CSE 311 Section 3

Proof Techniques

Administrivia

Announcements & Reminders

- HW1
  - If you think something was graded incorrectly, submit a regrade request!
  - You will have the chance to re-submit two questions!

- HW2 due yesterday 7/9 @ 11:59PM on Gradescope
  - Use late days if you need them!

- HW3
  - Due Wednesday 7/16 @ 11:59pm

References

- Helpful reference sheets can be found on the course website!
  - https://courses.cs.washington.edu/courses/cse311/25su/resources/
- How to LaTeX (found on Assignments page of website):
  - https://courses.cs.washington.edu/courses/cse311/25su/assignments/HowToLaTeX.pdf
- Proof Tips
  - https://courses.cs.washington.edu/courses/cse311/25su/resources/handout02-proof-tips.pdf
- Proof Style Guide
  - https://courses.cs.washington.edu/courses/cse311/25su/resources/styleguide.pdf
- Plus more!

English Proofs

Writing a Proof (symbolically or in English)

- Don’t just jump right in!
- Look at the **claim**, and make sure you know:
  - What every word in the claim means
  - What the claim as a whole means
- Translate the claim in predicate logic.
- Next, write down the **Proof Skeleton**:
  - **Where to start**
  - **What your target is**
- Then once you know what claim you are proving and your starting point and ending point, you can finally write the proof!

Helpful Tips for English Proofs

- Start by introducing your assumptions
  - Introduce variables with "let"
    - "Let x be an arbitrary prime number…"
  - Introduce assumptions with "suppose"
    - "Suppose that $y \in A \land y \notin B$…"

- When you supply a value for an existence proof, use "Consider"
  - "Consider $x = 2$…"

- **ALWAYS** state what type your variable is (integer, set, etc.)

- Universal Quantifier means variable must be arbitrary

- Existential Quantifier means variable can be specific

Problem 1 – Direct Proof

a) Let the domain of discourse be integers. Define the predicates $\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$, and $\mathrm{Even}(x) \coloneqq \exists k(x = 2k)$. Translate the following claim into predicate logic:

The sum of an even and odd integer is odd.

b) Prove that the claim holds.

Work on parts (a) and (b) of this problem with the people around you, and then we'll go over it together!

Problem 1 – Direct Proof

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$$\forall n \forall m\big((\text{Even}(n) \land \text{Odd}(m)\big) \rightarrow \text{Odd}(n + m))$$

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Lets walk through part (b) together!

Problem 1 – Direct Proof

b) Prove that the claim holds.

$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$
$\text{Even}(x) \coloneqq \exists k(x = 2k)$
Claim:
$\forall n \forall m((\text{Even}(n) \land \text{Odd}(m)) \rightarrow \text{Odd}(n + m))$

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Claim:
$\forall n \forall m((\text{Even}(n) \land \text{Odd}(m)) \rightarrow \text{Odd}(n + m))$

b) Prove that the claim holds.

Let n and m be arbitrary integers.

…

Since n and m were arbitrary, the sum of any even and odd integer is odd.

A good place to start is to set up arbitrary variables, and state your target!

Problem 1 – Direct Proof

$\mathrm{Odd}(x) := \exists k(x = 2k + 1)$
$\mathrm{Even}(x) := \exists k(x = 2k)$
Claim:
$\forall n \forall m((\mathrm{Even}(n) \wedge \mathrm{Odd}(m)) \rightarrow \mathrm{Odd}(n + m))$

b) Prove that the claim holds.

Let n and m be arbitrary integers. Suppose n is even and m is odd.

…

Thus by (some reasoning here), $n + m$ is odd. Since n and m were arbitrary, the sum of any even and odd integer is odd.

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b) Prove that the claim holds.

Let n and m be arbitrary integers. Suppose n is even and m is odd. Then by definition of even, $n = 2k$ for some integer k. By definition of odd, $m = 2j + 1$ for some integer j.

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Then $n + m$ is 2 times an integer plus 1. Thus by definition of odd, $n + m$ is odd.
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Let n and m be arbitrary integers. Suppose n is even and m is odd. Then by definition of even, $n = 2k$ for some integer k. By definition of odd, $m = 2j + 1$ for some integer j.

Then consider $n + m$:

$$n + m = 2k + 2j + 1$$

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Then $n + m$ is 2 times an integer plus 1. Thus by definition of odd, $n + m$ is odd. Since n and m were arbitrary, the sum of any even and odd integer is odd.

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Then consider $n + m$:

$$\begin{aligned} n + m &= 2k + 2j + 1 \\ &= 2(k + j) + 1 \end{aligned}$$

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Then $n + m$ is 2 times an integer plus 1. Thus by definition of odd, $n + m$ is odd.
Since n and m were arbitrary, the sum of any even and odd integer is odd.

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Then consider $n + m$:

$$\begin{aligned} n + m &= 2k + 2j + 1 \\ &= 2(k + j) + 1 \end{aligned}$$

Since k and j are integers, $k + j$ is an integer.
Then $n + m$ is 2 times an integer plus 1. Thus by definition of odd, $n + m$ is odd.
Since n and m were arbitrary, the sum of any even and odd integer is odd.

Problem 2 - Proof of Biconditional

(a) Let the domain of discourse be integers. Define the predicates Odd$(x) := \exists k(x = 2k+1)$, and Even$(x) := \exists k(x = 2k)$. Translate the following claim to predicate logic:

For all integers n, $n - 4$ is even if and only if $n + 17$ is odd.

Work on part (a) with the people around you, and then we'll go over it together!

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For all integers n, $n - 4$ is even if and only if $n + 17$ is odd.

$$\forall n(\text{Even}(n-4) \leftrightarrow \text{Odd}(n+17))$$

Problem 3 - Proof by Contrapositive

(a) Let the domain of discourse be integers. Define the predicates Odd$(x) := \exists k(x = 2k+1)$, and Even$(x) :=$ $\exists k(x = 2k)$. Translate the following claim to predicate logic:

For all integers x, if $7x + 9$ is even, then x is odd.

(b) Try to prove the claim directly. Do you get stuck?

Note that it is actually possible to write a direct proof, though it is slightly more difficult to see how.

(c) What is the contrapositive of the claim in predicate logic?

(d) Prove that the claim holds by proving the contrapostive.

Work on part (a), (b), (c), and (d) with the people around you, and then we'll go over it together!

Problem 3 - Proof by Contrapositive

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For all integers x, if $7x + 9$ is even, then x is odd.

$$\forall x(\text{Even}(7x + 9) \rightarrow \text{Odd}(x))$$

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Let x be an arbitrary integer. Suppose that $7x + 9$ is even.

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Since x was arbitrary, this shows that for all integers x, if $7x + 9$ is even, then x is odd.

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$\forall x(\text{Even}(7x + 9) \rightarrow \text{Odd}(x))$

Note that it is actually possible to write a direct proof, though it is slightly more difficult to see how.

Let x be an arbitrary integer. Suppose that $7x+9$ is even. Then by definition of even, there exists some integer k such that:

$$7x + 9 = 2k$$

This jump seems tricky!

...

So x is 2 times an integer plus 1. By definition of odd, x is odd. Since x was arbitrary, this shows that for all integers x, if $7x + 9$ is even, then x is odd.

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$\forall x(\text{Even}(7x+9) \rightarrow \text{Odd}(x))$

(b) Try to prove the claim directly. Do you get stuck?

Note that it is actually possible to write a direct proof, though it is slightly more difficult to see how.

Let x be an arbitrary integer. Suppose that $7x+9$ is even. Then by definition of even, there exists some integer k such that:

$$7x + 9 = 2k$$

Subtracting $6x + 9$ from both sides, we have:

$$\begin{aligned} x &= 2k - 6x - 9 \\ &= 2(k - 3x - 5) + 1 \end{aligned}$$

Since k and x are integers, $k - 3x - 5$ is an integer. So x is 2 times an integer plus 1. By definition of odd, x is odd. Since x was arbitrary, this shows that for all integers x, if $7x + 9$ is even, then x is odd.

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$$\forall x(\text{Even}(7x+9) \rightarrow \text{Odd}(x))$$

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$$\forall x(\text{Even}(x) \rightarrow \text{Odd}(7x+9))$$

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(d) Prove that the claim holds by proving the contrapostive.

Problem 3 - Proof by Contrapositive

(d) Prove that the claim holds by proving the contrapostive. $\forall x(\text{Even}(x) \to \text{Odd}(7x + 9))$

Let x be an arbitrary integer. Suppose that x is even.

...

Since x was arbitrary, this shows that for all integers x, if x is even then $7x + 9$ is odd.

Thus the contrapositive also holds: for all integers x, if $7x + 9$ is even, then x is odd.

Problem 3 - Proof by Contrapositive

(d) Prove that the claim holds by proving the contrapostive. $\forall x(\text{Even}(x) \rightarrow \text{Odd}(7x + 9))$

Let x be an arbitrary integer. Suppose that x is even. Then by definition of even, there exists some integer k such that:

$$x = 2k$$

Then consider $7x + 9$:

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So $7x + 9$ is 2 times an integer plus 1. By definition of odd, $7x + 9$ is odd. Since x was arbitrary, this shows that for all integers x, if x is even then $7x + 9$ is odd.

Thus the contrapositive also holds: for all integers x, if $7x + 9$ is even, then x is odd.

Problem 3 - Proof by Contrapositive

(d) Prove that the claim holds by proving the contrapostive. $\forall x(\text{Even}(x) \to \text{Odd}(7x+9))$

Let x be an arbitrary integer. Suppose that x is even. Then by definition of even, there exists some integer k such that:

$$x = 2k$$

Then consider $7x + 9$:

$$\begin{aligned} 7x + 9 &= 7(2k) + 9 \\ &= 14k + 9 \\ &= 2(7k + 4) + 1 \end{aligned}$$

Since k is an integer, $7k + 4$ is an integer. So $7x + 9$ is 2 times an integer plus 1. By definition of odd, $7x + 9$ is odd.

Since x was arbitrary, this shows that for all integers x, if x is even then $7x + 9$ is odd.

Thus the contrapositive also holds: for all integers x, if $7x + 9$ is even, then x is odd.

Did it feel easier to prove the claim in this direction?

Problem 4 - Proof by Cases

Prove by cases that for all integers n, $n^2 - 3n$ is even.

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Let n be an arbitrary integer.

We will remember that this line is at the top for the remainder of the proof!

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Prove by cases that for all integers n, $n^2 - 3n$ is even.

Case 1: n is even.

...

Thus we have shown that in this case, $n^2 - 3n$ is even.

Problem 4 - Proof by Cases

Prove by cases that for all integers n, $n^2 - 3n$ is even.

Case 1: n is even. Then by definition of even, $n = 2k$ for some integer k. Then consider $n^2 - 3n$:

...

Then, $n^2 - 3n$ is 2 times an integer. So, by definition of even, $n^2 - 3n$ is even. Thus we have shown that in this case, $n^2 - 3n$ is even.

Problem 4 - Proof by Cases

Prove by cases that for all integers n, $n^2 - 3n$ is even.

Case 1: n is even.

Then by definition of even, $n = 2k$ for some integer k. Then consider $n^2 - 3n$:

$$\begin{aligned} n^2 - 3n &= (2k)^2 - 3(2k) \\ &= 4k^2 - 6k \\ &= 2(2k^2 - 3k) \end{aligned}$$

Since k is an integer, $2k^2 - 3k$ is an integer. Then $n^2 - 3n$ is 2 times an integer. So by definition of even, $n^2 - 3n$ is even. Thus we have shown that in this case, $n^2 - 3n$ is even.

Problem 4 - Proof by Cases

Prove by cases that for all integers n, $n^2 - 3n$ is even.

Case 2: n is odd.

...

Thus we have shown that in this case, $n^2 - 3n$ is even.

Problem 4 - Proof by Cases

Prove by cases that for all integers n, $n^2 - 3n$ is even.

Case 2: n is odd.

Then by definition of odd, $n = 2k + 1$ for some integer k. Then consider $n^2 - 3n$:

...

Then $n^2 - 3n$ is 2 times an integer. So by definition of even, $n^2 - 3n$ is even. Thus we have shown that in this case, $n^2 - 3n$ is even.

Problem 4 - Proof by Cases

Prove by cases that for all integers n, $n^2 - 3n$ is even.

Case 2: n is odd.

Then by definition of odd, $n = 2k + 1$ for some integer k. Then consider $n^2 - 3n$:

$$\begin{aligned} n^2 - 3n &= (2k+1)^2 - 3(2k+1) \\ &= 4k^2 + 4k + 1 - 6k - 3 \\ &= 4k^2 - 2k - 2 \\ &= 2(2k^2 - k - 1) \end{aligned}$$

Since k is an integer, $2k^2 - k - 1$ is an integer. Then $n^2 - 3n$ is 2 times an integer. So by definition of even, $n^2 - 3n$ is even. Thus we have shown that in this case, $n^2 - 3n$ is even.

Problem 4 - Proof by Cases

Prove by cases that for all integers n, $n^2 - 3n$ is even.

Conclusion: Thus we have shown that in all cases, $n^2 - 3n$ is even. Since n was arbitrary, for all integers n, $n^2 - 3n$ is even.

Problem 5 - Disproving a “For All” Claim

Disprove the following claim:

For all integers a, b, c if $ac = bc$ then $a = b$.

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Step 1

Since we are disproving a ∀ claim, we just need to find one example!

Problem 5 - Disproving a “For All” Claim

Disprove the following claim:

For all integers a, b, c if $ac = bc$ then $a = b$.

Step 1

Since we are disproving a $\forall$ claim, we just need to find one example!

Step 2

Consider $a = 1, b = 2, c = 0$. Then $ac = 0 = bc$ but $a \neq b$.

Since we found one counterexample to the $\forall$ claim, we have disproved the claim.

That's All, Folks!

Thanks for coming to section this week!
Any questions?