1. Sheep and Mountain Goats [15 points]

For the rest of the page, let the domain of discourse be mammals. Interpret all sentences below as being in "mathematical English."

You may use the following predicates; the definition for the predicate is given after the colon in the list below.

- `MountainGoat`(x) : x is a mountain goat
- `Sheep`(x) : x is a sheep
- `Climb`(x) : x is climbing (Treat "climbs" and "is climbing" as equivalent ideas in this problem)
- `SummitsEverest`(x) : x summits Mt. Everest
- `ReachesTop`(x) : x reaches the top
- `Falls`(x) : x falls
- `Knows`(x, y): x knows y
- `Strong`(x) : x is strong

Translate the following English sentences into **PREDICATE LOGIC**.

(a) Every mountain goat that climbs and summits Mt. Everest is strong.

Solution:

$\forall x((\text{MountainGoat}(x) \land \text{Climb}(x) \land \text{SummitsEverest}(x)) \to \text{Strong}(x))$

(b) For every mammal that summits Mt. Everest, there is a Mountain Goat that knows it. **Solution:**

$\forall x \exists y(\text{SummitsEverest}(x) \to (\text{MountainGoat}(y) \land \text{Knows}(y, x))$

Write a **PREDICATE LOGIC** statement equivalent to the English statement below by taking the contrapositive of the implication inside it. Be sure to write an equivalent statement to the whole sentence, not just the implication inside.

(c) For every mammal, if it falls, then it is a sheep.
Solution:

The original sentence in predicate logic is $\forall x(\text{Falls}(x) \to \text{Sheep}(x))$. Taking the contrapositive of the implication inside it yields

$$\forall x(\neg\text{Sheep}(x) \to \neg\text{Falls}(x))$$

Write the negation of the following sentences **IN ENGLISH**. Your English sentences must have negations applied only to individual predicates, but you do not need to use domain restriction.

(d) There is a mountain goat that is climbing and reaches the top.
Solution:

The original sentence in predicate logic is $\exists x(\text{MountainGoat}(x) \land \text{Climb}(x) \land \text{ReachesTop}(x)$. The negation would be $\forall x(\neg\text{MountainGoat}(x) \lor \neg\text{Climb}(x) \lor \neg\text{ReachesTop}(x)$.

The negated translation in English would be "Every mammal is not a mountain goat or is not climbing or does not reach the top".
Alternatively: "Every mountain goat that is climbing does not reach the top"
Alternatively: "Every mountain goat that reaches the top is not climbing"

(e) There is a sheep that summits Mt. Everest that every mountain goat knows. **Solution:**

The original sentence in predicate logic is $\exists x \forall y(\text{Sheep}(x) \wedge \text{SummitsEverest}(x) \wedge (\text{MountainGoat}(y) \rightarrow \text{Knows}(y, x)))$. The negation would be

$$\begin{aligned}
&\neg(\exists x \forall y(\text{Sheep}(x) \wedge \text{SummitsEverest}(x) \wedge (\text{MountainGoat}(y) \rightarrow \text{Knows}(y, x))))\\
&= \forall x \exists y(\neg\text{Sheep}(x) \vee \neg\text{SummitsEverest}(x) \vee \neg(\text{MountainGoat}(y) \rightarrow \text{Knows}(y, x)))\\
&= \forall x \exists y(\neg\text{Sheep}(x) \vee \neg\text{SummitsEverest}(x) \vee \neg(\neg\text{MountainGoat}(y) \vee \text{Knows}(y, x)))\\
&= \forall x \exists y(\neg\text{Sheep}(x) \vee \neg\text{SummitsEverest}(x) \vee (\text{MountainGoat}(y) \wedge \neg\text{Knows}(y, x)))
\end{aligned}$$

The negated translation in English would be "For every mammal: it is not a sheep or does not summit Mt. Everest or there is a mountain goat that does not know that mammal."

Alternatively: "For every sheep that summits Mt. Everest, there is a mountain goat that does not know it.

2. A Proof! [13 points]

Let A, B be arbitrary sets. We have the following claim:

$$(A \cap \overline{B}) \cup (\overline{A} \cap B) \subseteq \overline{(A \cap B)}$$

(a) Translate the claim into predicate logic. [3 points] **Solution:**

$$\forall x(x \in (A \cap \overline{B}) \cup (\overline{A} \cap B) \rightarrow x \in \overline{(A \cap B)})$$

(b) Write an **English** proof for this claim. Do not write a logical equivalences proof, or part of a logical equivalences proof—your proof must be in English, though you can still use symbols appropriately. [10 points]

Hint: Parts of this proof are very similar to the set proofs you did on the homework!

Hint: If you get stuck on this proof (or another one), it's ok to skip a problem for a bit and come back. It's also a good idea to write the start and end of the proof at least, even if you don't know what happens in the middle.

Solution:

Let x be an arbitrary element in $(A \cap \overline{B}) \cup (\overline{A} \cap B)$. By definition of set Union, $x \in (A \cap \overline{B})$ or $x \in (\overline{A} \cap B)$. Then there are 2 cases:

Case 1: $x \in (A \cap \overline{B})$. Then by definition of set Intersection, $x \in A$ and $x \in \overline{B}$. Then x is not in B by definition of set complement. Using the intro OR rule, we have $x \notin A \vee x \notin B$. Applying DeMorgan's Law, we have $\neg(x \in A \wedge x \in B)$. Then by the definition of set intersection, it's not true that x is in $A \cap B$. Then by definition of set complement, $x \in \overline{A \cap B}$.

Case 2: $x \in (\overline{A} \cap B)$. Then by definition of set Intersection, $x \in \overline{A}$ and $x \in B$. Then x is not in A by definition of set Complement. Since it is true that $x \notin A$, we certainly have: $x \notin A \vee x \notin B)$. Applying DeMorgan's law, we have $\neg(x \in A \wedge x \in B)$ Then by the definition of set intersection, it's not the case that x is in $A \cap B$. Then by definition of set complement, $x \in \overline{A \cap B}$.

Since these cases are exhaustive, $x \in \overline{A \cap B}$. Since x is arbitrary, by definition of subset, we've proven that $(A \cap \overline{B}) \cup (\overline{A} \cap B) \subseteq \overline{(A \cap B)}$.

3. Another Proof! [10 points]

Show for all integers a, b, n where $n > 0$ that if $a \equiv b \pmod{n}$ then $a + 3n \equiv b - 2n \pmod{n}$.

In this problem, you **may** use

- The definition of equivalence (mod n) (i.e., that $x \equiv y \pmod{n}$ if and only if $n|(y - x)$.
- The definition of divides (i.e., that $x|y$ if and only if there is an integer z such that $xz = y$.)
- The theorem that if $x \equiv y \pmod{n}$ then $y \equiv x \pmod{n}$
- Algebra

You **may not** use other theorems proven or provided in class materials (for example you **may not** use that "if $a \equiv b \pmod{n}$ then $a + c \equiv b + c \pmod{n}$", or "if $a \equiv b \pmod{n}$ then $a \equiv b + n \pmod{n}$"), unless you reprove them.

Solution:

Let a, b, n be arbitrary integers with $n > 0$ and suppose $a \equiv b \pmod{n}$.

By definition of equivalence mod n, we have $n|(b - a)$. Then, by definition of divides, we can rewrite that as $nk = b - a$ for some integer k. Adding $-5n$ to each side, we get

$$nk - 5n = b - a + (-3n - 2n)$$

Rearranging each side we have

$$n(k - 5) = (b - 2n) - (a + 3n)$$

Noting that $k - 5$ is an integer, we have that $n|(b - 2n) - (a + 3n)$

Then by definition of equivalence mod n, we have $a + 3n \equiv b - 2n \pmod{n}$

4. Induction! [20 points]

Consider the following code snippet.

```
int Mystery(int n) {
    if n == 0:
        return 5
    if n == 1:
        return 16
    return 7 * Mystery(n - 1) - 10 * Mystery(n - 2)
}
```

In this problem, we will use `Mystery(n)` to refer to the value returned by the code snippet above when run on input `n`. For example, `Mystery(2)` $= (7 \cdot 16) - (10 \cdot 5) = 62$.

Use induction to show that `Mystery(n)` $= 3 \cdot 2^n + 2 \cdot 5^n$ for all integers $n \geq 0$.

Solution:

Let $P(n)$ be "`Mystery(n)` $= 3 \cdot 2^n + 2 \cdot 5^n$." We will show $P(n)$ holds for all integers $n \geq 0$ by induction on n.

Base cases: For $n = 0$, we have `Mystery(0)` $= 5 = 3 \cdot 2^0 + 2 \cdot 5^0 = 3 \cdot 1 + 2 \cdot 1$, so $P(0)$ holds. For $n = 1$, we have `Mystery(1)` $= 16 = 3 \cdot 2^1 + 2 \cdot 5^1 = 3 \cdot 2 + 2 \cdot 5$, so $P(1)$ holds.

Inductive Hypothesis: Suppose $P(j)$ holds for $0 \leq j \leq k$ for some arbitrary integer $k \geq 1$, i.e. `Mystery(k)` $= 3 \cdot 2^k + 2 \cdot 5^k$.

Inductive step: We will show that $P(k+1)$ holds, i.e. `Mystery(k + 1)` $= 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1}$.

$$
\begin{aligned}
\texttt{Mystery(k + 1)} &= 7 \cdot \texttt{Mystery(k + 1 - 1)} - 10 \cdot \texttt{Mystery(k + 1 - 2)} && \text{Recursive case of Mystery; } k \geq 1 \text{ so } k+1 > 1 \\
&= 7(3 \cdot 2^k + 2 \cdot 5^k) - 10(3 \cdot 2^{k-1} + 2 \cdot 5^{k-1}) && \text{IH} \\
&= 21 \cdot 2^k + 14 \cdot 5^k - 30 \cdot 2^{k-1} - 20 \cdot 5^{k-1} && \text{Algebra} \\
&= 21 \cdot 2^k + 14 \cdot 5^k - 15 \cdot 2^k - 4 \cdot 5^k && \text{Algebra} \\
&= 6 \cdot 2^k + 10 \cdot 5^k && \text{Algebra} \\
&= 3 \cdot 2^{k+1} + 2 \cdot 5^{k+1} && \text{Algebra}
\end{aligned}
$$

Therefore, `Mystery(n)` $= 3 \cdot 2^n + 2 \cdot 5^n$ for all integers $n \geq 0$ by induction.

5. Short Answer [12 points]

(a) You wish to show “For every integer, if it is divisible by 4 then it is even” with a proof by contradiction. Complete **just the first sentence** of the proof (be sure to assume everything you can, and to explicitly include any quantifiers) [4 points]

Suppose, for the sake of contradiction, that...

Solution:

”there is some integer divisible by 4 that is odd.”

(b) Which of the following is a multiplicative inverse of $3 \pmod 5$ [2 points]
Hint: You don’t have to use the process to find a multiplicative inverse if you remember what a multiplicative inverse is!

- ◯ 1
- ◯ 2
- ◯ 3
- ◯ 4

Solution:

2

(c) Which of the following expressions are equivalent to $(p \vee q) \wedge v$. Mark ALL that apply [2 points]

- ☐ $(\neg p \rightarrow q) \wedge v$
- ☐ $(q \rightarrow \neg p) \wedge v$
- ☐ $(p \rightarrow q) \wedge v$
- ☐ $p \vee (q \wedge v)$

Solution:

First option only

(d) A reliable source tells you the following statement is true: “If a dog likes purple and gold then it’s a husky.” What can you conclude about the statement “If a dog is a husky, then it likes purple and gold.” ? [2 points]

- ◯ The second statement must be true.
- ◯ The second statement might or might not be true.
- ◯ The second statement cannot be true.

Solution:

Answer choice 2

(e) Which of these is the Extended Euclidean algorithm good for. Mark ALL that apply [2 points]

- ☐ Finding the gcd of two integers
- ☐ Finding multiplicative inverse of an integer (mod n)
- ☐ Finding $a\%b$ (i.e., the remainder when you divide a by b)
- ☐ Determining how many prime numbers exist between two given numbers.

Solution:

First two options