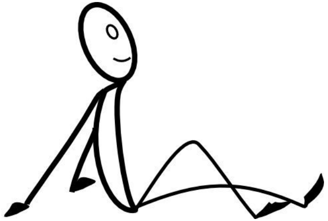
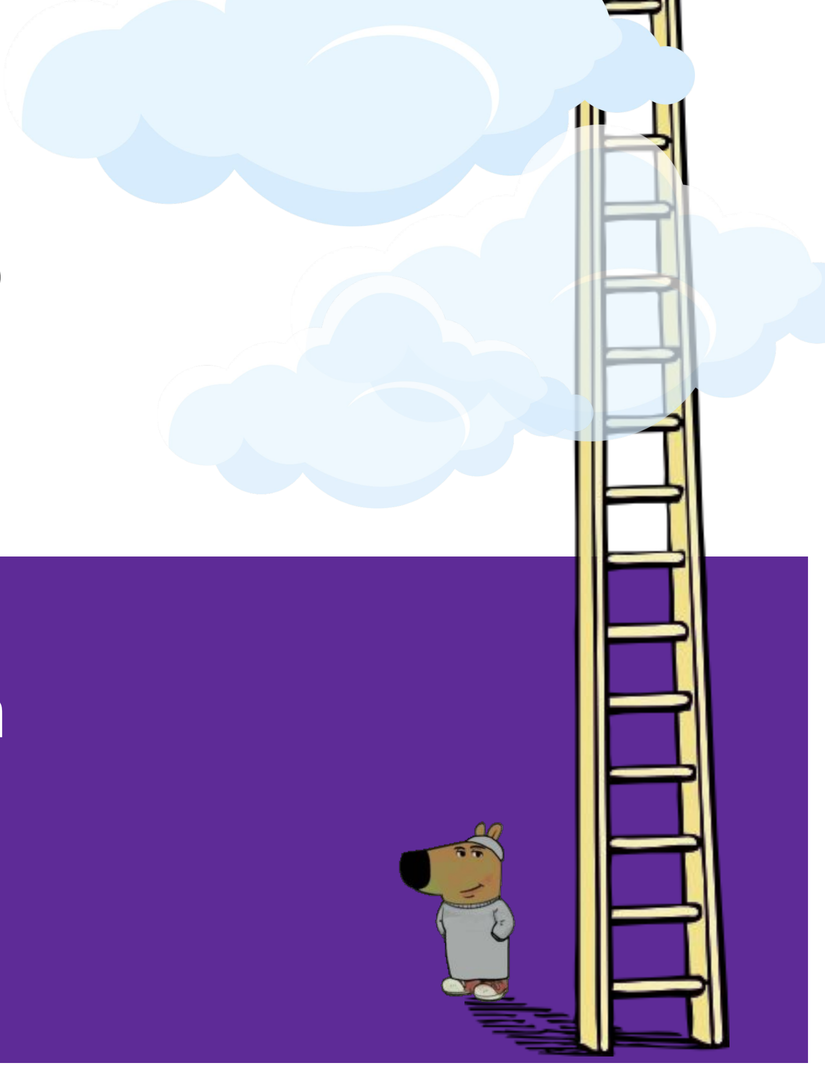
CSE 311 Section 5





Number Theory & Induction

Announcements & Reminders

- HW4 due yesterday @ 11:00PM on Gradescope
  - Use late days if you need to!
  - Make sure you tagged pages on gradescope correctly

- HW5
  - Releases tonight
  - Due Wednesday 2/12 @11:00 PM
- Book 1-on-1s on the ed message board!

Extended Euclid

Problem 2 – Extended Euclidean Algorithm

a) Find the multiplicative inverse y of $7 \bmod 33$. That is, find y such that $7y \equiv 1\ (mod\ 33)$. You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \leq y < 33$.

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a) Find the multiplicative inverse y of $7 \bmod 33$. That is, find y such that $7y \equiv 1 \ (mod\ 33)$. You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \leq y < 33$.

First, we find the gcd:

```
gcd(33,7)   = gcd(7,5)      33 = 4 • 7 + 5
            = gcd(5,2)      7  = 1 • 5 + 2
            = gcd(2,1)      5  = 2 • 2 + 1
            = gcd(1,0)      2  = 2 • 1 + 0
```

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$$\gcd(33,7) = \gcd(7,5) = \gcd(5,2) = \gcd(2,\mathbf{1}) = \gcd(\mathbf{1},0)$$

$33 = 4 \bullet 7 + 5$

$7 = 1 \bullet 5 + 2$

$5 = 2 \bullet 2 + \mathbf{1}$

$2 = 2 \bullet \mathbf{1} + 0$

Next, we re-arrange the equations by solving for the remainder:

$\mathbf{1} = 5 - 2 \bullet 2$

$2 = 7 - 1 \bullet 5$

$5 = 33 - 4 \bullet 7$

Problem 2 – Extended Euclidean Algorithm

a) Find the multiplicative inverse y of 7 mod 33. That is, find y such that $7y \equiv 1 \ (mod\ 33)$. You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \le y < 33$.

First, we find the gcd:

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Next, we re-arrange the equations by solving for the remainder:

```
1 = 5 - 2 • 2
2 = 7 - 1 • 5
5 = 33 - 4 • 7
```

Now, we backward substitute into the boxed numbers using the equations:

$$
\begin{aligned}
\mathbf{1} &= 5 - 2 \cdot 2 \\
&= 5 - 2 \cdot (7 - 1 \cdot 5) \\
&= 3 \cdot 5 - 2 \cdot 7 \\
&= 3 \cdot (33 - 4 \cdot 7) - 2 \cdot 7 \\
&= 3 \cdot 33 + -14 \cdot 7
\end{aligned}
$$

Problem 2 – Extended Euclidean Algorithm

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$$\begin{aligned} 33 &= 4 \cdot 7 + 5 \\ 7 &= 1 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + \mathbf{1} \\ 2 &= 2 \cdot \mathbf{1} + 0 \end{aligned}$$

Next, we re-arrange the equations by solving for the remainder:

$$\begin{aligned} \mathbf{1} &= 5 - 2 \cdot 2 \\ 2 &= 7 - 1 \cdot 5 \\ 5 &= 33 - 4 \cdot 7 \end{aligned}$$

Now, we backward substitute into the boxed numbers using the equations:

$$\begin{aligned} \mathbf{1} &= 5 - 2 \cdot 2 \\ &= 5 - 2 \cdot (7 - 1 \cdot 5) \\ &= 3 \cdot 5 - 2 \cdot 7 \\ &= 3 \cdot (33 - 4 \cdot 7) - 2 \cdot 7 \\ &= 3 \cdot 33 + -14 \cdot 7 \end{aligned}$$

So, $\mathbf{1} = \mathbf{3} \cdot 33 + \mathbf{-14} \cdot 7$. Thus, $33 \mathbf{- 14} = \mathbf{19}$ is the multiplicative inverse of 7 mod 33

Problem 2 – Extended Euclidean Algorithm

a) Find the multiplicative inverse y of $7 \bmod 33$. That is, find y such that $7y \equiv 1 \ (mod\ 33)$. You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \le y < 33$.

b) Now, solve $7z \equiv 2 \pmod{33}$ for all of its integer solutions z.

Try this problem with the people around you, and then we'll go over it together!

Problem 2 – Extended Euclidean Algorithm

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If we have $7z \equiv 2 \pmod{33}$, multiplying both sides by 19, we get:

$z \equiv 2 \cdot 19 \pmod{33} \equiv 5 \pmod{33}$.

This means that the set of solutions is $\{5 + 33k \mid k \in Z\}$

Introducing Induction (kind of)

Climb the ladder!

You are scared of heights and there is a prize at the top of a very very tall ladder.

You do not want to climb this ladder…

Climb the ladder!

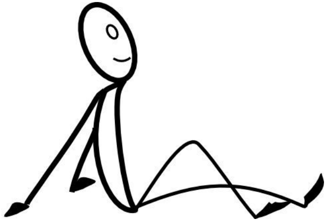
You are scared of heights and there is a prize at the top of a very very tall ladder.

You do not want to climb this ladder…

Lets convince your friend to climb it instead!!!

Climb the ladder!

You Claim: “There are k steps in the ladder. After k steps you will reach the top!”



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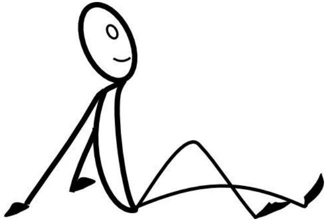
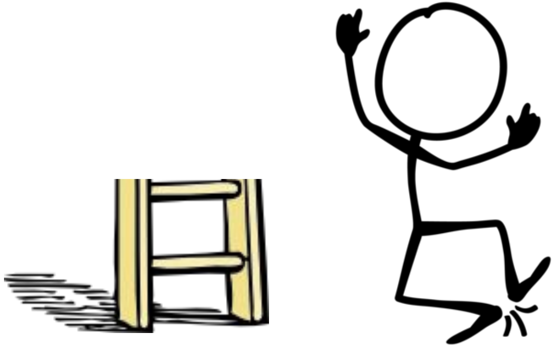
You Claim: “There are k steps in the ladder. After k steps you will reach the top!”

“If we have a ladder with **1** step. <u>I know you can lift your foot</u> so after 1 step you will reach the top of a 1 step ladder!”

“So my claim holds for 1 step!”





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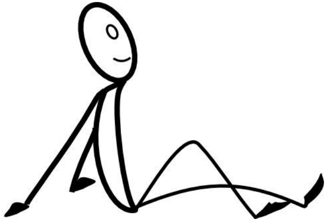

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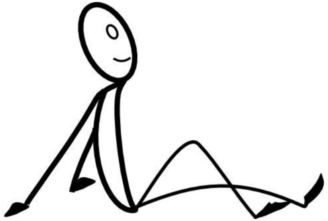
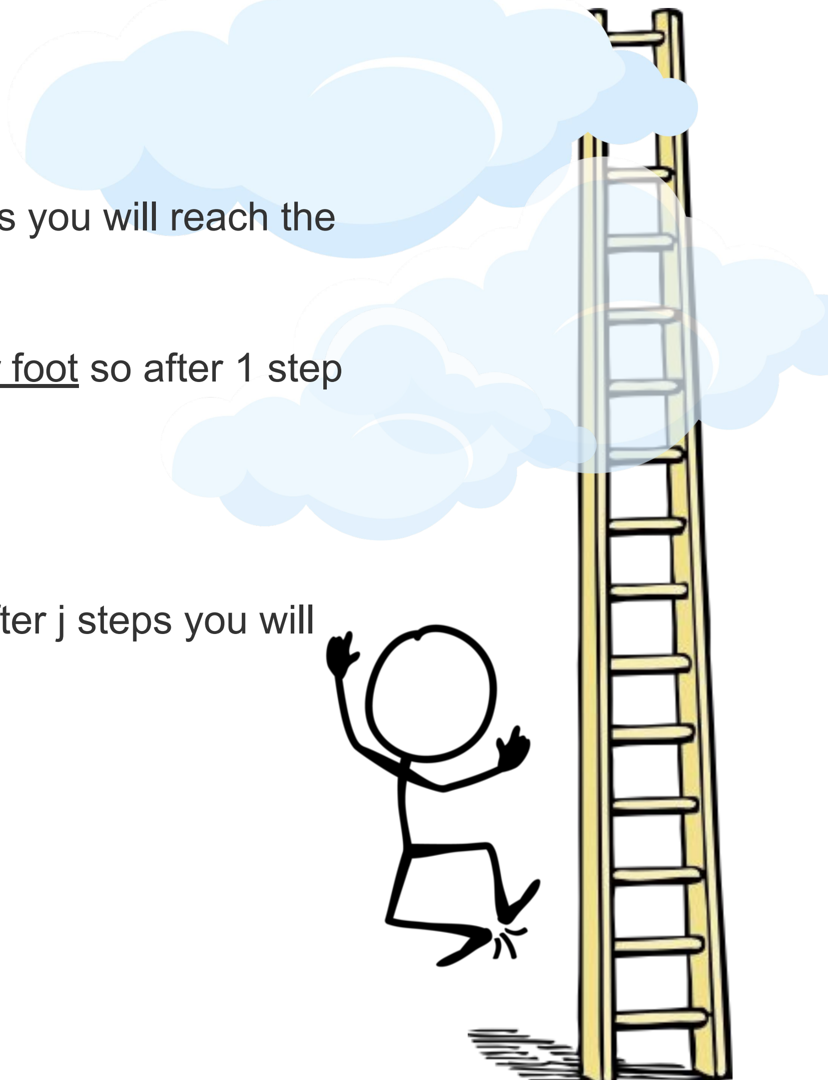
Let’s **suppose** that for an arbitrary number of steps j, after j steps you will reach the top.





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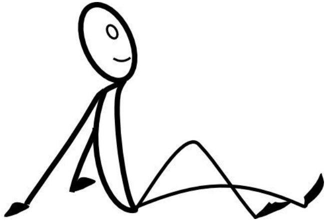
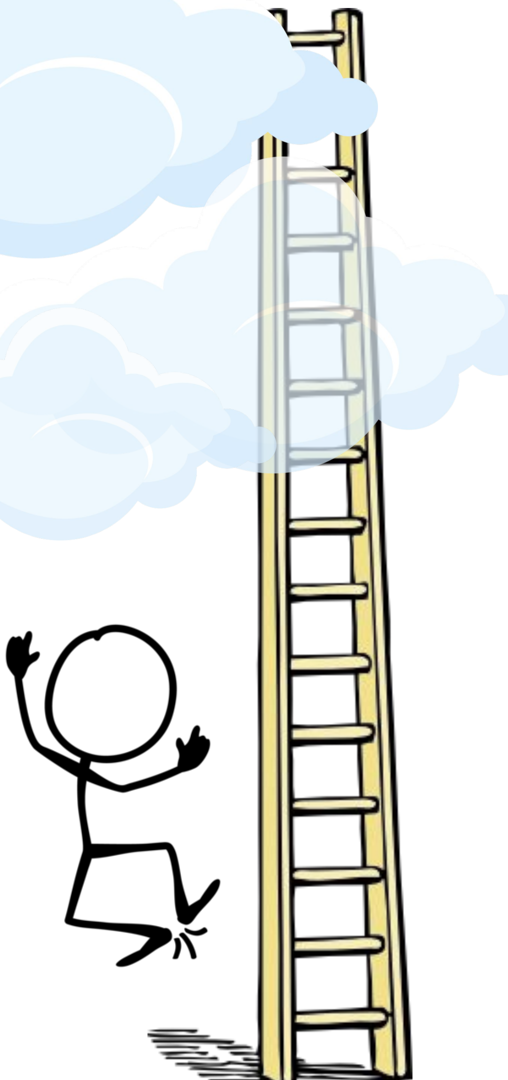
Let’s **suppose** that for an arbitrary number of steps j, after j steps you will reach the top.

I can prove to you that this claim will still hold for j+1 steps!

Goal: Prove that for **j+1** steps in the ladder, after **j+1** steps you will reach the top!





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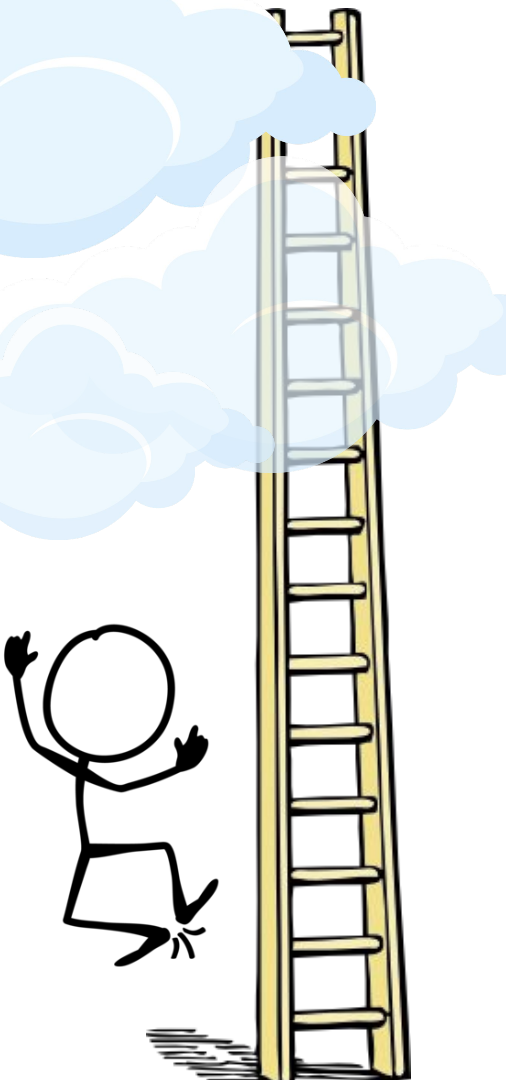

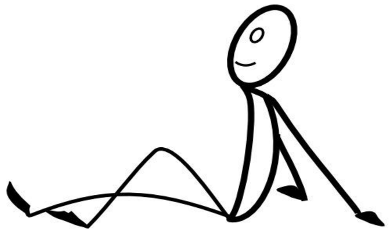

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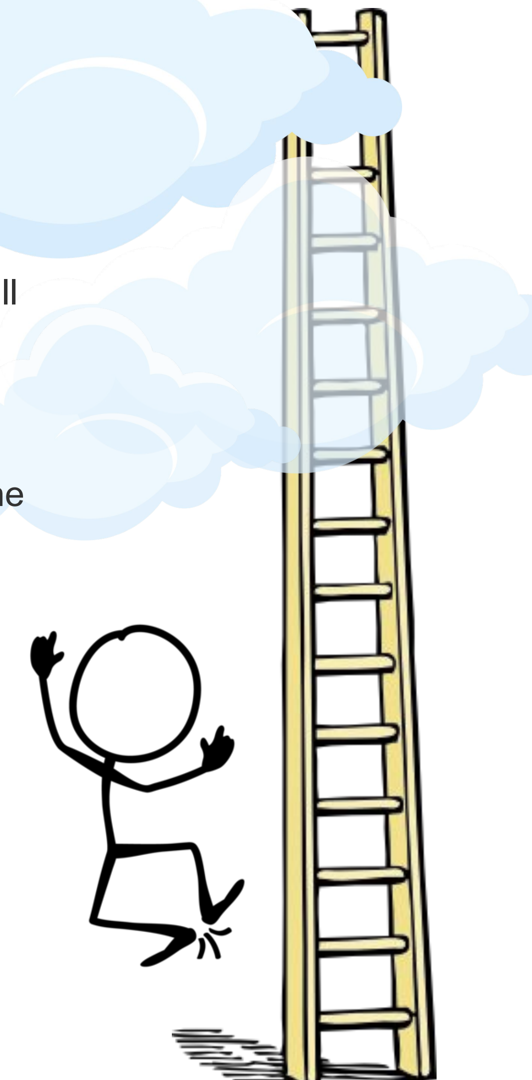

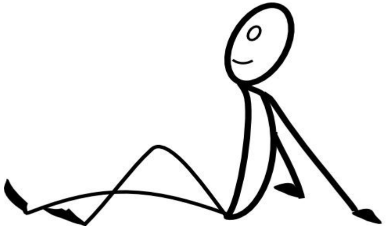

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THE CLAIM HOLDS YOUR FRIEND IS CLIMBING THE LADDER

WELCOME TO PROOF BY INDUCTION

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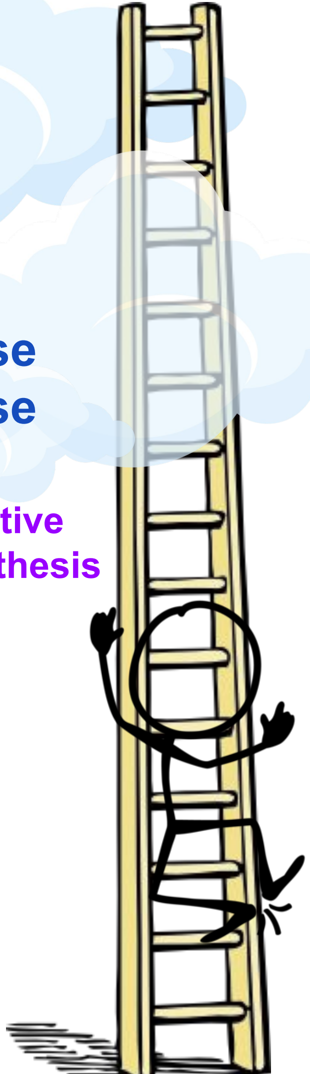

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Inductive Step P(j+1)

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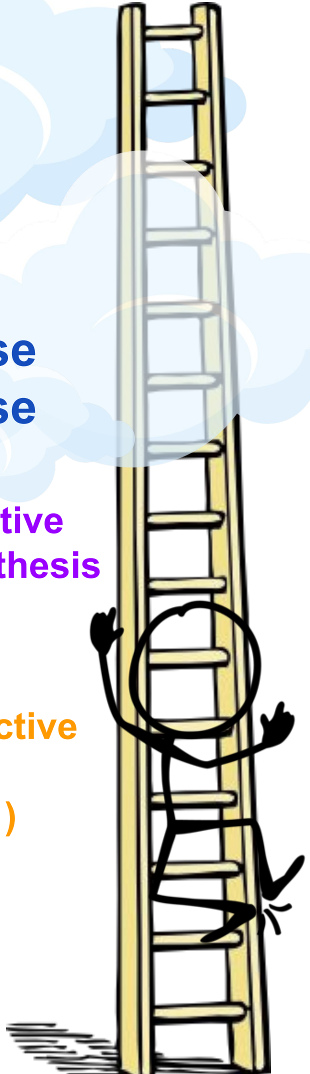

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Inductive Step

Using the IH

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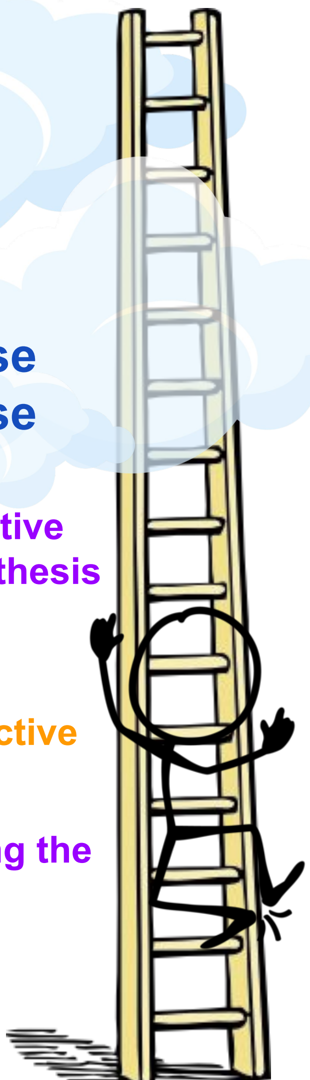

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P(n) holds!

Induction: How it actually works

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds for all n ∈ N by induction on n

Base Case: Show $P(b)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds for all n by the principle of induction.

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds **for all n** $\in$ N by induction on n

Note: often you will condition n here, like "all natural numbers n" or "$n \geq 0$"

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Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds **for all n** by the principle of induction.

Match the earlier condition on n in your conclusion!

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P(n) IS A PREDICATE, IT HAS A BOOLEAN VALUE NOT A NUMERICAL ONE

<u>Base Case:</u> Show $P(b)$ is true.

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YOU MUST INTRODUCE AN ARBITRARY VARIABLE IN YOUR IH

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START WITH LHS OF K + 1 ONLY AND WORK TOWARD RHS

Conclusion: Therefore, $P(n)$ holds **for all n** by the principle of induction.

Weak Induction

Task 3

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$.

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

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Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$.

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Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

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Induction Hypothesis: Assume that $9 \mid j^3 + (j+1)^3 + (j+2)^3$ for an arbitrary integer $j > 1$. Note that this is equivalent to assuming that $j^3 + (j+1)^3 + (j+2)^3 = 9k$ for some integer k by the definition of divides.

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Induction Step: Goal: Show $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$

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Induction Step: Goal: Show $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$

$$(j+1)^3 + (j+2)^3 + (j+3)^3 =$$

Task 3

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$.

Let $P(n)$ be "$9 \mid n^3 + (n+1)^3 + (n+2)^3$". We will prove $P(n)$ for all integers $n > 1$ by induction.

Base Case $(n = 2)$**:** $2^3 + (2+1)^3 + (2+2)^3 = 8 + 27 + 64 = 99 = 9 \cdot 11$, so $9 \mid 2^3 + (2+1)^3 + (2+2)^3$, so $P(2)$ holds.

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$(j+1)^3 + (j+2)^3 + (j+3)^3 = (j+3)^3 + 9k - j^3$ for some integer k [Induction Hypothesis]

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$$\begin{aligned}(j+1)^3 + (j+2)^3 + (j+3)^3 &= (j+3)^3 + 9k - j^3 \text{ for some integer } k \quad \text{[Induction Hypothesis]} \\ &= j^3 + 9j^2 + 27j + 27 + 9k - j^3\end{aligned}$$

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&= j^3 + 9j^2 + 27j + 27 + 9k - j^3 \\
&= 9j^2 + 27j + 27 + 9k \\
&= 9(j^2 + 3j + 3 + k)
\end{aligned}$$

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Since j is an integer, $j^2 + 3j + 3 + k$ is also an integer. Therefore, by the definition of divides, $9 \mid (j+1)^3 + (j+2)^3 + (j+3)^3$, so $P(j) \to P(j+1)$ for an arbitrary integer $j > 1$.

Conclusion:

Task 3

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Conclusion: $P(n)$ holds for all integers $n > 1$ by induction.

Task 3

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$.

Cozy walkthrough!

https://tinyurl.com/section5t3

Strong Induction

Task 7

Consider the function $a(n)$ defined for $n \geqslant 1$ recursively as follows.

$$a(1) = 1$$

$$a(2) = 3$$

$$a(n) = 2a(n-1) - a(n-2) \text{ for } n \geqslant 3$$

Use strong induction to prove that $a(n) = 2n - 1$ for all $n \geqslant 1$.

Let $P(n)$ be "$a(n) = 2n - 1$". We will show that $P(n)$ is true for all $n \geqslant 1$ by strong induction.

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Base Cases $(n = 1, n = 2)$:
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$$a(k+1) = 2a(k) - a(k-1) \qquad \text{[Definition of } a\text{]}$$

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$$\begin{aligned} a(k+1) &= 2a(k) - a(k-1) && \text{[Definition of } a\text{]} \\ &= 2(2k-1) - (2(k-1) - 1) && \text{[Inductive Hypothesis]} \end{aligned}$$

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So, $P(k+1)$ holds.

Conclusion:
Therefore, $P(n)$ holds for all integers $n \geqslant 1$ by principle of strong induction.

That's All!

I hope you enjoyed it, because I know I did

Written by Aruna & Zareef