1. Regular Expressions & CFGs

Let $\Sigma = \{1, 2, 3, 4\}$ and let L be the language containing all strings w where w is **nondecreasing**.

We define a string w as **nondecreasing** if for all k, the character at index k is greater than or equal to all the numbers to the left, meaning the number at index k is greater than or equal to each of the numbers at indices 0 to $k-1$.

Some example **nondecreasing** strings are:

- ε (this string vacuously meets the definition of nondecreasing)
- 13 $(3 \geq 1)$
- 11223334

An example on an invalid string (one not in the language) is 132, as $3 > 2$.

(a) Write a regular expression that matches L. (No explanation required).

(b) Write a CFG that generates L.
Be sure to tell us which symbol is the start symbol; also include a sentence or two of explanation of how your CFG works.

2. DFAs & NFAs

Let $\Sigma = \{0, 1\}$ for all problems on this page. For the following questions, show how to describe exactly the language specified using the given formalism (regular expression, DFA, NFA or CFG).

(a) Write a **regular expression** that matches all binary strings that do not end in 11.
Briefly (1-2 sentences) explain your solution.

(b) Draw a **DFA** that accepts all binary strings that meet both of the following conditions: the string has an odd length **and** the digits of the string sum to an odd number.
Briefly (1-2 sentences) explain your solution.

(c) Draw an **NFA** that accepts all binary strings with an odd length **or** end in 111 (or both).
Briefly (1-2 sentences) explain your solution.

(d) Write a context-free grammar that generates all binary strings that meet **both** of the following conditions:

- The string contains strictly more 0s than 1s
- In the string, all 0s appear before 1s **or** all 1s appear before 0s.

For example, $0, 00, 100, 001$ should be in the language but $\varepsilon, 1, 01, 010$ should not be in the language.

If your grammar has multiple non-terminal symbols, describe the purpose of each non-terminal (1 sentence per non-terminal symbol), and tell us which is one is the start symbol.

3. Treeshake

We define simple binary trees as the recursive set $\mathcal{B}$:

Basis Step: $\bullet \in \mathcal{B}$.
Recursive Step: If $L, R \in \mathcal{B}$, then $(L, \bullet, R) \in \mathcal{B}$.

Define the following functions on simple binary trees:

$$\texttt{edges}(t) = \begin{cases} 0 & \text{if } t = \bullet \\ 2 + \texttt{edges}(L) + \texttt{edges}(R) & \text{if } t = (L, \bullet, R) \end{cases}$$

$$\texttt{degree}(t) = \begin{cases} 1 & \text{if } t = \bullet \\ 3 & \text{if } t = (L, \bullet, R) \end{cases}$$

$$\texttt{sum}(t) = \begin{cases} \texttt{degree}(t) & \text{if } t = \bullet \\ \texttt{degree}(t) + \texttt{sum}(L) + \texttt{sum}(R) & \text{if } t = (L, \bullet, R) \end{cases}$$

Prove that for all $t \in \mathcal{B}$, $\texttt{sum}(t) = 2 \cdot \texttt{edges}(t) + 1$.
You must use structural induction for this problem.

4. Training Wheels

Let your domain of discourse be integers. Define the following predicates

- $\texttt{PerfectSquare}(x)$ returns true if and only if x is a perfect square (that is $x = z^2$ for some $z \in \mathbb{Z}$).
- $\texttt{Divides}(x, y)$ returns true if and only if $x|y$.
- $\texttt{Even}(x)$ returns true if and only if x is even.

(a) Translate this sentence into predicate logic notation.

If x is even and x is a perfect square then $4|x$.

(b) Translate this sentence into predicate logic notation.

Every even number is divisible by two different integers.

(c) Negate the following sentence (leave your answer as symbols; negations must be applied to single predicates, but you do not need to reformat your answer to show domain restriction).

$$\forall x \forall y([\texttt{Even}(x) \wedge \texttt{Even}(y)] \rightarrow [\texttt{Divides}(x, y) \vee \texttt{Divides}(y, x)])$$

(d) State **in English** the contrapositive of the **original** statement in the last part (not your answer). Your English contrapositive must use domain restriction.

(e) The (original) statement in (c) is [1 point]

◯ True
◯ False

(f) The contrapositive of the statement in (c) [i.e., the correct answer to part (d)] has: [1 point]

◯ The same truth value as part (e).
◯ The negation of the truth value in part (e).
◯ It depends on what x and y are.

Additional space to work on problems