Section 03: Solutions

1. Direct Proof

(a) Let the domain of discourse be integers. Define the predicates $Odd(x) := \exists k(x = 2k + 1)$, and $Even(x) := \exists k(x = 2k)$. Translate the following claim to predicate logic:

The sum of an even and odd integer is odd.

Solution:

$\forall n \forall m((Even(n) \wedge Odd(m)) \rightarrow Odd(n + m))$

(b) Prove that the claim holds.
Solution:

Let n and m be arbitrary integers. Suppose n is even and m is odd. Then by definition of even, $n = 2k$ for some integer k. By definition of odd, $m = 2j + 1$ for some integer j. Then consider $n + m$:

$$n + m = 2k + 2j + 1 = 2(k + j) + 1$$

Since k and j are integers, $k + j$ is an integer.

Then $n + m$ is 2 times an integer plus 1. Thus by definition of odd, $n + m$ is odd. Since n and m were arbitrary, the sum of any even and odd integer is odd.

2. Proof of Biconditional

(a) Let the domain of discourse be integers. Define the predicates $Odd(x) := \exists k(x = 2k + 1)$, and $Even(x) := \exists k(x = 2k)$. Translate the following claim to predicate logic:

For all integers n, $n - 4$ is even if and only if $n + 17$ is odd.

Solution:

$\forall n(Even(n - 4) \leftrightarrow Odd(n + 17))$

(b) For each direction, write the first few sentences and last few sentences of the English proof.
Solution:

$\Rightarrow$ Let n be an arbitrary integer. Suppose that $n - 4$ is even. Then by definition of even, $n - 4 = 2k$ for some integer k.
...
Thus by definition of odd, $n + 17$ is odd. Since n was arbitrary, we have shown that for all integers n if $n - 4$ is even, then $n + 17$ is odd.

$\Leftarrow$ Let n be an arbitrary integer. Suppose $n + 17$ is odd. Then by definition of odd, $n + 17 = 2k + 1$ for some integer k.
...
So by definition even, $n - 4$ is even. Since n was arbitrary, we have shown that for all integers n, if $n + 17$ is odd, then $n - 4$ is even.

(c) Prove that the claim holds.
Solution:

$\Rightarrow$ Let n be an arbitrary integer. Suppose that $n-4$ is even. Then by definition of even, $n-4=2k$ for some integer k. Then observe that:

$$n-4=2k$$
$$n+17=2k+21 \qquad \text{Adding 21 to both sides}$$
$$n+17=2(k+10)+1 \qquad \text{Factoring}$$

Thus $n+17=2(k+10)+1$. Since k is an integer, $k+10$ is an integer. So $n+17$ is 2 times an integer plus 1. Thus by definition of odd, $n+17$ is odd. Since n was arbitrary, we have shown that for all integers n if $n-4$ is even, then $n+17$ is odd.

$\Leftarrow$ Let n be an arbitrary integer. Suppose $n+17$ is odd. Then by definition of odd, $n+17=2k+1$ for some integer k. Then observe that:

$$n+17=2k+1$$
$$n-4=2k+1-21 \qquad \text{Subtracting 21 from both sides}$$
$$n-4=2(k-10) \qquad \text{Factoring}$$

Thus $n-4=2(k-10)$. Since k is an integer, $k-10$ is an integer. So $n-4$ is 2 times an integer. So by definition even, $n-4$ is even. Since n was arbitrary, we have shown that for all integers n, if $n+17$ is odd, then $n-4$ is even.

3. Proof by Contrapositive

(a) Let the domain of discourse be integers. Define the predicates $Odd(x) := \exists k(x=2k+1)$ and $Even(x) := \exists k(x=2k)$. Translate the following claim to predicate logic:

For all integers x, if $7x+9$ is even, the x is odd.

Solution:

$\forall x(Even(7x+9) \rightarrow Odd(x))$

(b) Try to prove the claim directly. Do you get stuck?
Note that it is actually possible to write a direct proof, though it is slightly more difficult to see how.
Solution:

You probably tried something like this:
Let x be an arbitrary integer. Suppose that $7x+9$ is even. Then by definition of even, there exists some integer k such that $7x+9=2k$. Then observe that

$$7x+9=2k$$
$$7x=2k-9 \qquad \text{Subtracting 9 from both sides}$$
$$x=(2k-9)/7 \qquad \text{Dividing both sides by 7}$$

Now what? Mathemagic?
It's easier to start with x and multiply by 7 than to start with $7x$ and divide by 7. If we use the contrapositive, we can do exactly that.

It is actually possible to write a direct proof. If you're curious, here's how.
Let x be an arbitrary integer. Suppose that $7x + 9$ is even. Then by the definition of even, there exists some integer k such that $7x + 9 = 2k$. Then subtracting $6x + 9$ from both sides, we have:

$$x = 2k - 6x - 9 = 2(k - 3x - 5) + 1$$

Since k and x are integers, $k - 3x - 5$ is an integer. So x is 2 times an integer plus 1. So by definition of odd, x is odd. Since x was arbitrary, this shows that for all integers x, if $7x + 9$ is even then x is odd.

(c) What is the contrapositive of the claim in predicate logic?
Solution:

$\forall x(Even(x) \rightarrow Odd(7x + 9))$

(d) Prove that the claim holds by proving the contrapositive.
Solution:

We prove by contrapositive. Let x be an arbitrary integer. Suppose that x is even. Then by definition of even, there exists some integer k such that $x = 2k$. Then consider $7x + 9$:

$$7x + 9 = 7(2k) + 9 = 14k + 9 = 2(7k + 4) + 1$$

Since k is an integer, $7k + 4$ is an integer. So $7x + 9$ is 2 times an integer plus 1. So by definition of odd, $7x + 9$ is odd. Since x was arbitrary, this shows that for all integers x, if x is even then $7x + 9$ is odd. Thus the contrapositive also holds: for all integers x, if $7x + 9$ is even, then x is odd.

4. Proof by Cases

Prove by cases that for all integers n, $n^2 - 3n$ is even.
Solution:

Let n be an arbitrary integer.

Case 1: n is even.
Then by definition of even, $n = 2k$ for some integer k. Then consider $n^2 - 3n$:

$$n^2 - 3n = (2k)^2 - 3(2k) = 4k^2 - 6k = 2(2k^2 - 3k)$$

Since k is an integer, $2k^2 - 3k$ is an integer. Then $n^2 - 3n$ is 2 times an integer. So by definition of even, $n^2 - 3n$ is even. Thus we have shown that in this case, $n^2 - 3n$ is even.

Case 2: n is odd
Then by definition of odd, $n = 2k + 1$ for some integer k. Then consider $n^2 - 3n$:

$$n^2 - 3n = (2k + 1)^2 - 3(2k + 1) = (4k^2 + 4k + 1) - (6k + 3) = 4k^2 - 2k - 2 = 2(2k^2 - k - 1)$$

Since k is an integer, $2k^2 - k - 1$ is an integer. Then $n^2 - 3n$ is 2 times an integer. So by definition of even, $n^2 - 3n$ is even. Thus we have shown that in this case, $n^2 - 3n$ is even.

Conclusion: We have shown that in all cases, $n^2 - 3n$ is even. Since n was arbitrary, for all integers n, $n^2 - 3n$ is even.

5. Disproving a For All Claim

Disprove the following claim:

For all integers a, b, c if $ac = bc$ then $a = b$.

Solution:

Consider $a = 1, b = 2, c = 0$. Then $ac = 0 = bc$ but $a \neq b$. This is a counterexample. Therefore, the claim is false.

6. Disproving a There Exists Claim

Consider the following claim:

There exists an integer x such that x is even and x^2 is odd.

(a) This claim is false. Without using any formal reasoning, what does your intuition say about how to disprove this claim?
Solution:

Show that if an integer x is even, then x^2 must also be even.

(b) Let the domain of discourse be integers. Define the predicates $Odd(x) := \exists k(x = 2k + 1)$ and $Even(x) = \exists k(x = 2k)$. Translate the above claim to predicate logic.
Solution:

$\exists x(Even(x) \wedge Odd(x^2))$

(c) Negate the predicate logic translation. Then use a chain of logical equivalences to show that your negation is equivalent to $\forall x(Even(x) \rightarrow Even(x^2))$.

Hint: You may use the fact that $\neg Odd(a) \equiv Even(a)$.
Solution:

$$\begin{aligned}
\neg\exists x(Even(x) \wedge Odd(x^2)) &\equiv \forall x \neg(Even(x) \wedge Odd(x^2)) && \text{[DeMorgan's Law for Quantifiers]}\\
&\equiv \forall x(\neg Even(x) \vee \neg Odd(x^2)) && \text{[DeMorgan's Law]}\\
&\equiv \forall x(\neg Even(x) \vee Even(x^2)) && \text{[Definition of Odd and Even]}\\
&\equiv \forall x(Even(x) \rightarrow Even(x^2)) && \text{[Law of Implication]}
\end{aligned}$$

(d) Recall that to disprove a claim, we must prove its negation. Part (c) shows us that to disprove the above claim, we should prove that if an integer x is even, then x^2 is also even. Does this match your intuition?
Solution:

Yes!!

(e) Write a proof of the fact that if an integer x is even, then x^2 is also even.
Solution:

Let x be an arbitrary integer. Suppose that x is even. Then by definition of even, there exists some integer k such that $x = 2k$.
Squaring both sides, we see that:

$$x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$$

Since k is an integer, $2k^2$ is also an integer. So by definition of even, x^2 is even.
Since x was an arbitrary integer, we can conclude that for all integers x, if x is even, then x^2 is even.

(f) Celebrate! You have successfully disproved the claim!