Problem Set 6

Due: Wednesday, February 26th by 11:00pm

Instructions

Solutions submission. You must submit your solution via Gradescope. In particular:

- The (a) parts of Tasks 1, 2, & 5 should be submitted **first** on Cozy. You must **also** include your formal proofs in the PDF you submit on Gradescope so that the grader can confirm that your English proof properly translates your formal proof. If you are using LaTeX, you can copy Cozy's "Show LaTeX" output. If you are not using LaTeX, a screenshot is fine!
- Submit a *single* PDF file containing your solutions to Tasks 1–6 (and optionally 7). Follow the prompt on Gradescope to link tasks to your pages. (There, we will only grade your English translations in Tasks 1, 2, & 5, but your formal proof needs to be included for reference.)
- The instructions for submitting Task 1(a), 2(a), and 5(a) in Cozy appear below those problems.

Task 1 – The Dude Divides [16 pts]

Let A and B be the following sets:

$$A := \{n \in \mathbb{Z} : 6 \mid n\}$$
$$B := \{n \in \mathbb{Z} : \text{Even}(n)\}$$

Now, consider the following claim:

$$A \subseteq B$$

a) Write a **formal proof** that the claim holds.

In Cozy, you can replace "$n \in A$" by "$6 \mid n$" with "`defof A`" and the reverse with "`undef A`".[1] You can replace "$A \subseteq B$" by "$\forall x\, (x \in A \to x \in B)$" with "`defof Subset`" and the reverse with "`undef Subset`".

Submit and check your formal proof here:
http://cozy.cs.washington.edu
You **must also** include your solution (as a screenshot, typeset LaTeX, or rewritten by hand) in the PDF you submit to **Gradescope**.

b) Translate your formal proof to an **English proof**.

Follow the structure of the template for subset proofs from lecture.

Keep in mind that your proof will be read by a *human*, not a computer, so you should explain the algebra steps in more detail, whereas some of the predicate logic steps (e.g., Elim $\exists$) can be skipped.

[1] Those commands should work as stated when reasoning backward. For forward reasoning, you need to add the line number at the end, e.g., `defof A 1.1.1`.

Task 2 – The Best is Set to Come [16 pts]

Let A, B, C, and $\mathcal{S}$ be sets. Consider the following claim:

$$\mathcal{S} \subseteq B \cap C \text{ follows from } \mathcal{S} \subseteq A,\ \mathcal{S} \subseteq C,\ \text{and } A \subseteq B$$

a) Write a **formal proof** that the claim holds.

> Submit and check your formal proof here:
>
> http://cozy.cs.washington.edu
>
> You **must also** include your solution (as a screenshot, typeset LaTeX, or rewritten by hand) in the PDF you submit to **Gradescope**.

b) Translate your formal proof to an **English proof**.

Follow the structure of the template for subset proofs from lecture.

Note: given all the steps we are allowed to skip in English, your proof may be quite short!

Task 3 – Keeping Up With the Cartesians [16 pts]

Let A, B, and C be sets. Consider the following claim:

$$A \times B \subseteq B \times C$$

a) Suppose that $A = \{1\}$, $B = \{1, 2\}$, and $C = \{1, 2, 3\}$.

Calculate the values of the sets $A \times B$ and $B \times C$. Check whether the claim holds.

b) Suppose that $A = \{1, 2\}$, $B = \{1\}$, and $C = \{1, 3\}$.

Calculate the values of the sets $A \times B$ and $B \times C$. Check whether the claim holds.

c) Write an **English proof** that the claim holds given that $A \subseteq B$ and $B \subseteq C$.

(This updated claim describes the situation in part (a) but not part (b).)

Follow the structure of the template for subset proofs from lecture.

Task 4 – Our Finest Power [16 pts]

Let A, B, and C be sets. Consider the following claim:

$$\mathcal{P}(A) \cap \mathcal{P}(C) \subseteq \mathcal{P}(B \cap C)$$

a) Suppose that $A = \{1, 2\}$, $B = \{1, 3\}$, and $C = \{1, 2, 3\}$.

Calculate the values of the sets $\mathcal{P}(A) \cap \mathcal{P}(C)$ and $\mathcal{P}(B \cap C)$. Check whether the claim holds.

b) Suppose that $A = \{1\}$, $B = \{1, 3\}$, and $C = \{1, 2, 3\}$.

Calculate the values of the sets $\mathcal{P}(A) \cap \mathcal{P}(C)$ and $\mathcal{P}(B \cap C)$. Check whether the claim holds.

c) Write an **English proof** that the claim holds given that $A \subseteq B$.

(This updated claim describes the situation in part (b) but not part (a).)

Follow the structure of the template for subset proofs from lecture.

Note: feel free to cite Task 2 in your proof!

Task 5 – Blood, Set, and Tears [20 pts]

Let A and B be sets. Consider the following claim

$$A \setminus \overline{B} = A \cap B$$

a) Write a **formal proof** that the claim holds.

In Cozy, you can replace "$x \in \overline{A}$" by "$\neg(x \in A)$" with "`defof Complement`" and the reverse with "`undef Complement`". You can replace "$x \in A \cap B$" by "$(x \in A) \wedge (x \in B)$" with "`defof Intersection`" and the reverse with "`undef Intersection`". And you can replace "$x \in A \setminus B$" by "$(x \in A) \wedge \neg(x \in B)$" with "`defof SetDifference`" and the reverse with "`undef SetDifference`".[2]

> Submit and check your formal proof here:
>
> `http://cozy.cs.washington.edu`
>
> You **must also** include your solution (as a screenshot, typeset LaTeX, or rewritten by hand) in the PDF you submit to **Gradescope**.

b) Translate your formal proof to an **English proof**.

Follow the structure of the Meta-Theorem template from lecture.

[2]Those commands should work as stated when reasoning backward. For forward reasoning, you need to add the line number at the end, e.g., `defof Intersection 1.1.1`.

Task 6 – Parmesian, Romano, and Meta [16 pts]

For each of the claims below, say which of the following categories describes the given proof:

Proof The proof is correct.

Goof The claim is true but the proof is wrong.

Spoof The claim is false.

Then, if it is a goof, point out all errors in the proof and explain how to correct them, and if it is a spoof, point out the *first* error in the proof and then show that the claim is false by giving a counterexample. (If it is a correct proof, then skip this part.)

a) Claim: $(A \setminus B) \cup C \subseteq (A \cup C) \setminus B$

Proof or Spoof: Let x be arbitrary. Suppose that $x \in (A \setminus B) \cup C$. The definition of set difference gives us $x \in A$ and $x \notin B$. Since $x \in A$, we must know $x \in A$ or $x \in C$. Therefore, $x \in A \cup C$ follows from applying the definition of set union. Combining this with $x \notin B$, this means that $x \in (A \cup C) \setminus B$ by the definition of intersection. Since x was arbitrary, we have shown that the subset claim holds.

b) Claim: $(C \setminus A) \cap (C \setminus B) \subseteq C \setminus (A \cup B)$

Proof or Spoof: Let x be arbitrary. Suppose $x \in (C \setminus A) \cap (C \setminus B)$. The definition of intersection tells us that $x \in C \setminus A$ and $x \in C \setminus B$. Applying the definition of set difference to the first part gives $x \in C$ and $x \notin A$. Since $x \notin A$, we have $x \notin A$ or $x \notin B$, which is equivalent to $x \notin A \cup B$ by the definition of set union. Combining this with $x \in C$, we have $x \in C \setminus (A \cup B)$ by the defintion of set difference. Hence, x is in $(C \setminus A) \cap (C \setminus B)$ iff it is in $C \setminus (A \cup B)$.

Since x was arbitrary, we have shown that the two sets are equal. The desired subset relationship follows from the definition of set equality, showing that the claim holds.

c) Claim: $(\overline{A} \cap \overline{B}) \setminus (A \cup B) = \overline{A \cup B}$

Proof or Spoof: Let x be arbitrary. Then, we can see that the two conditions—being an element of the left set and being an element of the right set—are equivalent as follows:

$$x \in (\overline{A} \cap \overline{B}) \setminus (A \cup B)$$

$\equiv (x \in \overline{A} \cap \overline{B}) \land \neg(x \in A \cup B)$	Def of Difference
$\equiv ((x \in \overline{A}) \land (x \in \overline{B})) \land \neg(x \in A \cup B)$	Def of Intersection
$\equiv (\neg(x \in A) \land \neg(x \in B)) \land \neg(x \in A \cup B)$	Def of Complement
$\equiv \neg((x \in A) \lor (x \in B)) \land \neg(x \in A \cup B)$	De Morgan
$\equiv \neg(x \in A \cup B) \land \neg(x \in A \cup B)$	Def of Union
$\equiv \neg(x \in A \cup B)$	Idempotence
$\equiv x \in \overline{A \cup B}$	Def of Complement

Since x was arbitrary, we have shown that two sets contain the same elements.

Task 7 – Extra Credit: Stone By the Company He Keeps [0 pts]

Consider an infinite sequence of positions $1, 2, 3, \ldots$ and suppose we have a stone at position 1 and another stone at position 2. In each step, we choose one of the stones and move it according to the following rule: Say we decide to move the stone at position i; if the other stone is not at any of the positions $i+1, i+2, \ldots, 2i$, then it goes to $2i$, otherwise it goes to $2i+1$.

For example, in the first step, if we move the stone at position 1, it will go to 3 and if we move the stone at position 2 it will go to 4. Note: no matter how we move the stones, they will never be at the same position.

Use induction to prove that, for any given positive integer n, it is possible to move one of the stones to position n. For example, if $n = 7$ first we move the stone at position 1 to 3. Then, we move the stone at position 2 to 5 Finally, we move the stone at position 3 to 7.