CSE 311 Section 2

Equivalences and Predicate Logic

Administrivia & Introductions

Announcements & Reminders

- Sections are Graded
  - You will be graded on section participation, so please try to come ☺

- HW1 due YESTERDAY on Gradescope
  - Remember, you have 6 late days to use throughout the quarter
  - You can use up to 3 late days on any 1 assignment
  - You don't get extra credit for having any unused late days, so feel free to use them if you need them!
  - Let us know if you have questions about HW!

- Check the course website for OH times!
  - There are office hours every day, so come visit if you have questions!

References

- Helpful reference sheets can be found on the course website!
  - https://courses.cs.washington.edu/courses/cse311/25su/resources/
- How to LaTeX (found on Assignments page of website):
  - https://courses.cs.washington.edu/courses/cse311/25su/assignments/HowToLaTeX.pdf
- Equivalence Reference Sheet
  - https://courses.cs.washington.edu/courses/cse311/25su/resources/reference-logical_equiv.pdf
- Boolean Algebra Reference Sheet
  - https://courses.cs.washington.edu/courses/cse311/25su/resources/reference-boolean-alg.pdf
- Plus more!

Typesetting

- You are STRONGLY ENCOURAGED to use LaTeX for your assignments.
- We have lots of resources available to help you get started typesetting with LaTeX!
- Come to office hours and we are happy to answer any questions!!

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\begin{align*}
  \neg p \rightarrow (q \rightarrow r) &\equiv \neg \neg p \vee (q \rightarrow r) &&\text{Law Of Impl.} \\
  &\equiv p \vee (q \rightarrow r) &&\text{Double Neg} \\
  &\equiv p \vee (\neg q \vee r) &&\text{Law of Impl.} \\
  &\equiv (p \vee \neg q) \vee r &&\text{Assoc.} \\
  &\equiv (\neg q \vee p) \vee r &&\text{Comm.} \\
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  &\equiv q \rightarrow (p \vee r) &&\text{Law of Imp.}
\end{align*}
```

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\end{aligned}$$

Equivalence Proofs

Equivalence Proof Review

$p \wedge (p \rightarrow q) \equiv p \wedge (\neg p \vee q)$	[Law of Implication]
$\equiv (p \wedge \neg p) \vee (p \wedge q)$	[Distributivity]
$\equiv F \vee (p \wedge q)$	[Negation]
$\equiv (p \wedge q) \vee F$	[Commutativity]
$\equiv p \wedge q$	[Identity]

Notice: You may be tempted to use the **Identity** property immediately if you see **F ∨ p** but you need to use **Commutativity** first!

Problem 2 – Equivalences

Prove that each of the following pairs of propositional formulae are equivalent using propositional equivalences.

a) $p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$
You may use the rule: $p \leftrightarrow q \equiv (p \rightarrow q) \land (q \rightarrow p)$

b) $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \lor r)$

Work on part (b) with the people around you, and then we'll go over it together!

Problem 2 – Equivalences

b) $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \vee r)$

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$\neg p \rightarrow (q \rightarrow r) \equiv \ldots$

$\equiv q \rightarrow (p \vee r)$

Problem 2 – Equivalences

b) $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \vee r)$

$\neg p \rightarrow (q \rightarrow r) \equiv \neg\neg p \vee (q \rightarrow r)$ [Law of Implication]

$\equiv \neg q \vee (p \vee r)$

$\equiv q \rightarrow (p \vee r)$ [Law of Implication]

Problem 2 – Equivalences

b) $\neg p \to (q \to r) \equiv q \to (p \vee r)$

$\neg p \to (q \to r) \equiv \neg\neg p \vee (q \to r)$ [Law of Implication]

$\equiv p \vee (q \to r)$ [Double Negation]

$\equiv \neg q \vee (p \vee r)$

$\equiv q \to (p \vee r)$ [Law of Implication]

Problem 2 – Equivalences

b) $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \lor r)$

$$
\begin{aligned}
\neg p \rightarrow (q \rightarrow r) &\equiv \neg\neg p \lor (q \rightarrow r) && \text{[Law of Implication]} \\
&\equiv p \lor (q \rightarrow r) && \text{[Double Negation]} \\
&\equiv p \lor (\neg q \lor r) && \text{[Law of Implication]} \\
&\equiv \neg q \lor (p \lor r) && \\
&\equiv q \rightarrow (p \lor r) && \text{[Law of Implication]}
\end{aligned}
$$

Problem 2 – Equivalences

b) $\neg p \to (q \to r) \equiv q \to (p \lor r)$

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&\equiv p \lor (q \to r) && \text{[Double Negation]} \\
&\equiv p \lor (\neg q \lor r) && \text{[Law of Implication]} \\
&\equiv (p \lor \neg q) \lor r && \text{[Associativity]} \\
\\
&\equiv \neg q \lor (p \lor r) \\
&\equiv q \to (p \lor r) && \text{[Law of Implication]}
\end{aligned}$$

Problem 2 – Equivalences

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&\equiv (p \lor \neg q) \lor r && \text{[Associativity]} \\
&\equiv (\neg q \lor p) \lor r && \text{[Commutativity]} \\
&\equiv \neg q \lor (p \lor r) && \\
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Problem 2 – Equivalences

b) $\neg p \rightarrow (q \rightarrow r) \equiv q \rightarrow (p \vee r)$

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$\equiv p \vee (q \rightarrow r)$	[Double Negation]
$\equiv p \vee (\neg q \vee r)$	[Law of Implication]
$\equiv (p \vee \neg q) \vee r$	[Associativity]
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$\equiv q \rightarrow (p \vee r)$	[Law of Implication]

Predicates & Quantifiers

Predicates & Quantifiers Review

- **Predicate**: a function that outputs true or false
  - Cat(x) := "x is a cat"
  - LessThan(x, y) := "$x < y$"

- **Domain of Discourse**: the types of inputs allowed in predicates
  - Numbers, mammals, cats and dogs, people in this class, etc.

- **Quantifiers**
  - Universal Quantifier $\forall x$: for all x, for every x
  - Existential Quantifier $\exists x$: there is an x, there exists an x, for some x

\forall x : Cat(x)

- **Domain Restriction**
  - Universal Quantifier ∀x: add the restriction as the hypothesis to an **implication**
  - Existential Quantifier $\exists x$: **AND** in the restriction

Problem 6 – Domain Restriction

Translate each of the following sentences into logical notation. These translations require some of our quantifier tricks. You may use the operators + and · which take two numbers as input and evaluate to their sum or product, respectively.

a) Domain: Positive integers; Predicates: *Even, Prime, Equal*
"There is only one positive integer that is prime and even."

b) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"There are two different prime numbers that sum to an even number."

c) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"The product of two distinct prime numbers is not prime."

d) Domain: Real numbers; Predicates: *Even, Prime, Equal, Positive, Greater, Integer*
"For every positive integer, there is a greater even integer"

Work on parts (a) and (b) with the people around you, and then we'll go over it together!

Problem 6 – Domain Restriction

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We can start out with:

$$\exists x(\text{Prime}(x) \land \text{Even}(x))$$

Problem 6 – Domain Restriction

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We can start out with:

$$\exists x(\text{Prime}(x) \land \text{Even}(x))$$

But now we need to add in the restriction that this x is the ONLY positive integer that is prime and even. This is a technique you'll use whenever you need to have only one of something:

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$$\exists x(\text{Prime}(x) \land \text{Even}(x) \land \forall y[\neg\text{Equal}(x, y) \rightarrow \neg(\text{Even}(y) \land \text{Prime}(y))])$$

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$$\exists x(\text{Prime}(x) \land \text{Even}(x) \land \forall y[\neg\text{Equal}(x, y) \rightarrow \neg(\text{Even}(y) \land \text{Prime}(y))])$$

Or, we could use the contrapositive:

$$\exists x(\text{Prime}(x) \land \text{Even}(x) \land \forall y[(\text{Even}(y) \land \text{Prime}(y) \rightarrow \text{Equal}(x, y)])$$

Problem 6 – Domain Restriction

b) Domain: Real numbers; Predicates: *Even, Prime, Equal*
“There are two different prime numbers that sum to an even number.”

Problem 6 – Domain Restriction

b) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"There are two different prime numbers that sum to an even number."

Seems like maybe we should be able to say something like:

$$\exists x \exists y(\text{Prime}(x) \land \text{Prime}(y) \land \text{Even}(x + y))$$

Problem 6 – Domain Restriction

b) Domain: Real numbers; Predicates: *Even, Prime, Equal*
"There are two **different** prime numbers that sum to an even number."

Seems like maybe we should be able to say something like:

$$\exists x \exists y(\text{Prime}(x) \wedge \text{Prime}(y) \wedge \text{Even}(x + y))$$

But this leaves open the possibility of x and y being equal (so they won't be two DIFFERENT numbers). So, we need to explicitly add in that x and y are not equal:

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But this leaves open the possibility of x and y being equal (so they won't be two DIFFERENT numbers). So, we need to explicitly add in that x and y are not equal:

$$\exists x \exists y (\text{Prime}(x) \wedge \text{Prime}(y) \wedge \text{Even}(x + y) \wedge \neg \text{Equal}(x, y))$$

Problem 7 – There Exists An Implication

Implications are uncommon under existential quantifiers. Consider this expression (which we'll call "the original expression"): $\exists x(P(x) \rightarrow Q(x))$

a) Suppose that P(x) is not always true (i.e. there is an element in the domain for which P(x) is false). Explain why the original expression is true in this case.

b) Suppose that P(x) is always true (i.e. $\forall x\, P(x)$). There is a simpler statement which conveys the meaning of the original expression (i.e. is equivalent to it for all domains and predicates. By simpler, we mean "uses fewer symbols").

We'll go over a) and b) together!

Problem 7 – There Exists An Implication

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P(x)	**Q(x)**	**P(x) → Q(x)**
T	T	T
T	F	F
F	T	T
F	F	T

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T	T	T
T	F	F
F	T	T
F	F	T

If P(x) ever false, $P(x) \rightarrow Q(x)$ is true

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P(x)	**Q(x)**	**P(x) → Q(x)**
T	T	T
T	F	F
F	T	T
F	F	T

If P(x) always true,
P(x) → Q(x) is Q(x)

Problem 7 – There Exists An Implication

Takeaway

You rarely want to see $\exists x(P(x) \rightarrow Q(x))$ in your final answer!

$\exists x(P(x) \rightarrow Q(x))$ is weird!

Problem 8 – Quantifier Switch

Consider the following pairs of sentences. For each pair, determine if one implies the other, if they are equivalent, or neither.

a) $\forall x\, \forall y\, P(x,y)$ $\quad$ $\forall y\, \forall x\, P(x,y)$

b) $\exists x\, \exists y\, P(x,y)$ $\quad$ $\exists y\, \exists x\, P(x,y)$

c) $\forall x\, \exists y\, P(x,y)$ $\quad$ $\forall y\, \exists x\, P(x,y)$

d) $\forall x\, \exists y\, P(x,y)$ $\quad$ $\exists x\, \forall y\, P(x,y)$

e) $\forall x\, \exists y\, P(x,y)$ $\quad$ $\exists y\, \forall x\, P(x,y)$

Work on parts (c), (d) and (e) with the people around you, and then we'll go over it together!

Problem 8 – Quantifier Switch

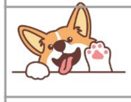
c) $\forall x\, \exists y\, P(x, y)$ $\qquad$ $\forall y\, \exists x\, P(x, y)$

Problem 8 – Quantifier Switch

c) $\forall x\, \exists y\, P(x,y)$ $\qquad$ $\forall y\, \exists x\, P(x,y)$

Different! It would only be true if P was **symmetric** (argument order didn't matter)
For all x, there is a y **vs** for all y, there is an x

"All people own a dog"

	Robbie	Aruna	Anna	Jacob
	X			
			X	
		X		
				X

"All dogs have an owner"

	Robbie	Aruna	Anna	Jacob
	X			
	X			
		X		
			X	

Problem 8 – Quantifier Switch

d) $\forall x \, \exists y \, P(x, y)$ $\exists x \, \forall y \, P(x, y)$

Problem 8 – Quantifier Switch

d) $\forall x\, \exists y\, P(x,y)$ $\exists x\, \forall y\, P(x,y)$

Different!
For all x, there is a y **vs** there exists an x, such that, for all y

"All people own a dog"

	Robbie	Aruna	Anna	Jacob
	X			
				X
		X		
			X	

VS

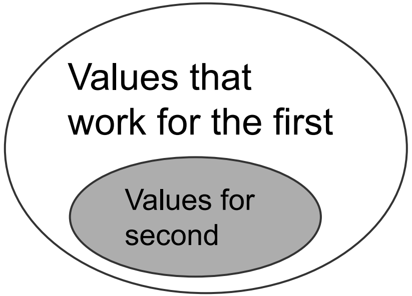
"There is person that owns all dogs"

	Robbie	Aruna	Anna	Jacob
	X			
	X			
	X			
	X			

Problem 8 – Quantifier Switch

e) $\forall x \, \exists y \, P(x, y)$ $\qquad$ $\exists y \, \forall x \, P(x, y)$

Problem 8 – Quantifier Switch



e) $\forall x\, \exists y\, P(x,y)$ $\exists y\, \forall x\, P(x,y)$

The second implies the first

For all x, there is a y **vs** there exists a y, such that, for all x

The second is **stronger** since a **specific y** must work **for all x** whereas for the first, the y value **does not** have to be the same **for every x**

"All people own a dog"

	Robbie	Aruna	Anna	Jacob
	X			
				X
		X		
			X	

VS

"There is a dog owned by all people"

	Robbie	Aruna	Anna	Jacob
				
				
	X	X	X	X
				

That's All, Folks!

Thanks for coming to section this week!
Any questions?

Problem 5 – ctrl-z

Translate these logical expressions to English. For each of the translations, assume that domain restriction is being used and take that into account in your English versions.

Let your domain be all UW Students. Predicates $\text{143Student}(x)$ and $\text{311Student}(x)$ mean the student is in CSE 143 and 311, respectively. $\text{BioMajor}(x)$ means x is a bio major, $\text{DidHomeworkOne}(x)$ means the student did homework 1 (of 311). Finally, $\text{KnowsJava}(x)$ and $\text{KnowsDeMorgan}(x)$ mean x knows Java and knows DeMorgan's Laws, respectively.

a) $\forall x\big(\text{143Student}(x) \to \text{KnowsJava}(x)\big)$

b) $\exists x\big(\text{143Student}(x) \wedge \text{BioMajor}(x)\big)$

c) $\forall x([\text{311Student}(x) \wedge \text{DidHomeworkOne}(x)] \to \text{KnowsDeMorgan}(x))$

Work on part (c) with the people around you, and then we'll go over it together!

Problem 5 – ctrl-z

c) $\forall x([\text{311Student}(x) \land \text{DidHomeworkOne}(x)] \rightarrow \text{KnowsDeMorgan}(x))$

Problem 5 – ctrl-z

c) $\forall x([311\text{Student}(x) \land \text{DidHomeworkOne}(x)] \rightarrow \text{KnowsDeMorgan}(x))$

Every 311 students who did Homework 1 know DeMorgan's Laws.

"If a UW student is a 311 student and did Homework 1, then they know DeMorgan's Laws" is a valid translation of the original sentence, but it is not taking advantage of the domain restriction.