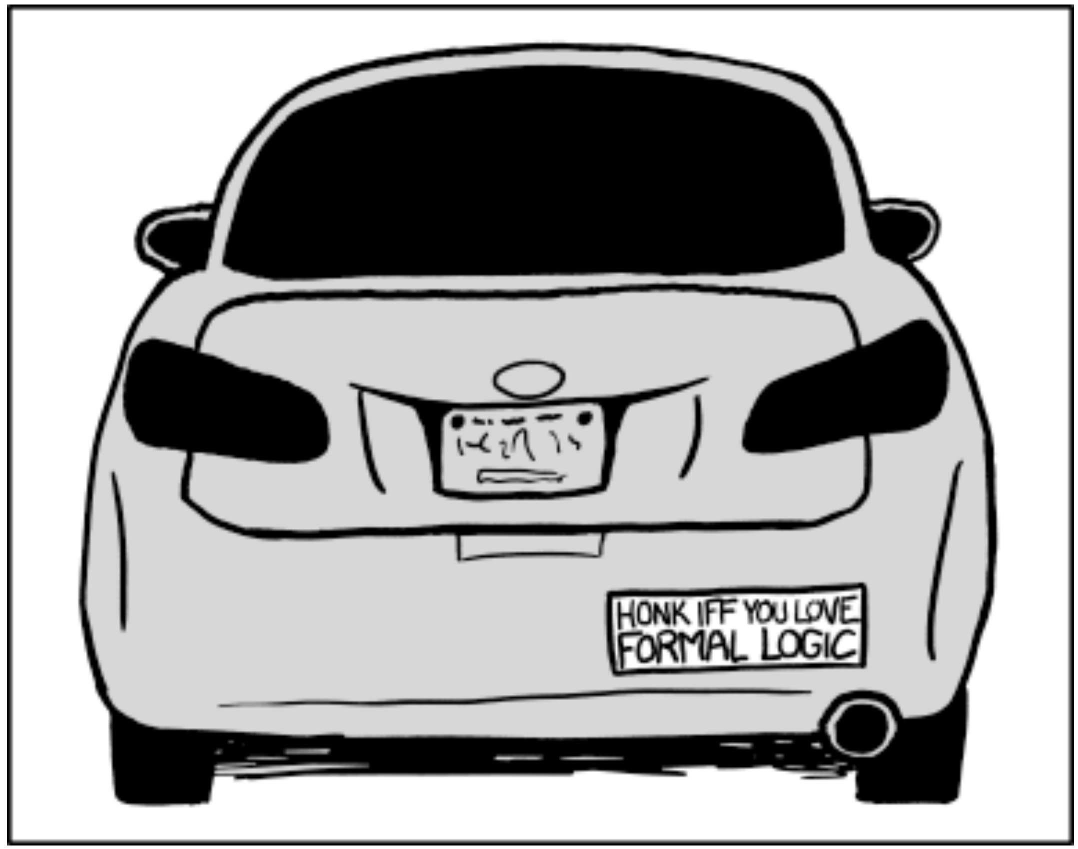
CSE 311: Foundations of Computing I

Topic 1: Propositional Logic



What is logic and why do we need it?

Logic is a language, like English or Java, with its own

- words and rules for combining words into sentences (syntax)
- ways to assign meaning to words and sentences (semantics)

Compared to English, Logic is more

- concise (useful)
- precise (critical!)

Importantly, Logic comes with its own formal toolkit

Why not use English?

– Turn right here…

  Does “right” mean the direction or now?

– We saw her duck

  Does “duck” mean the animal or crouch down?

– Buffalo buffalo Buffalo buffalo buffalo buffalo Buffalo buffalo

  This means “Bison from Buffalo, that bison from Buffalo bully, themselves bully bison from Buffalo.

Natural languages can be unclear / imprecise

Propositions: building blocks of logic

A ***proposition*** is a statement that

– is "well-formed"

– is either true or false

Propositions: building blocks of logic

A ***proposition*** is a statement that

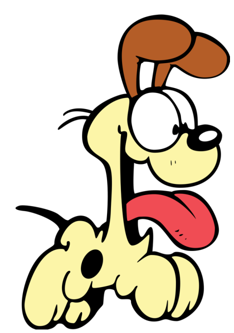
- is “well-formed”
- is either true or false

Garfield is a mammal and Garfield is a cat

true

Odie is a mammal and Odie is a cat

false



Are These Propositions?

$2 + 2 = 5$

This is a proposition. It's okay for propositions to be false.

$x + 2 = 5389$, where x is my PIN number

This is a proposition. We don't need to know what x is.

Akjsdf!

Not a proposition because it's gibberish.

Who are you?

This is a question which means it doesn't have a truth value.

Every positive even integer can be written as the sum of two primes.

This is a proposition. We don't know if it's true or false, but we know it's one of them!

Propositions

We need a way of talking about *arbitrary* ideas...

Propositional Variables: $p, q, r, s, \ldots$

Truth Values:

- **T** for **true**
- **F** for **false**

Familiar from Java

Java `boolean` represents a truth value

– constants `true` and `false`

– variables hold *unknown* values

Operators calculate new values from given ones

– unary: not (`!`)

– binary: and (&&), or (`||`)

Logical Connectives

Negation (not)	$\neg p$
Conjunction (and)	$p \wedge q$
Disjunction (or)	$p \vee q$

con	with	p with q (i.e., both)
dis-	apart from	not necessarily both

Logical Connectives

Negation (not)	$\neg p$
Conjunction (and)	$p \wedge q$
Disjunction (or)	$p \vee q$
Exclusive Or	$p \oplus q$

$p \vee q$ at least one of p or q

$p \oplus q$ exactly one of p or q

Logic forces us to distinguish $\vee$ from $\oplus$

Logical Connectives

Negation (not)	$\neg p$
Conjunction (and)	$p \wedge q$
Disjunction (or)	$p \vee q$
Exclusive Or	$p \oplus q$
Implication	$p \longrightarrow r$
Biconditional	$p \leftrightarrow q$

Syntax of Logical Connectives

		Precedence
Negation (not)	$\neg p$	highest
Conjunction (and)	$p \land q$	↓
Disjunction (or)	$p \lor q$	↓
Exclusive Or	$p \oplus q$	↓
Implication	$p \longrightarrow r$	↓
Biconditional	$p \leftrightarrow q$	lowest

$p \lor q \land r \longrightarrow t$ **means** $(p \lor (q \land r)) \longrightarrow t$

Syntax of Logical Connectives

Connective	Notation	Associativity
Conjunction (and)	$p \land q$	left-to-right
Disjunction (or)	$p \lor q$	left-to-right
Exclusive Or	$p \oplus q$	left-to-right
Implication	$p \longrightarrow r$	right-to-left
Biconditional	$p \leftrightarrow q$	right-to-left

$p \lor q \lor r \lor t$ means $((p \lor q) \lor r) \lor t$

$p \longrightarrow q \longrightarrow r$ means $p \longrightarrow (q \longrightarrow r)$

Some Truth Tables

p	$\neg p$
T	
F	

p	q	$p \wedge q$
T	T	
T	F	
F	T	
F	F	

p	q	$p \vee q$
T	T	
T	F	
F	T	
F	F	

p	q	$p \oplus q$
T	T	
T	F	
F	T	
F	F	

Some Truth Tables

p	$\neg p$
T	F
F	T

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Truth Table

- Example of a "**case analysis**":
  - list off all possible cases
  - analyze each one individually

- Truth table: one case for each setting of variables
  - with n variables, we get 2^n cases (rows)

- Useful tool for many kinds of problems
  - will see more examples in the homework...

Another Truth Table

p	r	$p \rightarrow r$
T	T	
T	F	
F	T	
F	F	

With implication ($\rightarrow$), p is called the "premise" and r is called the "conclusion".

The implication is true when p and r are true.

The implication is true ("vacuously") when p is false.

Another Truth Table

p	r	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

With implication ($\rightarrow$), *p* is called the "premise" and r is called the "conclusion".

The implication is true when p and r are true.

The implication is true ("vacuously") when p is false.

Implication

"If it was raining, then I had my umbrella"

p	r	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

It's useful to think of implications as promises. That is "Was I ***wrong****?"*

	It's raining	It's not raining
I have my umbrella		
I do not have my umbrella		

Implication

"If it was raining, then I had my umbrella"

p	r	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

*It's useful to think of implications as promises. That is "Was I **wrong**?"*

	It's raining	It's not raining
I have my umbrella	No	No
I do not have my umbrella	**Yes**	No

I am only wrong when:

(a) It's raining AND

(b) I don't have my umbrella

Implication

"If the Seahawks won,
then I was at the game."

p	r	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

In what scenario was I wrong?

	I was at the game	I wasn't at the game
Seahawks won		
Seahawks lost		

Implication

"If the Seahawks won,
then I was at the game."

p	r	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

In what scenario was I wrong?

	I was at the game	I wasn't at the game
Seahawks won	Ok	**Doh!**
Seahawks lost	Ok	Ok

Implication

"If it's raining, then I have my umbrella"

p	r	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

Are these true?

2 + 2 = 4 → earth is a planet

The fact that these are unrelated doesn't make the statement false! "2 + 2 = 4" is true; "earth is a planet" is true. T→T is true. So, the statement is true.

2 + 2 = 5 → 26 is prime

Again, these statements may or may not be related. "2 + 2 = 5" is false; so, the implication is true. (Whether 26 is prime or not is irrelevant).

Implication is not a causal relationship!

$p \to r$

---

(1) "I have collected all 151 Pokémon if I am a Pokémon master"

(2) "I have collected all 151 Pokémon only if I am a Pokémon master"

In English, the "if" can be written at the end of the sentence rather than at the beginning of the sentence (followed by a ",").

$p \to r$

(1) "I have collected all 151 Pokémon if I am a Pokémon master"

(2) "I have collected all 151 Pokémon only if I am a Pokémon master"

These sentences are implications in opposite directions:

(1) "Pokémon masters have all 151 Pokémon"

(2) "People who have 151 Pokémon are Pokémon masters"

So, the implications are:

(1) If I am a Pokémon master, *then* I have collected all 151 Pokémon.

(2) If I have collected all 151 Pokémon, *then* I am a Pokémon master.

$p \rightarrow r$

p	r	$p \rightarrow r$
T	T	T
T	F	F
F	T	T
F	F	T

Implication:

- *p* implies *r*
- whenever *p* is true, *r* must be true
- if *p*, then *r*
- *r* if *p*
- *p* only if *r*
- *p* is sufficient for *r*
- *r is necessary for p*

Biconditional: $p \leftrightarrow q$

- *p* if and only if *q*
- *p* "iff" *q*
  - *p and q have the same value truth value*

p	q	$p \leftrightarrow q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**F**
F	**F**	**T**

A Compound Proposition (Practical Example)

"Show the notification to the user if its their second login or they've used it for two weeks and haven't tried the feature X unless they did use the feature Y."

Not at all clear what exactly this means!

Can use logic to understand exactly when to show it

A Compound Proposition (Silly Example)

"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"

We'd like to *understand* what this proposition means.

A Compound Proposition

“Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna”

We’d like to *understand* what this proposition means.

First find the simplest (**atomic) propositions**:

- q “Garfield has black stripes”
- r “Garfield is an orange cat”
- s “Garfield likes lasagna”

(q if (r and s)) and (r or (not s))

Logical Connectives

Negation (not)	$\neg p$
Conjunction (and)	$p \wedge q$
Disjunction (or)	$p \vee q$
Exclusive Or	$p \oplus q$
Implication	$p \longrightarrow r$
Biconditional	$p \leftrightarrow q$

q "Garfield has black stripes"
r "Garfield is an orange cat"
s "Garfield likes lasagna"

"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"

↓

(q if (r and s)) and (r or (not s))

Logical Connectives

Negation (not)	$\neg p$
Conjunction (and)	$p \land q$
Disjunction (or)	$p \lor q$
Exclusive Or	$p \oplus q$
Implication	$p \longrightarrow r$
Biconditional	$p \leftrightarrow r$

q	"Garfield has black stripes"
r	"Garfield is an orange cat"
s	"Garfield likes lasagna"

"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"

↓

(q if (r and s)) and (r or (not s))

↓

$((r \land s) \longrightarrow q) \land (r \lor \neg s)$

Analyzing the Garfield Sentence with a Truth Table

q	r	s	$((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$
F	F	F	
F	F	T	
F	T	F	
F	T	T	
T	F	F	
T	F	T	
T	T	F	
T	T	T	

subexpressions are not (yet) columns in this table

we will always include **all subexpressions** (easiest to verify)

Analyzing the Garfield Sentence with a Truth Table

q	r	s	$r \vee \neg s$	$(r \wedge s) \to q$	$((r \wedge s) \to q) \wedge (r \vee \neg s)$
F	F	F			
F	F	T			
F	T	F			
F	T	T			
T	F	F			
T	F	T			
T	T	F			
T	T	T			

Analyzing the Garfield Sentence with a Truth Table

q	r	s	$\neg s$	$r \vee \neg s$	$r \wedge s$	$(r \wedge s) \rightarrow q$	$((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$
F	F	F					
F	F	T					
F	T	F					
F	T	T					
T	F	F					
T	F	T					
T	T	F					
T	T	T					

Analyzing the Garfield Sentence with a Truth Table

q	r	s	$\neg s$	$r \vee \neg s$	$r \wedge s$	$(r \wedge s) \rightarrow q$	$((r \wedge s) \rightarrow q) \wedge (r \vee \neg s)$
F	F	F	T	T	F	T	T
F	F	T	F	F	F	T	F
F	T	F	T	T	F	T	T
F	T	T	F	T	T	F	F
T	F	F	T	T	F	T	T
T	F	T	F	F	F	T	F
T	T	F	T	T	F	T	T
T	T	T	F	T	T	T	T

Understanding Garfield Claim

"Garfield has black stripes if he is an orange cat and likes lasagna, and he is an orange cat or does not like lasagna"

Black Stripes	Orange	Likes Lasagna	Claim
F	F	F	**T**
F	F	T	F
F	T	F	**T**
F	T	T	F
T	F	F	**T**
T	F	T	F
T	T	F	**T**
T	T	T	**T**

Propositional Logic makes clear exactly what is being claimed.

Understanding Garfield Claim

Black Stripes	Orange	Likes Lasagna	Claim
F	F	F	T
...	...	...	...
T	T	T	T

Consistent with

but also

Last Time on CSE 311

- Saw how to formalize logical statements in the language of Propositional Logic (PL)

- Saw our first formal tool for analyzing them
  - a truth table
  - (useful but very large if many variables)

- Next, will see two more tools for analyzing PL
  - one more that uses a truth table
  - two that analyze expressions *without* truth tables

Converse, Contrapositive

Implication:

$p \rightarrow r$

Converse:

$r \rightarrow p$

Contrapositive:

$\neg r \rightarrow \neg p$

Inverse:

$\neg p \rightarrow \neg r$

Consider

p: 6 is divisible by 2

r: 6 is divisible by 4

$p \rightarrow r$	
$r \rightarrow p$	
$\neg r \rightarrow \neg p$	
$\neg p \rightarrow \neg r$	

Converse, Contrapositive

Implication:

$p \rightarrow r$

Converse:

$r \rightarrow p$

Contrapositive:

$\neg r \rightarrow \neg p$

Inverse:

$\neg p \rightarrow \neg r$

<u>Consider</u>

p: 6 is divisible by 2

r: 6 is divisible by 4

$p \rightarrow r$	F
$r \rightarrow p$	T
$\neg r \rightarrow \neg p$	F
$\neg p \rightarrow \neg r$	T

Converse, Contrapositive

Implication:

$p \rightarrow r$

Contrapositive:

$\neg r \rightarrow \neg p$

Converse:

$r \rightarrow p$

Inverse:

$\neg p \rightarrow \neg r$

How do these relate to each other?

p	r	$p \rightarrow r$	$r \rightarrow p$	$\neg p$	$\neg r$	$\neg p \rightarrow \neg r$	$\neg r \rightarrow \neg p$
T	T						
T	F						
F	T						
F	F						

Converse, Contrapositive

Implication:

$p \to r$

Contrapositive:

$\neg r \to \neg p$

Converse:

$r \to p$

Inverse:

$\neg p \to \neg r$

An implication and its contrapositive have the same truth value!

p	r	$p \to r$	$r \to p$	$\neg p$	$\neg r$	$\neg p \to \neg r$	$\neg r \to \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Converse, Contrapositive

Implication:

$p \rightarrow r$

Contrapositive:

$\neg r \rightarrow \neg p$

Converse:

$r \rightarrow p$

Inverse:

$\neg p \rightarrow \neg r$

An implication and its inverse
<u>do not</u> have the same truth value!

p	r	$p \rightarrow r$	$r \rightarrow p$	$\neg p$	$\neg r$	$\neg p \rightarrow \neg r$	$\neg r \rightarrow \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Equivalence

- **Propositional Logic expressions with the same truth table are called "equivalent"**

- **Examples:**
  - **implication and its contrapositive are equivalent**

    e.g., $(p \vee q) \to (q \wedge r)$ is equivalent to $\neg(q \wedge r) \to \neg(p \vee q)$

  - **implication and its inverse are not equivalent**

    e.g., $(p \vee q) \to (q \wedge r)$ is **not** equivalent to $\neg(p \vee q) \to \neg(q \wedge r)$

    assuming they are the same is the "fallacy of the inverse"

- **Greatly expand on equivalence next week**
  - **prove equivalence without a truth table**

Satisfiability (SAT)

Problem: Given a Propositional Logic expression, is there a way to set the values of the variables to make the expression evaluate to T?

- if yes, the expression is "satisfiable"
- if not, the expression is "unsatisfiable"

- Many problems can be stated as SAT problems
  - e.g., many "puzzle" type problems

    see HW1 for an example
  - lots of important & useful problems in this category

    e.g., verifying correctness of hardware

SAT Solvers

<u>**Problem**</u>: Given a Propositional Logic expression, is there a way to set the values of the variables to make the expression evaluate to T?

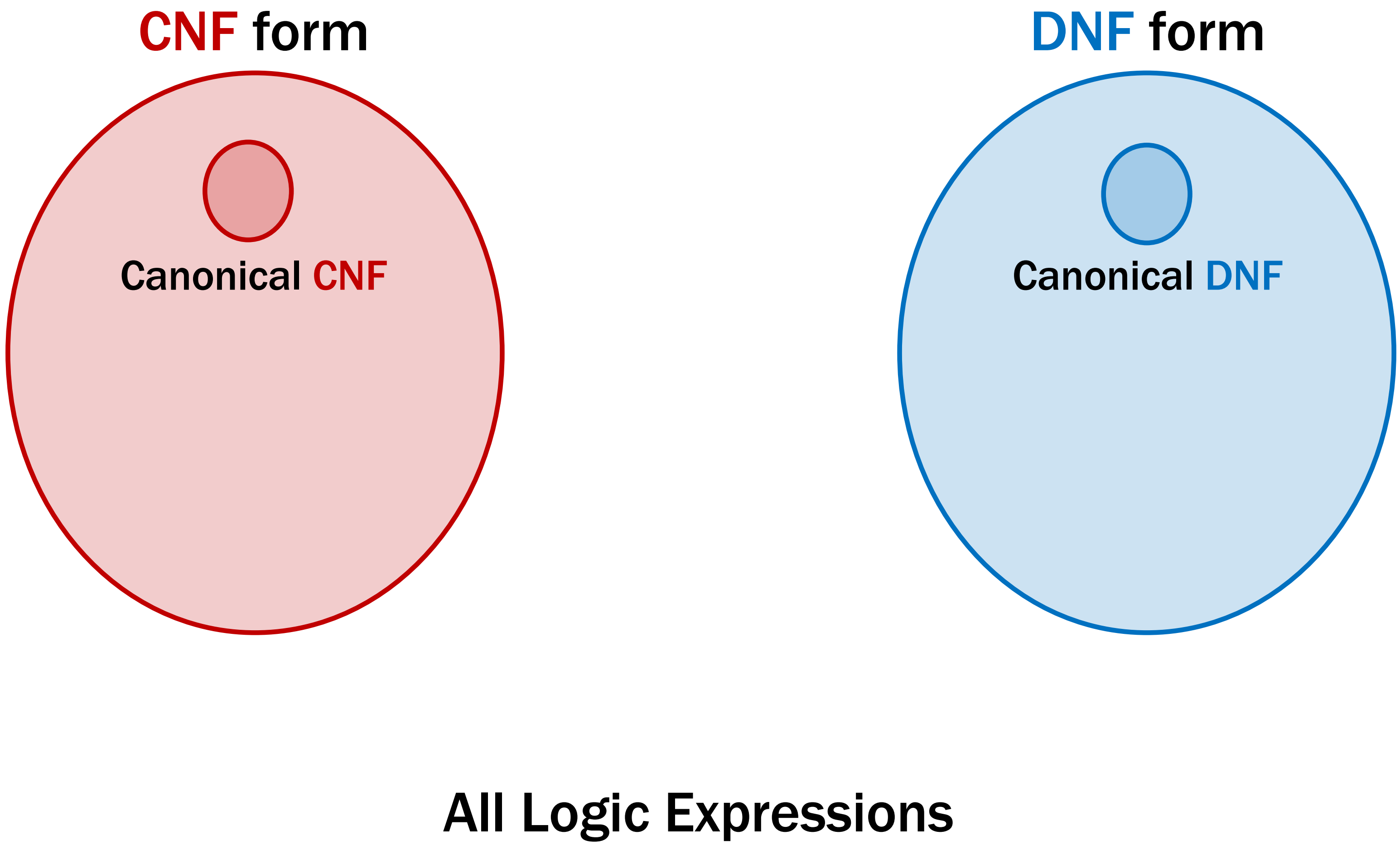
- if yes, the expression is "satisfiable"
- if not, the expression is "unsatisfiable"

- Brute force is doesn't get you far…
  - 2^{264} ≈ # atoms in the observable universe
- Modern SAT solvers handle *millions* of variables
  - would be nice to have access to these!

SAT Solvers

- Usually, do not accept *arbitrary* Logic expressions
  - require the expression to come in a simpler form
- Typically, require the expression in "CNF" form
  - one of the two common forms (other is "DNF")
  - see notes on the website for more on "Why CNF?"
- Once we understand CNF, we can use a SAT solver

CNF / DNF Forms



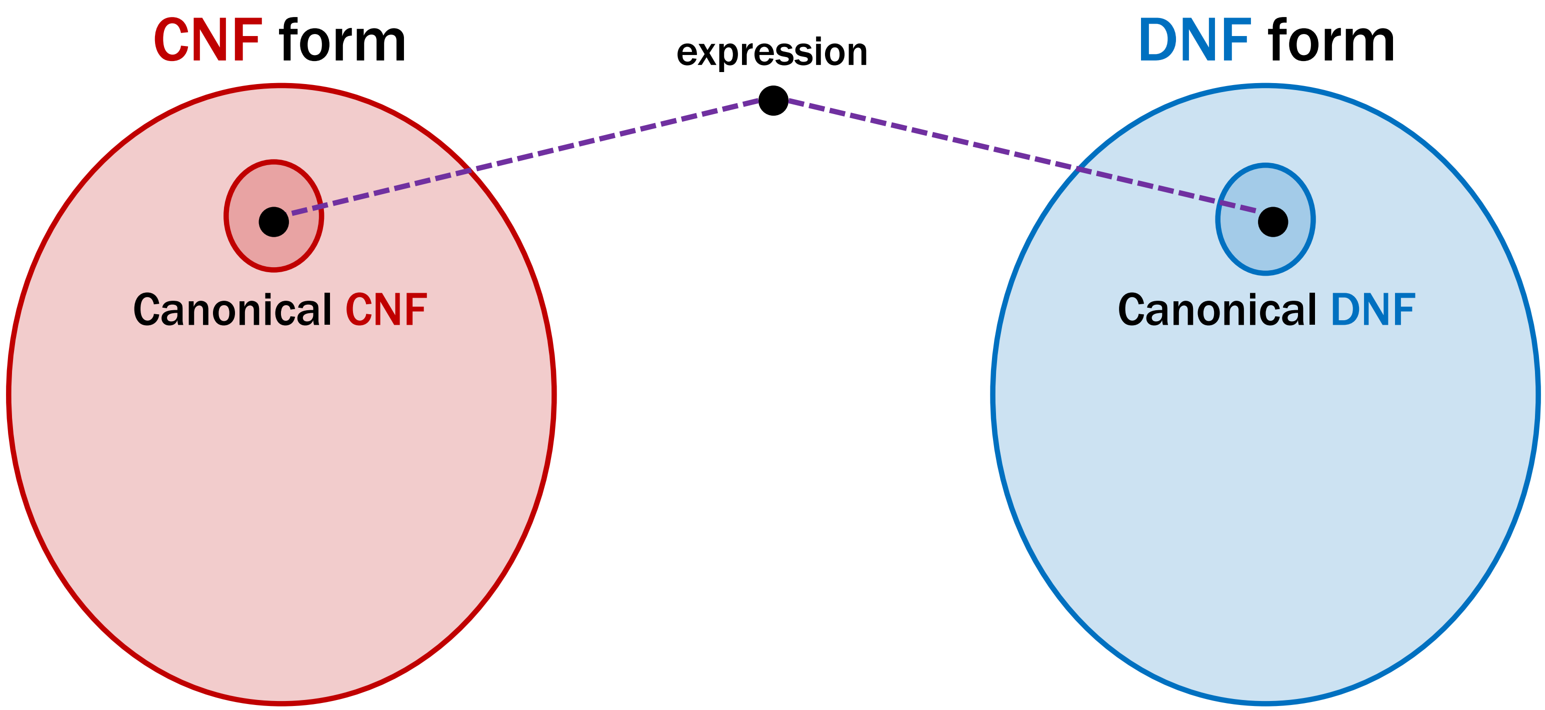
Canonical Forms

- Canonical is from Latin "canon" (ruler)
  - compare against to see if equivalent

- We saw one way to do this already: truth table

- Canonical forms are a second way…

CNF / DNF Forms



equivalent to exactly one in canonical CNF form (up to reordering)

if our expressions are in canonical CNF form,
then they are equivalent iff they are the same

DNF Canonical Form

① Find the T rows in the truth table

Suppose F is an expression using the variables a, b, c

a	b	c	F
F	F	F	F
F	F	**T**	**T**
F	**T**	F	F
F	**T**	**T**	**T**
T	F	F	F
T	F	**T**	**T**
T	**T**	F	**T**
T	**T**	**T**	**T**

DNF Canonical Form

① Find the T rows in the truth table

② For each T row, write an expression that is T in that row but *no others* ("min term")

a	b	c	F	
F	F	F	F	
F	F	**T**	**T**	
F	**T**	F	F	
F	**T**	**T**	**T**	
T	F	F	F	
T	F	**T**	**T**	
T	**T**	F	**T**	
T	**T**	**T**	**T**	

$\neg a \land \neg b \land c$

This is only T if a = F, b = F, and c = T
(AND requires *all* arguments to be T)

DNF Canonical Form

① Find the T rows in the truth table

② For each T row, write an expression that is T in that row but *no others* ("min term")

a	b	c	F	
F	F	F	F	
F	F	**T**	**T**	
F	**T**	F	F	
F	**T**	**T**	**T**	
T	F	F	F	
T	F	**T**	**T**	
T	**T**	F	**T**	
T	**T**	**T**	**T**	

$\neg a \land \neg b \land c$

$\neg a \land b \land c$

This is only T if a = F, b = T, and c = T

DNF Canonical Form

① Find the T rows in the truth table

② For each T row, write an expression that is T in that row but *no others* ("min term")

a	b	c	F	
F	F	F	F	
F	F	**T**	**T**	¬a ∧ ¬b ∧ c
F	**T**	F	F	
F	**T**	**T**	**T**	¬a ∧ b ∧ c
T	F	F	F	
T	F	**T**	**T**	a ∧ ¬b ∧ c
T	**T**	F	**T**	a ∧ b ∧ ¬c
T	**T**	**T**	**T**	a ∧ b ∧ c

A **min term** includes every variable exactly once, either negated or unnegated, AND-ed together

DNF Canonical Form

a	b	c	F
F	F	F	F
F	F	**T**	**T**
F	**T**	F	F
F	**T**	**T**	**T**
T	F	F	F
T	F	**T**	**T**
T	**T**	F	**T**
T	**T**	**T**	**T**

① **Find the T rows in the truth table**

② **For each T row, write an expression that is T in that row but *no others* ("min term")**

$\neg a \land \neg b \land c$

$\neg a \land b \land c$

$a \land \neg b \land c$

$a \land b \land \neg c$

$a \land b \land c$

③ **Form the disjunction (OR) of the min terms**

$$(\neg a \land \neg b \land c) \lor (\neg a \land b \land c) \lor (a \land \neg b \land c) \lor (a \land b \land \neg c) \lor (a \land b \land c)$$

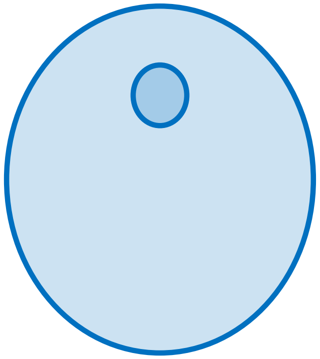
DNF Form: Canonical and Non

- Stands for "Disjunctive Normal Form"
  - **outermost** operation is disjunction (OR)
  - operands are conjunctions (ANDs) of variables or their negations

$(a \land c) \lor (\neg a) \lor (\neg a \land \neg b)$ non-canonical DNF

$(a \land b \land \neg c) \lor (\neg a \land b \land c) \lor (a \land \neg b \land \neg c)$ canonical DNF

(every disjunct is a min term)



CNF Canonical Form

① **Find the F rows in the truth table**

a	b	c	F
F	F	F	F
F	F	**T**	**T**
F	**T**	F	F
F	**T**	**T**	**T**
T	F	F	F
T	F	**T**	**T**
T	**T**	F	**T**
T	**T**	**T**	**T**

CNF Canonical Form

① Find the F rows in the truth table

② For each F row, write an expression that is T in *every row but that one* ("max term")

a	b	c	F	
F	F	F	F	
F	F	T	T	
F	T	F	F	
F	T	T	T	
T	F	F	F	
T	F	T	T	
T	T	F	T	
T	T	T	T	

$a \vee b \vee c$

This is only F if a = F, b = F, and c = F
(OR is T if *any* arguments is a T)

CNF Canonical Form

1. Find the F rows in the truth table

2. For each F row, write an expression that is T in *every row but that one* ("max term")

a	b	c	F	
F	F	F	F	
F	F	T	T	
F	T	F	F	
F	T	T	T	
T	F	F	F	
T	F	T	T	
T	T	F	T	
T	T	T	T	

$a \lor b \lor c$

$a \lor \neg b \lor c$

This is only F if a = F, b = T, and c = F

CNF Canonical Form

① **Find the F rows in the truth table**

② **For each F row, write an expression that is T in *every row but that one* ("max term")**

a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

$a \lor b \lor c$

$a \lor \neg b \lor c$

$\neg a \lor b \lor c$

This is only F if a = T, b = F, and c = F

CNF Canonical Form

a	b	c	F
F	F	F	F
F	F	T	T
F	T	F	F
F	T	T	T
T	F	F	F
T	F	T	T
T	T	F	T
T	T	T	T

1. **Find the F rows in the truth table**

2. **For each F row, write an expression that is T in *every row but that one* ("max term")**

   $a \lor b \lor c$

   $a \lor \neg b \lor c$

   $\neg a \lor b \lor c$

3. **Form the conjunction (AND) of the max terms**

   $(a \lor b \lor c) \land (a \lor \neg b \lor c) \land (\neg a \lor b \lor c)$

CNF Form: Canonical and Non

- Stands for "Conjunctive Normal Form"
  - **outermost** operation is conjunction (AND)
  - operands (conjuncts) are disjunctions (ORs) of variables or their negations

$(a \lor b \lor \neg c) \land (\neg a \lor b \lor c) \land (a \lor \neg b \lor \neg c)$ canonical CNF

$(a \lor c) \land (\neg a) \land (\neg a \lor \neg b)$ non-canonical CNF

Comparing DNF and CNF

	DNF	**CNF**
operation	disjunction (OR)	conjunction (AND)
operands	conjunctions (ANDs)	disjunctions (ORs)
	(of only variables or their negations)	
canonical iff	all conjunctions are **min** terms	all disjunctions are **max** terms

Comparing Min and Max Terms

- **Min/Max term if *every* variable appears <u>exactly</u> once**

	Min Term	**Max Term**
operation	conjunction (AND)	disjunction (OR)
	(of every variables or its negation)	
result	only one T row	only one F row

Important Corollary of DNF Construction

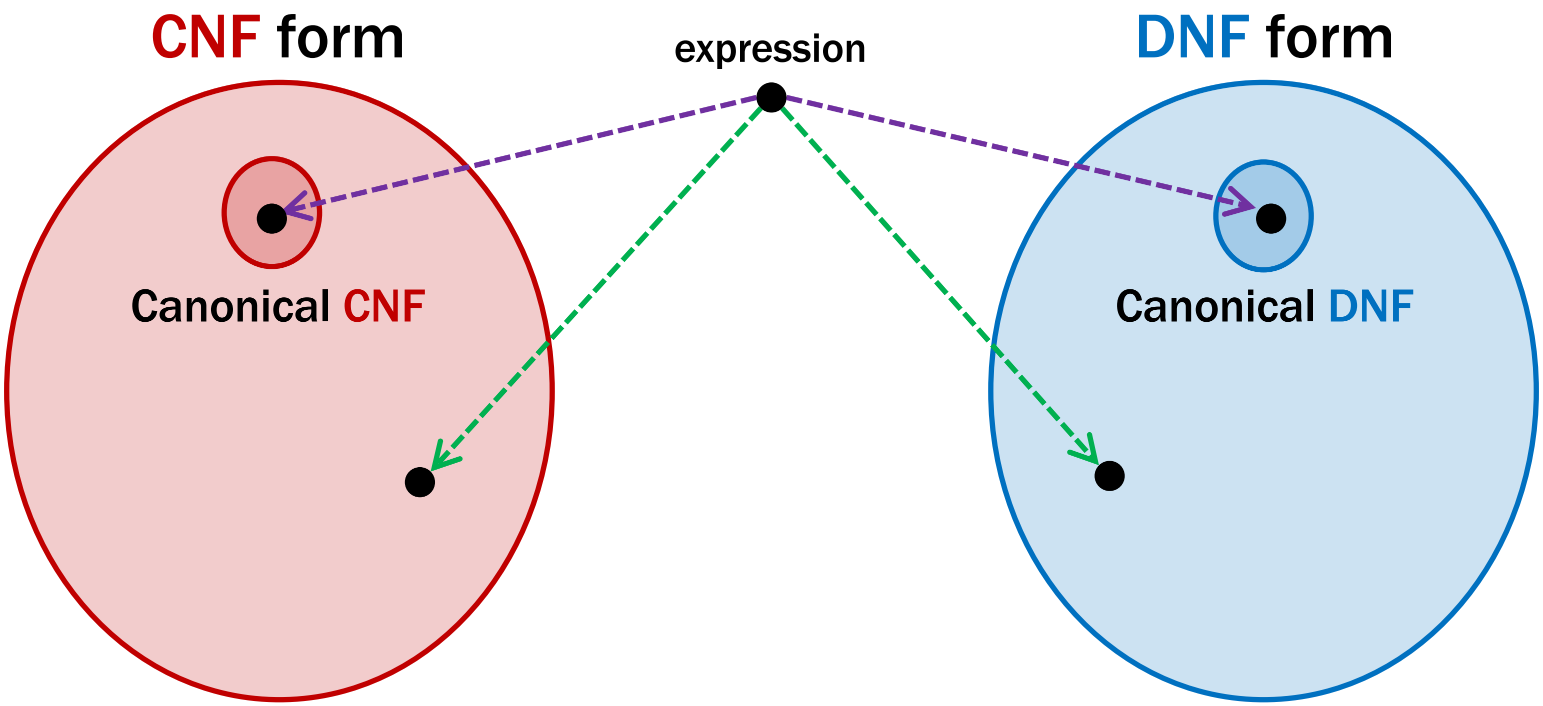
$\neg$, $\wedge$, $\vee$ can implement any Boolean function!

no need for anything else

Why? Because this construction only uses $\neg$, $\wedge$, $\vee$

DNF conversion works for any boolean function

CNF / DNF Forms



equivalent but slow conversion

fast conversion but only "equi-satisfiable"
(and outside the scope of 311)