \land q)$



p	q	out
0	0	1
0	1	1
1	0	1
1	1	0

NOR

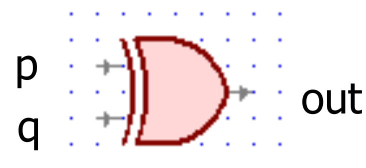
$\neg(p \lor q)$



p	q	out
0	0	1
0	1	0
1	0	0
1	1	0

XOR

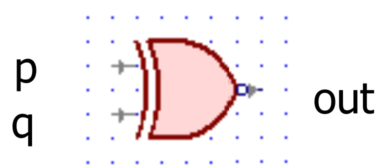
$p \oplus q$



p	q	out
0	0	0
0	1	1
1	0	1
1	1	0

XNOR

$p \leftrightarrow q$

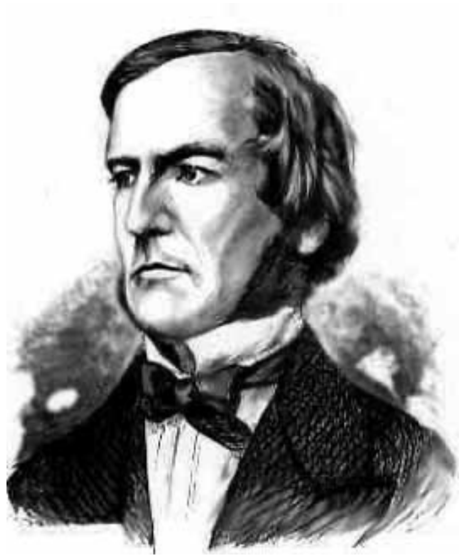


p	q	out
0	0	1
0	1	0
1	0	0
1	1	1

Boolean Algebra

- Usual notation used in circuit design

- Boolean algebra
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { a' } or { $\bar{a}$ }



Write these in Boolean Algebra:

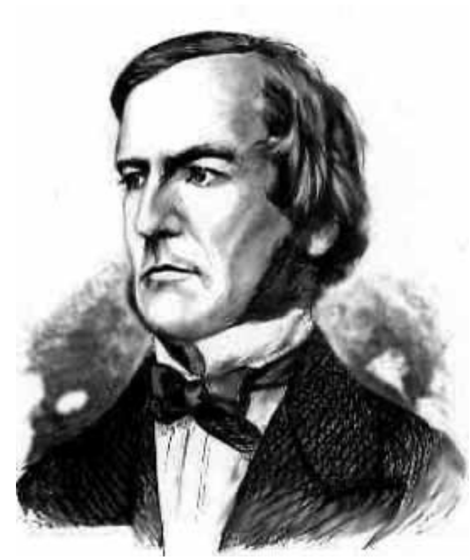
$\neg p \wedge (\neg q \wedge (r \vee s))$

$(p \wedge \neg q) \vee (\neg q \wedge r)$

Boolean Algebra

- **Usual notation used in circuit design**

- **Boolean algebra**
  - a set of elements B containing {0, 1}
  - binary operations { + , • }
  - and a unary operation { a' } or { $\bar{a}$ }



Write these in Boolean Algebra:

$\neg p \land (\neg q \land (r \lor s))$

$(p \land \neg q) \lor (\neg q \land r)$

$p'q'(r+s)$

$pq' + q'r$

A Combinational Logic Example

Sessions of Class:

We would like to compute the number of lectures or quiz sections remaining *at the start* of a given day of the week.

- **Inputs:** Day of the Week, Lecture/Section flag
- **Output:** Number of sessions left

Examples: Input: (Wednesday, Lecture) Output: 2

Input: (Monday, Section) Output: 1

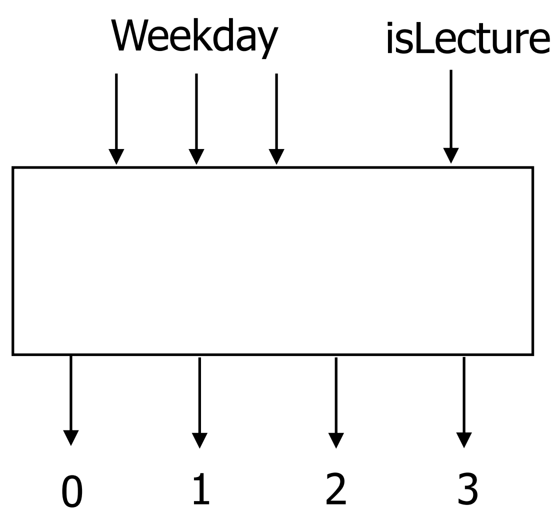
Implementation in Software

```
public int classesLeftInMorning(int weekday, boolean isLecture) {
    switch (weekday) {
        case SUNDAY:
        case MONDAY:
            return isLecture ? 3 : 1;
        case TUESDAY:
        case WEDNESDAY:
            return isLecture ? 2 : 1;
        case THURSDAY:
            return isLecture ? 1 : 1;
        case FRIDAY:
            return isLecture ? 1 : 0;
        case SATURDAY:
            return isLecture ? 0 : 0;
   }
}
```

Implementation with Hardware

Encoding:

- How many bits for each input/output?
- Binary number for weekday
- One bit for each possible output



Defining Our Inputs!

Weekday Input:

– Binary number for weekday

– Sunday = 0, Monday = 1, …

– We care about these in binary:

Weekday	Number	Binary
Sunday	0	000
Monday	1	001
Tuesday	2	010
Wednesday	3	011
Thursday	4	100
Friday	5	101
Saturday	6	110

Converting to a Truth Table!

```
case SUNDAY or MONDAY:
   return isLecture ? 3 : 1;
case TUESDAY or WEDNESDAY:
   return isLecture ? 2 : 1;
case THURSDAY:
   return isLecture ? 1 : 1;
case FRIDAY:
   return isLecture ? 1 : 0;
case SATURDAY:
   return isLecture ? 0 : 0;
```

Weekday		isLecture	c_0	c_1	c_2	c_3
SUN	000	0				
SUN	000	1				
MON	001	0				
MON	001	1				
TUE	010	0				
TUE	010	1				
WED	011	0				
WED	011	1				
THU	100	-				
FRI	101	0				
FRI	101	1				
SAT	110	-				

Converting to a Truth Table!

```
case SUNDAY or MONDAY:
   return isLecture ? 3 : 1;
case TUESDAY or WEDNESDAY:
   return isLecture ? 2 : 1;
case THURSDAY:
   return isLecture ? 1 : 1;
case FRIDAY:
   return isLecture ? 1 : 0;
case SATURDAY:
   return isLecture ? 0 : 0;
```

Weekday		isLecture	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

Truth Table to Logic

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

Let's begin by finding an expression for $\mathbf{c_3}$. To do this, we look at the rows where $\mathbf{c_3} = 1$ (true).

Truth Table to Logic

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

DAY == SUN && L == 1

DAY == MON && L == 1

Truth Table to Logic

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

$d_2d_1d_0$ == 000 && L == 1

$d_2d_1d_0$ == 001 && L == 1

Substituting DAY for the binary representation.

Truth Table to Logic

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

d_2 == 0 && d_1 == 0 && d_0 == 0 && L == 1

d_2 == 0 && d_1 == 0 && d_0 == 1 && L == 1

Splitting up the bits of the day; so, we can write a formula.

Truth Table to Logic

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

SUN, L = 1 → $d_2' \cdot d_1' \cdot d_0' \cdot L$

MON, L = 1 → $d_2' \cdot d_1' \cdot d_0 \cdot L$

Replacing with Boolean Algebra...

Truth Table to Logic

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

$d_2' \bullet d_1' \bullet d_0' \bullet L$

$d_2' \bullet d_1' \bullet d_0 \bullet L$

How do we combine them?

Truth Table to Logic

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

$d_2' \cdot d_1' \cdot d_0' \cdot L$

$d_2' \cdot d_1' \cdot d_0 \cdot L$

Either situation causes c_3 to be true. So, we "or" them.

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

Truth Table to Logic (Part 2)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$

Now, we do c_2.

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

For c_1, let's look at the 0s:

$d_2 + d_1 + d_0 + L'$

$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$

$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3	
SUN	000	0	0	1	0	0	
SUN	000	1	0	0	0	1	→ $d_2 + d_1 + d_0 + L'$
MON	001	0	0	1	0	0	
MON	001	1	0	0	0	1	→ $d_2 + d_1 + d_0' + L'$
TUE	010	0	0	1	0	0	
TUE	010	1	0	0	1	0	→ $d_2 + d_1' + d_0 + L'$
WED	011	0	0	1	0	0	
WED	011	1	0	0	1	0	→ $d_2 + d_1' + d_0' + L'$
THU	100	-	0	1	0	0	
FRI	101	0	1	0	0	0	→ $d_2' + d_1 + d_0' + L$
FRI	101	1	0	1	0	0	
SAT	110	-	1	0	0	0	→ ???

For c_1, let's look at the 0s:

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

For c_1, let's look at the 0s:

- SUN 000, L=1 → $d_2 + d_1 + d_0 + L'$
- MON 001, L=1 → $d_2 + d_1 + d_0' + L'$
- TUE 010, L=1 → $d_2 + d_1' + d_0 + L'$
- WED 011, L=1 → $d_2 + d_1' + d_0' + L'$
- FRI 101, L=0 → $d_2' + d_1 + d_0' + L$
- SAT 110 → $d_2' + d_1' + d_0$

No matter what L is, we always say it's 1.
So, we don't need L in the expression.

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

For c_1, let's look at the 0s:

$d_2 + d_1 + d_0 + L'$

$d_2 + d_1 + d_0' + L'$

$d_2 + d_1' + d_0 + L'$

$d_2 + d_1' + d_0' + L'$

$d_2' + d_1 + d_0' + L$

$d_2' + d_1' + d_0$

How do we combine them?

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

For c_1, let's look at the 0s:

- SUN 000, L = 1 → $d_2 + d_1 + d_0 + L'$
- MON 001, L = 1 → $d_2 + d_1 + d_0' + L'$
- TUE 010, L = 1 → $d_2 + d_1' + d_0 + L'$
- WED 011, L = 1 → $d_2 + d_1' + d_0' + L'$
- FRI 101, L = 0 → $d_2' + d_1 + d_0' + L$
- SAT 110, L = - → $d_2' + d_1' + d_0$

$$c_1 = (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')(d_2 + d_1' + d_0 + L')(d_2 + d_1' + d_0' + L')(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)$$

Truth Table to Logic (Part 3)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0
-	111	-	1	0	0	0

$$c_1 = (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')$$
$$(d_2 + d_1' + d_0 + L')(d_2 + d_1' + d_0' + L')$$
$$(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)$$

$$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$$

$$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$$

Is c_1 still in CNF form?

Yes, but not canonical CNF

Truth Table to Logic (Part 4)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

$$c_1 = (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')(d_2 + d_1' + d_0 + L')(d_2 + d_1' + d_0' + L')(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)$$

$$c_2 = d_2' \bullet d_1 \bullet d_0' \bullet L + d_2' \bullet d_1 \bullet d_0 \bullet L$$

$$c_3 = d_2' \bullet d_1' \bullet d_0' \bullet L + d_2' \bullet d_1' \bullet d_0 \bullet L$$

Finally, we do c_0:

Truth Table to Logic (Part 4)

	$d_2d_1d_0$	L	c_0	c_1	c_2	c_3
SUN	000	0	0	1	0	0
SUN	000	1	0	0	0	1
MON	001	0	0	1	0	0
MON	001	1	0	0	0	1
TUE	010	0	0	1	0	0
TUE	010	1	0	0	1	0
WED	011	0	0	1	0	0
WED	011	1	0	0	1	0
THU	100	-	0	1	0	0
FRI	101	0	1	0	0	0
FRI	101	1	0	1	0	0
SAT	110	-	1	0	0	0

$c_1 = (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')(d_2 + d_1' + d_0 + L')(d_2 + d_1' + d_0' + L')(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)$

$c_2 = d_2' \cdot d_1 \cdot d_0' \cdot L + d_2' \cdot d_1 \cdot d_0 \cdot L$

$c_3 = d_2' \cdot d_1' \cdot d_0' \cdot L + d_2' \cdot d_1' \cdot d_0 \cdot L$

Finally, we do c_0:

FRI 101 0 → $d_2 \cdot d_1' \cdot d_0 \cdot L'$

SAT 110 - → $d_2 \cdot d_1 \cdot d_0'$

Truth Table to Logic (Part 4)

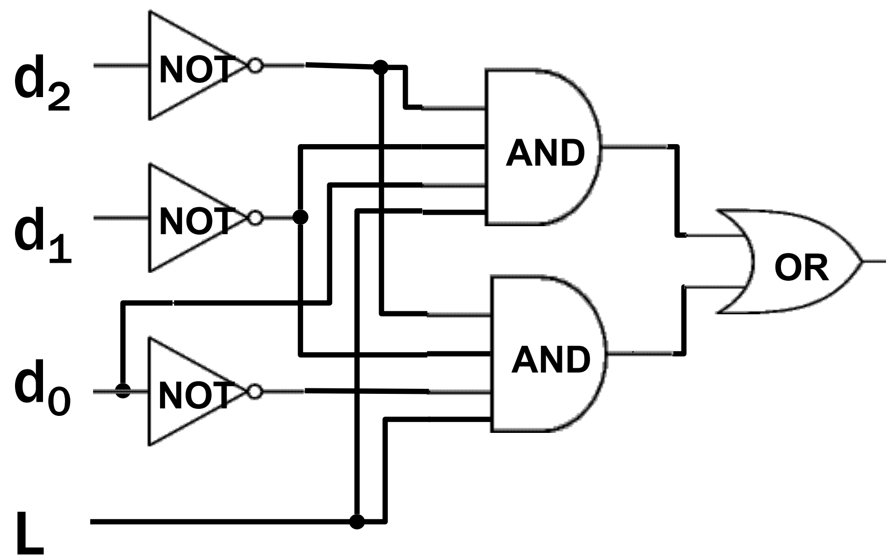
$c_0 = d_2 \bullet d_1' \bullet d_0 \bullet L' + d_2 \bullet d_1 \bullet d_0'$

$c_1 = (d_2 + d_1 + d_0 + L')(d_2 + d_1 + d_0' + L')(d_2 + d_1' + d_0 + L')(d_2 + d_1' + d_0' + L')(d_2' + d_1 + d_0' + L)(d_2' + d_1' + d_0)$

$c_2 = d_2' \bullet d_1 \bullet d_0' \bullet L + d_2' \bullet d_1 \bullet d_0 \bullet L$

$c_3 = d_2' \bullet d_1' \bullet d_0' \bullet L + d_2' \bullet d_1' \bullet d_0 \bullet L$

Here's c_3 as a circuit:



Mapping Truth Tables to Logic Gates

Given a truth table:

1. Write the output in a table
2. Write the Boolean expression
3. Draw as gates
4. Map to available gates

Mapping Truth Tables to Logic Gates

Given a truth table:

1. Write the output in a table
2. Write the Boolean expression
3. Draw as gates
4. Map to available gates

This will give us *some* circuit.
But is it the <u>best</u> circuit?

Equivalence

Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

a	***b***	**…**	**T**	**F**
T	T	…	**T**	**F**
F	T	…	**T**	**F**
T	F	…	**T**	**F**
F	F	…	**T**	**F**
…	…	…	…	…

Tautologies!

Terminology: A compound proposition is a…

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$p \vee \neg p$

$p \oplus p$

$(p \rightarrow r) \wedge p$

Tautologies!

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- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

$p \vee \neg p$

This is a tautology. It's called the "law of the excluded middle".
If p is true, then $p \vee \neg p$ is true. If p is false, then $p \vee \neg p$ is true.

$p \oplus p$

This is a contradiction. It's always false no matter what truth value p takes on.

$(p \rightarrow r) \wedge p$

This is a contingency. When p=T, r=T, $(T \rightarrow T) \wedge T$ is true.
When p=T, r=F, $(T \rightarrow F) \wedge T$ is false.

Tautologies!

Terminology: A compound proposition is a…

- *Tautology* if it is always true
- *Contradiction* if it is always false
- *Contingency* if it can be either true or false

SAT Problem: is it not a contradiction?

- every row is F in a contradiction
- not a contradiction means *some* row is T

Logical Equivalence

A = B means A and B are the same thing written twice:

- $p \wedge r = p \wedge r$

- $p \wedge r \neq r \wedge p$

Logical Equivalence

A = B means A and B are the same thing written twice:

- $p \wedge r = p \wedge r$

  These are equal, because they are character-for-character identical.

- $p \wedge r \neq r \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

in more detail, "=" means same parse tree (see week 8), so we can ignore differences in whitespace etc.

Logical Equivalence

A = B means A and B are the same thing written twice:

- $p \wedge r = p \wedge r$

  These are equal, because they are character-for-character identical.

- $p \wedge r \neq r \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

A $\equiv$ B means A and B have identical truth values:

- $p \wedge r \equiv p \wedge r$
- $p \wedge r \equiv r \wedge p$
- $p \wedge r \not\equiv r \vee p$

Logical Equivalence

A = B means A and B are the same thing written twice:

- $p \wedge r = p \wedge r$

  These are equal, because they are character-for-character identical.

- $p \wedge r \neq r \wedge p$

  These are NOT equal, because they are different sequences of characters. They "mean" the same thing though.

A ≡ B means A and B have identical truth values:

- $p \wedge r \equiv p \wedge r$

  Two formulas that are **equal** also are equivalent.

- $p \wedge r \equiv r \wedge p$

  These two formulas have the same truth table!

- $p \wedge r \not\equiv r \vee p$

  When p=T and r=F, $p \wedge r$ is false, but $p \vee r$ is true!

$A \leftrightarrow B$ vs. $A \equiv B$

$A \leftrightarrow B$ is a **proposition** that may be true or false depending on the truth values of A and B.

$A \equiv B$ is an **assertion** over all possible truth values that A and B always have the same truth values.

$A \equiv B$ and $(A \leftrightarrow B) \equiv T$ have the same meaning as does "$A \leftrightarrow B$ is a tautology"

Logical Equivalence $A \equiv B$

$A \equiv B$ is an assertion that ***two propositions*** A and B always have the same truth values.

$A \equiv B$ and $(A \leftrightarrow B) \equiv T$ have the same meaning.

$p \wedge r \equiv r \wedge p$

p	r	$p \wedge r$	$r \wedge p$	$(p \wedge r) \leftrightarrow (r \wedge p)$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

De Morgan's Laws

$$\neg(p \wedge r) \equiv \neg p \vee \neg r$$
$$\neg(p \vee r) \equiv \neg p \wedge \neg r$$

Negate the statement:

"My code compiles or there is a bug."

To negate the statement,

ask "when is the original statement false".

It's false when not(my code compiles) AND not(there is a bug).

Translating back into English, we get:

My code doesn't compile and there is not a bug.

De Morgan's Laws

Example: $\neg(p \wedge r) \equiv \neg p \vee \neg r$

p	r	$\neg p$	$\neg r$	$\neg p \vee \neg r$	$p \wedge r$	$\neg(p \wedge r)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

De Morgan's Laws

$$\neg(p \land r) \equiv \neg p \lor \neg r$$

$$\neg(p \lor r) \equiv \neg p \land \neg r$$

```
if (!(front != null && value > front.data)) {
    front = new ListNode(value, front);
} else {
    ListNode current = front;
    while (current.next != null && current.next.data < value))
        current = current.next;
    current.next = new ListNode(value, current.next);
}
```

De Morgan's Laws

$$\neg(p \wedge r) \equiv \neg p \vee \neg r$$

$$\neg(p \vee r) \equiv \neg p \wedge \neg r$$

```
!(front != null && value > front.data)
```

$$\equiv$$

```
front == null || value <= front.data
```

Law of Implication

$$p \rightarrow r \equiv \neg p \vee r$$

p	r	$p \rightarrow r$	$\neg p$	$\neg p \vee r$
T	T			
T	F			
F	T			
F	F			

Law of Implication

$$p \rightarrow r \equiv \neg p \vee r$$

p	r	$p \rightarrow r$	$\neg p$	$\neg p \vee r$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Biconditional: $p \leftrightarrow r$

- *p* if and only if *r* (*p* iff *r*)
- *p* implies *r* and *r* implies *p*
- *p* is necessary and sufficient for *r*

p	r	$p \leftrightarrow r$	$p \rightarrow r$	$r \rightarrow p$	$(p \rightarrow r) \wedge (r \rightarrow p)$
T	T	T	T	T	
T	F	F	F	T	
F	T	F	T	F	
F	F	T	T	T	

Biconditional: $p \leftrightarrow r$

- *p* if and only if *r* (*p* iff *r*)
- *p* implies *r* and *r* implies *p*
- *p* is necessary and sufficient for *r*

p	r	$p \leftrightarrow r$	$p \rightarrow r$	$r \rightarrow p$	$(p \rightarrow r) \wedge (r \rightarrow p)$
T	**T**	**T**	**T**	**T**	**T**
T	**F**	**F**	**F**	**T**	**F**
F	**T**	**F**	**T**	**F**	**F**
F	**F**	**T**	**T**	**T**	**T**

Some Familiar Properties of Arithmetic

- $x + y = y + x$ **(Commutativity)**

- $x \cdot (y + z) = x \cdot y + x \cdot z$ **(Distributivity)**

- $(x + y) + z = x + (y + z)$ **(Associativity)**

Important Equivalences

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- Commutative
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- Associative
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- Negation
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

Some Familiar Properties of Arithmetic

- $x \cdot 1 = x$ (Identity)
- $x + 0 = x$

- $x \cdot 0 = 0$ (Domination)

Important Equivalences

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- Commutative
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- Associative
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- Negation
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

Some Familiar Properties of Arithmetic

- Usual properties hold under relabeling:
  - 0, 1 becomes F, T
  - "+" becomes "$\vee$"
  - " $\cdot$ " becomes "$\wedge$"
- But there are some new facts:
  - Distributivity works for both "$\wedge$" and "$\vee$"
  - Domination works with T
- There are some other facts specific to logic…

Important Equivalences

- Identity
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- Domination
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- Idempotent
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- Commutative
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- Associative
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- Distributive
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- Absorption
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- Negation
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

Important Equivalences

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

Using Equivalences

- Note that p, q, and r can be **any** propositions (not just atomic propositions)

- Ex: $(r \rightarrow s) \wedge (\neg t) \equiv (\neg t) \wedge (r \rightarrow s)$

  – apply commutativity: $p \wedge q \equiv q \wedge p$
  with $p := r \rightarrow s$
  and $q := \neg t$

One more easy equivalence

Double Negation

$$p \equiv \neg\neg p$$

p	$\neg p$	$\neg\neg p$
T	F	T
F	T	F

Uses of Equivalence

- Working with logical formulas
  - simplification
- Working with circuits
  - hardware verification
- Software applications
  - query optimization and caching
  - artificial intelligence
  - program verification

Computing Equivalence

Given two propositions, can we write an algorithm to determine if they are equivalent?

What is the runtime of our algorithm?

Computing Equivalence

Given two propositions, can we write an algorithm to determine if they are equivalent?

Yes! Generate the truth tables for both propositions and check if they are the same for every entry.

What is the runtime of our algorithm?

Every atomic proposition has two possibilities (T, F). If there are n atomic propositions, there are 2^n rows in the truth table.

Another approach: Equivalence Chains

To show A is equivalent to B

– Apply a series of logical equivalences to sub-expressions to convert A to B

To show A is a tautology

– Apply a series of logical equivalences to sub-expressions to convert A to **T**

Another approach: Equivalence Chains

To show A is equivalent to B

– Apply a series of logical equivalences to sub-expressions to convert A to B

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general equivalence proof looks like:

$$p \vee (p \wedge p) \equiv$$
$$\equiv p$$

Another approach: Logical Equivalences

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \rightarrow q \equiv \neg p \vee q$$

Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general equivalence proof looks like:

$$p \vee (p \wedge p) \equiv$$
$$\equiv p$$

Logical Equivalences

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$

- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
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$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

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Contrapositive

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

Biconditional

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$p \vee (p \wedge p)$", and B be "p".

Our general equivalence proof looks like:

$p \vee (p \wedge p) \equiv p \vee p$ — **Idempotent**

$\equiv p$ — **Idempotent**

Logical Equivalences

To show A is a tautology

– Apply a series of logical equivalences to sub-expressions to convert A to **T**

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general equivalence proof looks like:

$$\begin{aligned} \neg p \vee (p \vee p) &\equiv \\ &\equiv \\ &\equiv \mathbf{T} \end{aligned}$$

Logical Equivalences

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
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  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
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  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
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  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \to q \equiv \neg p \vee q$$

Contrapositive

$$p \to q \equiv \neg q \to \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general equivalence proof looks like:

$$\neg p \vee (p \vee p) \equiv$$
$$\equiv$$
$$\equiv \mathbf{T}$$

Logical Equivalences

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
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- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
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  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
- **Negation**
  - $p \vee \neg p \equiv \mathrm{T}$
  - $p \wedge \neg p \equiv \mathrm{F}$

De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$
$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Law of Implication

$$p \to q \equiv \neg p \vee q$$

Contrapositive

$$p \to q \equiv \neg q \to \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \to q) \wedge (q \to p)$$

Double Negation

$$p \equiv \neg\neg p$$

Example:

Let A be "$\neg p \vee (p \vee p)$".

Our general equivalence proof looks like:

$\neg p \vee (p \vee p)$	$\equiv \neg p \vee p$	Idempotent
	$\equiv p \vee \neg p$	Commutative
	$\equiv \mathbf{T}$	Negation

Prove these propositions are equivalent: Option 1

Prove: $p \wedge (p \rightarrow r) \equiv p \wedge r$

Make a Truth Table and show:

$$(p \wedge (p \rightarrow r)) \leftrightarrow (p \wedge r) \equiv \mathbf{T}$$

p	r	$p \rightarrow r$	$(p \wedge (p \rightarrow r))$	$p \wedge r$	$(p \wedge (p \rightarrow r)) \leftrightarrow (p \wedge r)$
T	T				
T	F				
F	T				
F	F				

Prove these propositions are equivalent: Option 1

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Make a Truth Table and show:

$$(p \wedge (p \rightarrow r)) \leftrightarrow (p \wedge r) \equiv \mathbf{T}$$

p	r	$p \rightarrow r$	$(p \wedge (p \rightarrow r))$	$p \wedge r$	$(p \wedge (p \rightarrow r)) \leftrightarrow (p \wedge r)$
T	T	T	T	T	T
T	F	F	F	F	T
F	T	T	F	F	T
F	F	T	F	F	T

Prove these propositions are equivalent: Option 2

Prove: $p \wedge (p \rightarrow r) \equiv p \wedge r$

$$p \wedge (p \rightarrow r) \equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv p \wedge r$$

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
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$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Law of Implication

$p \rightarrow q \equiv \neg p \vee q$

Contrapositive

$p \rightarrow q \equiv \neg q \rightarrow \neg p$

Biconditional

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Double Negation

$p \equiv \neg\neg p$

Prove these propositions are equivalent: Option 2

Prove: $p \wedge (p \rightarrow r) \equiv p \wedge r$

$p \wedge (p \rightarrow r) \equiv p \wedge (\neg p \vee r)$	Law of Implication
$\equiv (p \wedge \neg p) \vee (p \wedge r)$	Distributive
$\equiv \mathbf{F} \vee (p \wedge r)$	Negation
$\equiv (p \wedge r) \vee \mathbf{F}$	Commutative
$\equiv p \wedge r$	Identity

- **Identity**
  - $p \wedge \mathrm{T} \equiv p$
  - $p \vee \mathrm{F} \equiv p$
- **Domination**
  - $p \vee \mathrm{T} \equiv \mathrm{T}$
  - $p \wedge \mathrm{F} \equiv \mathrm{F}$
- **Idempotent**
  - $p \vee p \equiv p$
  - $p \wedge p \equiv p$
- **Commutative**
  - $p \vee q \equiv q \vee p$
  - $p \wedge q \equiv q \wedge p$
- **Associative**
  - $(p \vee q) \vee r \equiv p \vee (q \vee r)$
  - $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
- **Distributive**
  - $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
  - $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
- **Absorption**
  - $p \vee (p \wedge q) \equiv p$
  - $p \wedge (p \vee q) \equiv p$
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  - $p \vee \neg p \equiv \mathrm{T}$
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Contrapositive

$$p \rightarrow q \equiv \neg q \rightarrow \neg p$$

Biconditional

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

Double Negation

$$p \equiv \neg\neg p$$

Prove this is a Tautology: Option 1

$$(p \wedge r) \rightarrow (r \vee p)$$

Make a Truth Table and show:

$$(p \wedge r) \rightarrow (r \vee p) \equiv \mathbf{T}$$

$\boldsymbol{p}$	$\boldsymbol{r}$	$\boldsymbol{p \wedge r}$	$\boldsymbol{r \vee p}$	$\boldsymbol{(p \wedge r) \rightarrow (r \vee p)}$
T	**T**			
T	**F**			
F	**T**			
F	**F**			

Prove this is a Tautology: Option 1

$$(p \wedge r) \rightarrow (r \vee p)$$

Make a Truth Table and show:

$$(p \wedge r) \rightarrow (r \vee p) \equiv \mathbf{T}$$

$\boldsymbol{p}$	$\boldsymbol{r}$	$\boldsymbol{p \wedge r}$	$\boldsymbol{r \vee p}$	$\boldsymbol{(p \wedge r) \rightarrow (r \vee p)}$
T	**T**	**T**	**T**	**T**
T	**F**	**F**	**T**	**T**
F	**T**	**F**	**T**	**T**
F	**F**	**F**	**F**	**T**

Prove this is a Tautology: Option 2

$$(p \wedge r) \rightarrow (r \vee p)$$

Use a series of equivalences like so:

$$(p \wedge r) \rightarrow (r \vee p) \equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv$$
$$\equiv \ \mathrm{T}$$

Associative
- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Distributive
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Absorption
- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

Negation
- $p \vee \neg p \equiv \mathrm{T}$
- $p \wedge \neg p \equiv \mathrm{F}$

Identity
- $p \wedge \mathrm{T} \equiv p$
- $p \vee \mathrm{F} \equiv p$

Domination
- $p \vee \mathrm{T} \equiv \mathrm{T}$
- $p \wedge \mathrm{F} \equiv \mathrm{F}$

Idempotent
- $p \vee p \equiv p$
- $p \wedge p \equiv p$

Commutative
- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

Prove this is a Tautology: Option 2

$$(p \wedge r) \rightarrow (r \vee p)$$

Use a series of equivalences like so:

$(p \wedge r) \rightarrow (r \vee p) \equiv \neg(p \wedge r) \vee (r \vee p)$		Law of Implication
$\equiv (\neg p \vee \neg r) \vee (r \vee p)$		De Morgan
$\equiv \neg p \vee (\neg r \vee (r \vee p))$		Associative
$\equiv \neg p \vee ((\neg r \vee r) \vee p)$		Associative
$\equiv \neg p \vee (p \vee (\neg r \vee r))$		Commutative
$\equiv (\neg p \vee p) \vee (\neg r \vee r)$		Associative
$\equiv (p \vee \neg p) \vee (r \vee \neg r)$		Commutative (twice)
$\equiv \mathbf{T} \vee \mathbf{T}$		Negation (twice)
$\equiv \mathbf{T}$		Domination/Identity

Associative
- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

Distributive
- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

Absorption
- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

Negation
- $p \vee \neg p \equiv \mathrm{T}$
- $p \wedge \neg p \equiv \mathrm{F}$

Identity
- $p \wedge \mathrm{T} \equiv p$
- $p \vee \mathrm{F} \equiv p$

Domination
- $p \vee \mathrm{T} \equiv \mathrm{T}$
- $p \wedge \mathrm{F} \equiv \mathrm{F}$

Idempotent
- $p \vee p \equiv p$
- $p \wedge p \equiv p$

Commutative
- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

Chains of Equivalence/Tautology

- Not smaller than truth tables when there are only a few propositional variables…
- …but usually ***much shorter*** than truth table proofs when there are many propositional variables
- A big advantage will be that we can extend them to a more in-depth understanding of logic for which truth tables don't apply.

Predicate Logic

Predicate Logic

- **Propositional Logic**
  - Allows us to analyze complex propositions in terms of their simpler constituent parts (a.k.a. atomic propositions) joined by connectives

- **Predicate Logic**
  - Lets us analyze them at a deeper level by expressing how those propositions depend on the objects they are talking about

"All positive integers x, y, and z satisfy $x^3 + y^3 \neq z^3$."

Predicate Logic

Adds two key notions to propositional logic

- Predicates

- Quantifiers

Predicates

Predicate

– **A function that returns a truth value, e.g.,**

Cat(x) := “x is a cat”

Prime(x) := “x is prime”

HasTaken(x, y) := “student x has taken course y”

LessThan(x, y) := “x < y”

Sum(x, y, z) := “x + y = z”

GreaterThan5(x) := “x > 5”

HasNChars(s, n) := “string s has length n”

Predicates can have varying numbers of arguments and input types.

Domain of Discourse

For ease of use, we define one "type"/"domain" that we work over. This non-empty set of objects is called the "**domain of discourse**".

For each of the following, what might the domain be?

(1) "x is a cat", "x barks", "x ruined my couch"

"mammals" or "sentient beings" or "cats and dogs" or ...

(2) "x is prime", "x = 0", "x > 0", "x is a power of two"

"numbers" or "integers" or "non-negative integers" or ...

(3) "student x has taken course y" "x is a pre-req for z"

"students and courses" or "university entities" or ...

Quantifiers

We use ***quantifiers*** to talk about collections of objects.

$\forall x\ P(x)$

$P(x)$ **is true for every** x **in the domain**

read as "for all x, P of x"

$\exists x\ P(x)$

There is an x **in the domain for which** $P(x)$ **is true**

read as "there exists x, P of x"

Quantifiers

We use ***quantifiers*** to talk about collections of objects.

Universal Quantifier (“for all”): $\forall x\ P(x)$

$P(x)$ **is true for every** x **in the domain**

read as “for all x, P of x”

Examples: Are these true?

- $\forall$x **Odd(x)**
- $\forall$x **LessThan4(x)**

Quantifiers

We use ***quantifiers*** to talk about collections of objects.

Universal Quantifier ("for all"): $\forall x\ P(x)$

$P(x)$ **is true for every** x **in the domain**

read as "for all x, P of x"

Examples:

Are these true? It depends on the domain. For example:

	{1, 3, -1, -27}	Integers	Odd Integers
• $\forall x$ **Odd(x)**	True	False	True
• $\forall x$ **LessThan4(x)**	True	False	False

Quantifiers

We use ***quantifiers*** to talk about collections of objects.

Existential Quantifier ("exists"): $\exists x\ P(x)$

There is an x in the domain for which P(x) is true

read as "there exists x, P of x"

Examples: Are these true?

- $\exists x$ Odd(x)
- $\exists x$ LessThan4(x)

Quantifiers

We use ***quantifiers*** to talk about collections of objects.

Existential Quantifier ("exists"): $\exists x\ P(x)$

There is an x in the domain for which $P(x)$ is true

read as "there exists x, P of x"

Examples:

Are these true? It depends on the domain. For example:

	{1, 3, -1, -27}	Integers	Positive Multiples of 5
• $\exists$x Odd(x)	True	True	True
• $\exists$x LessThan4(x)	True	True	False

Statements with Quantifiers

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) := "x is even"	Greater(x, y) := "x > y"
Odd(x) := "x is odd"	Equal(x, y) := "x = y"
Prime(x) := "x is prime"	Sum(x, y, z) := "x + y = z"

Determine the truth values of each of these statements:

$\exists x$ Even(x)	**T**	**e.g. 2, 4, 6, ...**
$\forall x$ Odd(x)	**F**	**e.g. 2, 4, 6, ...**
$\forall x$ (Even(x) $\vee$ Odd(x))	**T**	**every integer is either even or odd**
$\exists x$ (Even(x) $\wedge$ Odd(x))	**F**	**no integer is both even and odd**
$\forall x$ Greater(x+1, x)	**T**	**adding 1 makes a bigger number**
$\exists x$ (Even(x) $\wedge$ Prime(x))	**T**	**Even(2) is true and Prime(2) is true**

Syntax of Quantifiers

		Precedence
Negation (not)	$\neg p$	highest
For all	$\forall x\ P(x)$	
Exists	$\exists x\ P(x)$	
Conjunction (and)	$p \wedge q$	
Disjunction (or)	$p \vee q$	
Exclusive Or	$p \oplus q$	
Implication	$p \longrightarrow r$	
Biconditional	$p \leftrightarrow q$	lowest

$\forall x\ \neg P(x) \wedge Q(y)$ **means** $(\forall x\ \neg P(x)) \wedge Q(y)$

Syntax of Quantifiers

Negation (not)	$\neg p$
For all	$\forall x\, P(x)$
Exists	$\exists x\, P(x)$
Conjunction (and)	$p \wedge q$
Disjunction (or)	$p \vee q$
Exclusive Or	$p \oplus q$
Implication	$p \longrightarrow r$
Biconditional	$p \leftrightarrow q$

Not everyone uses this convention!

We will try to accommodate both approaches...

Syntax of Quantifiers (Two Conventions)

Negation (not)	$\neg p$	highest
For all	$\forall x\ P(x)$	
Exists	$\exists x\ P(x)$	
Conjunction (and)	$p \land q$	
Disjunction (or)	$p \lor q$	
Exclusive Or	$p \oplus q$	
Implication	$p \longrightarrow r$	
Biconditional	$p \longleftrightarrow q$	
For all	$\forall x, P(x)$	
Exists	$\exists x, P(x)$	lowest

Syntax of Quantifiers (Two Conventions)

Negation (not)	$\neg p$
For all	$\forall x\ P(x)$
Exists	$\exists x\ P(x)$
Conjunction (and)	$p \wedge q$
Disjunction (or)	$p \vee q$
Exclusive Or	$p \oplus q$
Implication	$p \longrightarrow r$
Biconditional	$p \longleftrightarrow q$
For all	$\forall x, P(x)$
Exists	$\exists x, P(x)$

$\forall x, \neg P(x) \wedge Q(y)$

means

$\forall x\ (\neg P(x) \wedge Q(y))$

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) := "x is even"	Greater(x, y) := "x > y"
Odd(x) := "x is odd"	Equal(x, y) := "x = y"
Prime(x) := "x is prime"	Sum(x, y, z) := "x + y = z"

Translate the following statements to English

$\forall x\ \exists y\ \text{Greater}(y, x)$

For every positive integer x, there is a positive integer y, such that y > x.

$\exists y\ \forall x\ \text{Greater}(y, x)$

There is a positive integer y such that, for every pos. int. x, we have y > x.

$\forall x\ \exists y\ (\text{Prime}(y) \wedge \text{Greater}(y, x))$

For every positive integer x, there is a pos. int. y such that y > x and y is prime.

$\forall x\ (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

For each positive integer x, if x is prime, then x = 2 or x is odd.

$\exists x\ \exists y\ (\text{Prime}(x) \wedge \text{Prime}(y) \wedge \text{Sum}(x, 2, y))$

There exist positive integers x and y such that x and y are prime and x + 2 = y.

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

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Even(x) := “x is even”	Greater(x, y) := “x > y”
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There is a positive integer y such that, for every pos. int. x, we have y > x.

$\forall x\ \exists y\ (\text{Prime}(y) \wedge \text{Greater}(y, x))$

For every positive integer x, there is a pos. int. y such that y > x and y is prime.

Statements with Quantifiers (Natural Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) := "x is even"	Greater(x, y) := "x > y"
Odd(x) := "x is odd"	Equal(x, y) := "x = y"
Prime(x) := "x is prime"	Sum(x, y, z) := "x + y = z"

Translate the following statements to English

$\forall x \exists y$ Greater(y, x)

For every positive integer, there is some larger positive integer.

$\exists y \forall x$ Greater(y, x)

There is a positive integer that is larger than every other positive integer.

$\forall x \exists y$ (Prime(y) $\wedge$ Greater(y, x))

For every positive integer, there is a prime that is larger.

Sound more natural without introducing variable names

Statements with Quantifiers (Literal Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) := "x is even"	Greater(x, y) := "x > y"
Odd(x) := "x is odd"	Equal(x, y) := "x = y"
Prime(x) := "x is prime"	Sum(x, y, z) := "x + y = z"

Translate the following statements to English

$\exists x\, \exists y\, (\text{Prime}(x) \wedge \text{Prime}(y) \wedge \text{Sum}(x, 2, y))$

There exist positive integers x and y such that x and y are prime and x + 2 = y.

$\forall x\, (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

For each positive integer x, if x is prime, then x = 2 or x is odd.

Statements with Quantifiers (Natural Translations)

Domain of Discourse
Positive Integers

Predicate Definitions	
Even(x) := "x is even"	Greater(x, y) := "x > y"
Odd(x) := "x is odd"	Equal(x, y) := "x = y"
Prime(x) := "x is prime"	Sum(x, y, z) := "x + y = z"

Translate the following statements to English

$\exists x\, \exists y\, (\text{Prime}(x) \wedge \text{Prime}(y) \wedge \text{Sum}(x, 2, y))$

There exist primes x and y such that x + 2 = y.

There exist prime numbers that are 2 apart.

$\forall x\, (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

Statements with Quantifiers (Natural Translations)

Domain of Discourse
Positive Integers

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Even(x) := "x is even"	Greater(x, y) := "x > y"
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Translate the following statements to English

$\exists x\, \exists y\, (\text{Prime}(x) \wedge \text{Prime}(y) \wedge \text{Sum}(x, 2, y))$

There exist primes x and y such that x + 2 = y.

There exist prime numbers that are 2 apart.

$\forall x\, (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \vee \text{Odd}(x)))$

Every prime number is either 2 or odd.

Spot the domain restriction patterns

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) := “x is a cat”
Red(x) := “x is red”
LikesTofu(x) := “x likes tofu”

“All red cats like tofu”

$$\forall x\ ((\text{Red}(x) \wedge \text{Cat}(x)) \rightarrow \text{LikesTofu}(x))$$

“Some red cats don’t like tofu”

$$\exists y\ ((\text{Red}(y) \wedge \text{Cat}(y)) \wedge \neg\text{LikesTofu}(y))$$

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) := “x is a cat”
Red(x) := “x is red”
LikesTofu(x) := “x likes tofu”

When putting two predicates together like this, we use an “and”.

“All Red cats like tofu”

When restricting to a smaller domain in a “for all” we use **implication.**

“Some red cats don’t like tofu”

When restricting to a smaller domain in an “exists” we use **and.**

“Some” means “there exists”.

English to Predicate Logic

Domain of Discourse
Mammals

Predicate Definitions
Cat(x) := "x is a cat"
Red(x) := "x is red"
LikesTofu(x) := "x likes tofu"

"All Red cats like tofu"

"Red cats like tofu"

When there's no leading quantification, it *usually* means "for all".

"Some red cats don't like tofu"

"A red cat doesn't like tofu"

"A" means "there exists".

Statements with Quantifiers (Natural Translations)

Translations often (not always) sound more natural if we

1. Notice "domain restriction" patterns

$\forall x\ (\text{Prime}(x) \rightarrow (\text{Equal}(x, 2) \lor \text{Odd}(x)))$

Every prime number is either 2 or odd.

2. Avoid introducing *unnecessary* variable names

$\forall x\ \exists y\ \text{Greater}(y, x)$

For every positive integer, there is some larger positive integer.

3. Can sometimes drop "all" or "there is"

$\neg\ \exists x\ (\text{Even}(x) \land \text{Prime}(x) \land \text{Greater}(x, 2))$

No even prime is greater than 2.

More English Ambiguity

Implicit quantifiers in English are often **ambiguous**

Three people that are all friends can form a raiding party $\forall$

Three people that I know are all friends with Bill Gates $\exists$

Formal logic removes this ambiguity

- quantifiers can always be specified
- unquantified variables that are not known constants (e.g, π) are implicitly $\forall$–quantified (mostly… one special case coming later)

Negations of Quantifiers

Domain of Discourse
Fruits

Predicate Definitions
Purple(x) := “x is a purple fruit”

(*) $\forall$x Purple(x) (“All fruits are purple”)

What is the negation of (*)?

(a) “there exists a purple fruit”
(b) “there exists a non-purple fruit”
(c) “all fruits are not purple”

Try your intuition! Which one seems right?

Negations of Quantifiers

Domain of Discourse
Fruits

Predicate Definitions
Purple(x) := "x is a purple fruit"

(*) $\forall$x Purple(x) ("All fruits are purple")

What is the negation of (*)?

(a) "there exists a purple fruit"
(b) "there exists a non-purple fruit"
(c) "all fruits are not purple"

Domain of Discourse
{plum, apple}

(*) Purple(plum) $\wedge$ Purple(apple)

(a) Purple(plum) $\vee$ Purple(apple)
(b) $\neg$ Purple(plum) $\vee$ $\neg$ Purple(apple)
(c) $\neg$ Purple(plum) $\wedge$ $\neg$ Purple(apple)

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$

$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x\, \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x\, \neg P(x)$$

There is no unicorn $\neg \exists x\, \text{Unicorn}(x)$

Every animal is not a unicorn $\forall x\, \neg \text{Unicorn}(x)$

These are **equivalent** but not **equal**

De Morgan's Laws for Quantifiers

$$\neg \forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg \exists x\, P(x) \equiv \forall x \neg P(x)$$

"There is no integer larger than every other integer"

$$\neg \exists x \forall y\ (x \geq y)$$
$$\equiv \forall x \neg \forall y\ (x \geq y)$$
$$\equiv \forall x\ \exists y \neg (x \geq y)$$
$$\equiv \forall x\ \exists y\ (y > x)$$

"For every integer, there is a larger integer"

These are **equivalent** but not **equal**

De Morgan's Laws for Quantifiers

$$\neg\forall x\, P(x) \equiv \exists x \neg P(x)$$
$$\neg\exists x\, P(x) \equiv \forall x \neg P(x)$$

"No even prime is greater than 2"

$$\neg\exists x\,(\text{Even}(x) \wedge \text{Prime}(x) \wedge \text{Greater}(x, 2))$$
$$\equiv \forall x\, \neg(\text{Even}(x) \wedge \text{Prime}(x) \wedge \text{Greater}(x, 2))$$
$$\equiv \forall x\, (\neg(\text{Even}(x) \wedge \text{Prime}(x)) \vee \neg\text{Greater}(x, 2))$$
$$\equiv \forall x\, (\neg(\text{Even}(x) \wedge \text{Prime}(x)) \vee \text{LessEq}(x, 2))$$
$$\equiv \forall x\, ((\text{Even}(x) \wedge \text{Prime}(x)) \rightarrow \text{LessEq}(x, 2))$$

"Every even prime is less than or equal to 2."

De Morgan's Laws for Quantifiers

We just saw that

$$\neg \exists x\, (P(x) \wedge R(x)) \equiv \forall x\, (P(x) \rightarrow \neg R(x))$$

Can similarly show that

$$\neg \forall x\, (P(x) \rightarrow R(x)) \equiv \exists x\, (P(x) \wedge \neg R(x))$$

De Morgan's Laws respect domain restrictions!
(It leaves them in place and only negates the other parts.)

Quantifiers in Java

- **Implementing quantifiers in Java…**

```
boolean forAll(Map<Integer, Boolean> P) {
  for (Integer x : P.keySet()) {
    if (!P.get(x)) return false;
  }
  return true;
}
```

$\forall x\ P(x)$

(Bound) variable names don't matter: $\forall x\ P(x) \equiv \forall a\ P(a)$

```
boolean exists(Map<Integer, Boolean> P) {
  for (Integer x : P.keySet()) {
    if (P.get(x)) return true;
  }
  return false;
}
```

$\exists x\ P(x)$

Scope of Quantifiers

$\exists x\ (P(x) \wedge Q(x))$ **vs.** $(\exists x\ P(x)) \wedge (\exists x\ Q(x))$

This one asserts P and Q of the *same* x.

This one asserts P and Q of potentially different x's.

Variables with the same name do not necessarily refer to the same object.

Scope of Quantifiers

Domain of Discourse
{1, 2, 3, …, 100}

Example: NotLargest(y) := $\exists$ x Greater (x, y)

$\equiv$ $\exists$ z Greater (z, y)

truth value:

doesn't depend on x **or** z **"bound variables"**

does depend on y **"free variable"**

Scope of Quantifiers

Domain of Discourse
{1, 2, 3, ..., 100}

Example: NotLargest(y) := $\exists$ x Greater (x, y)

$\equiv$ $\exists$ z Greater (z, y)

truth value:

doesn't depend on x or z "bound variables"

does depend on y "free variable"

quantifiers only act on free variables of the formula

$$\forall x\ \exists y\ (P(x,y) \rightarrow \forall x\ Q(y, x)))$$

Quantifier "Style"

$$\forall x\,(\exists y\,(P(x,y)\ \wedge\ \forall x\, Q(y, x)))$$

This isn't "wrong", it's just horrible style.
Don't confuse your reader by using the same variable multiple times…there are a lot of letters…

Nested Quantifiers

- **Bound variable names don't matter**

  $$\forall x\ \exists y\ P(x, y) \equiv \forall a\ \exists b\ P(a, b)$$

- **Positions of quantifiers can <u>sometimes</u> change**

  $$\forall x\ (Q(x) \wedge \exists y\ P(x, y)) \equiv \forall x\ \exists y\ (Q(x) \wedge P(x, y))$$

- **But: order is important...**

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) := "x ≥ y"

"There is a number greater than or equal to all numbers."

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

Quantifier Order Can Matter

Domain of Discourse
{1, 2, 3, 4}

Predicate Definitions
GreaterEq(x, y) := "x ≥ y"

"There is a number greater than or equal to all numbers."

$$\exists x\ \forall y\ \text{GreaterEq}(x, y)$$

"Every number has a number greater than or equal to it."

$$\forall y\ \exists x\ \text{GreaterEq}(x, y)$$

x \ y	1	2	3	4
1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

Quantifier Order Can Matter

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1	T	F	F	F
2	T	T	F	F
3	T	T	T	F
4	T	T	T	T

The purple statement requires **an entire row** to be true.
The red statement requires one entry in **each column** to be true.

Important: both include the case x = y

Different names does not imply different objects!

Quantification with Two Variables

expression	when true	when false
$\forall x \forall y P(x, y)$	Every pair is true.	At least one pair is false.
$\exists x \exists y P(x, y)$	At least one pair is true.	All pairs are false.
$\forall x \exists y P(x, y)$	We can find a specific y for each x. $(x_1, y_1), (x_2, y_2), (x_3, y_3)$	Some x doesn't have a corresponding y.
$\exists y \forall x P(x, y)$	We can find ONE y that works no matter what x is. $(x_1, y), (x_2, y), (x_3, y)$	For any candidate y, there is an x that it doesn't work for.