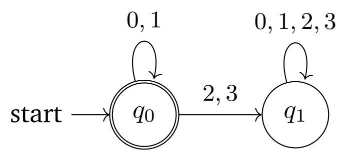
Section 08: Solutions

1. DFAs, Stage 1

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1, 2, 3\}$.

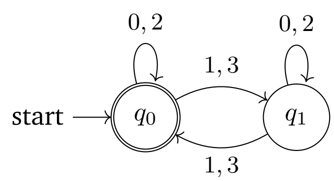
(a) All binary strings.
Solution:



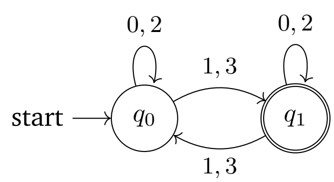
q_0: binary strings

q_1: strings that contain a character which is not 0 or 1.

(b) All strings whose digits sum to an even number.
Solution:



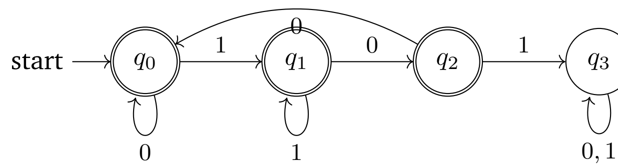
(c) All strings whose digits sum to an odd number.
Solution:



2. DFAs, Stage 2

Construct DFAs to recognize each of the following languages. Let $\Sigma = \{0, 1\}$.

(a) All strings which do not contain the substring 101.
Solution:



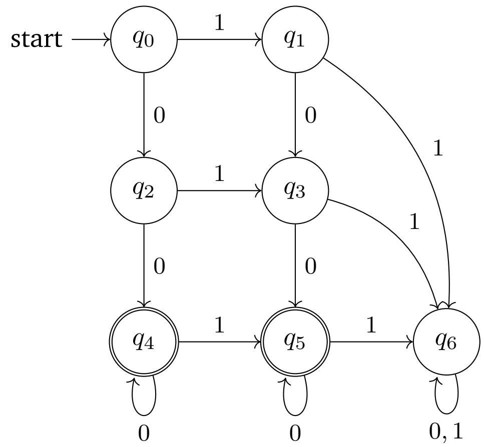
q_3: string that contain 101.

q_2: strings that don't contain 101 and end in 10.

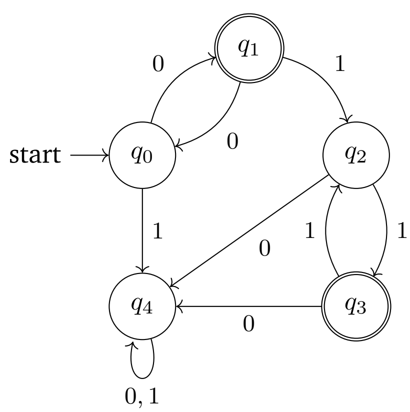
q_1: strings that don't contain 101 and end in 1.

q_0: ε, 0, strings that don't contain 101 and end in 00.

(b) All strings containing at least two 0's and at most one 1.
Solution:

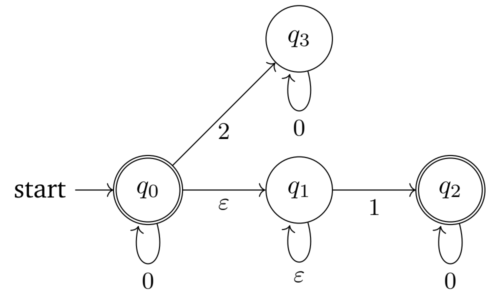


(c) All strings containing an even number of 1's and an odd number of 0's and not containing the substring 10.
Solution:



3. NFAs

(a) What language does the following NFA accept?

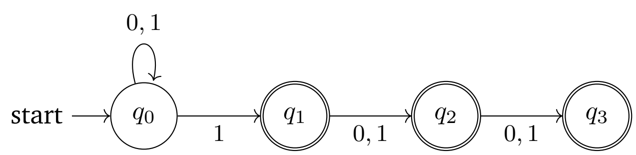


Solution:

Binary strings containing any number of 0s and at most one 1.

(b) Create an NFA for the language "all binary strings that have a 1 as one of the last three digits".
Solution:

The following is one such NFA:



4. All The Models

Construct a valid regular expression, CFG, and DFA for the following languages.

(a) All strings whose base-6 representation is divisible by 3 (leading zeros are ok). Let $\Sigma = \{0, 1, 2, 3, 4, 5\}$.

Solution:

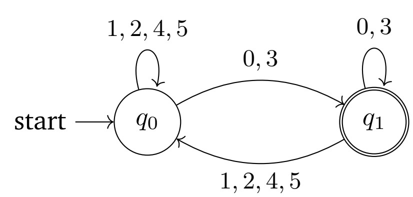
Regular Expression:

$$(0 \cup 1 \cup 2 \cup 3 \cup 4 \cup 5)^*(0 \cup 3)$$

CFG:

$$\mathbf{S} \to \mathbf{T}0 \mid \mathbf{T}3$$
$$\mathbf{T} \to 0\mathbf{T} \mid 1\mathbf{T} \mid 2\mathbf{T} \mid 3\mathbf{T} \mid 4\mathbf{T} \mid 5\mathbf{T} \mid \varepsilon$$

DFA:



(b) All binary strings of 0s capped by a 1 on either side.

Solution:

We are working with binary strings, therefore $\Sigma = \{0, 1\}$.

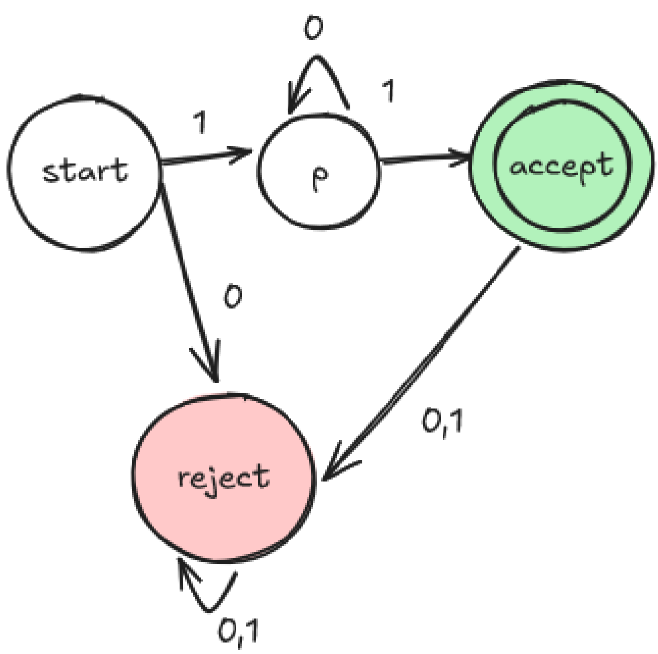
Regular Expression:

$$1(0)^*1$$

CFG:

$$\mathbf{S} \to 1\mathbf{T}1$$
$$\mathbf{T} \to 0\mathbf{T} \mid \varepsilon$$

DFA:



5. Irregularity

(a) Let $\Sigma = \{0, 1\}$. Prove that $\{0^n1^n0^n \ : \ n \geq 0\}$ is not regular.
Solution:

Suppose for the sake of contradiction that $L = \{0^n1^n0^n \ : \ n \geq 0\}$ is regular. Then there is some DFA M that accepts **exactly** L.
Let $S = \{0^n1^n \ : \ n \geq 0\}$
Because M is finite and S is infinite, there are two (different) strings, x, y in S such that x and y go to the same state when read by M. Since both are in S, we can write $x = 0^a1^a$ for some integer $a \geq 0$, and $y = 0^b1^b$ for some integer $b \geq 0$. Since $x \neq y$, $a \neq b$.
Consider the string $z = 0^a$

Since x, y both end up in the same state, and M is deterministic, xz and yz will also end up at the same state q in M. Observe that $xz = 0^a 1^a 0^a$, so $xz \in L$, but $yz = 0^b 1^b 0^a$ and $a \neq b$, so $yz \notin L$. So q is both an accept and reject state, which is a contradiction.
Therefore, $\{0^n 1^n 0^n \ : \ n \geq 0\}$ must be an irregular language.

(b) Let $\Sigma = \{0, 1, 2\}$. Prove that $\{0^n (12)^m \ : \ n \geq m \geq 0\}$ is not regular.
Solution:

Note: The key to this proof is that we have to have at least as many 0s as we do 12s. If we got rid of that requirement, the language would be regular.

Suppose for the sake of contradiction that $L = \{0^n (12)^m \ : \ n \geq m \geq 0\}$ is regular. Then there is a DFA M that accepts exactly L.
Consider $S = \{0^n : n \geq 0\}$.
Since M is finite and S is infinite, there are two (different) strings, x, y in S such that x and y go to the same state when read by M. Since both are in S, we can write $x = 0^a$ for some integer $a \geq 0$, and $y = 0^b$ for some integer $b \geq 0$. Since $x \neq y$, $a \neq b$. **Without loss of generality, suppose** $a > b$.
Consider the string $z = (12)^a$
Since x, y both end up in the same state, and M is deterministic, xz and yz will also end up at the same state q in M. Observe that $xz = 0^a (12)^a$, so $xz \in L$, but $yz = 0^b (12)^a$ with $b < a$, so $yz \notin L$. So q is both an accept and reject state, which is a contradiction.
Therefore $\{0^n (12)^m \ : \ n \geq m \geq 0\}$ must be an irregular language.