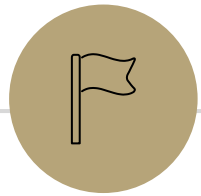


# Halting Problem

CSE 311 Summer 2025  
Lecture 23

# Announcements

- Midterm retakes are happening today and tomorrow
- HW7 is out and due this Friday, 8/15
- HW5 solutions are at the front of the class
- Final logistics are posted on the Exams page of the course website



# Review: Uncomputability



# A proof idea

A set is countable iff it can be listed (a list is a bijection with  $\mathbb{N}$ ).

We'll take advantage of that to find an uncountable set.

Claim  $\mathbb{R}$  is uncountable.

Actually, it's easier if we show  $[0,1)$  is uncountable (i.e. real numbers between 0 and 1).

# Uncountable

Suppose, for the sake of contradiction, that  $[0,1)$  is countable.

Then there is a bijection  $f: \mathbb{N} \rightarrow [0,1)$ .

Use that bijection to make the following table...

# Proof that $[0,1)$ is not countable

Suppose, for the sake of contradiction, that there is a list of them:

Number	Digits after decimal	0	1	2	3	4	5	6	7	...
$f(0)$	0.	3	3	3	3	3				
$f(1)$	0.	2	7	2	7	2				
$f(2)$	0.	1	4	1	5	9				
$f(3)$	0.	2	2	2	2	2				
$f(4)$	0.	1	2	3	4	5	6	7	8	...
$f(5)$	0.	9	8	7	6	5	4	3	2	...
$f(6)$	0.	8	2	7	6	4	5	7	4	...
$f(7)$	0.	5	9	4	2	7	5	1	7	...
				...	...	...	...	...	...	...

Flipping Rule: let's set the  $i^{\text{th}}$  column to:  
 7 if  $f(i)$ 's  $i^{\text{th}}$  column is not 7  
 3 if  $f(i)$ 's  $i^{\text{th}}$  column is 7.

0.73777733...

# Wrapping Up

0.73777733...

What is it?

It's a real number between 0 and 1(!!!)

Is the number on the list? Well it's not  $f(0)$ , they differ in column 0.

It's not  $f(1)$ , they differ in column 1.

It's not  $f(i)$ , they differ in column  $i$ .

But... $f$  was a bijection. That's a contradiction!

# Diagonalization

This proof technique is called **diagonalization**

Often “Cantor’s Diagonalization” (after Cantor, who developed it).

# Proof that $[0,1)$ set of binary-valued functions is not countable

Suppose, for the sake of contradiction, that there is a list of them:

$f$ bijection from $\mathbb{N}$ to function	Output on 0	Output on 1	Output on 2	Output on 3	Output on 4	Output on 5	Output on 6	Output on 7	...
$f(0)$	1	0	1	1					
$f(1)$	0	1	1	0					
$f(2)$	1	1	1	0					
$f(3)$	0	0	0	0					
$f(4)$	1	0	1	1	1	0	1	1	...
$f(5)$	0	0	0	1	0	1	1	1	...
$f(6)$	0	1	1	0	0	1	0	0	...
$f(7)$	1	0	1	0	1	0	1	0	...
...	...	...	...	...	...	...	...	...	...

How do we find a function not on our list?  
 Well to make sure it's not  $f(i)$  (the function in the  $i^{th}$  row)  
 Have  $g_{diag}(i) = 1 - f(i)(i)$

$$g_{diag}(x) = \begin{cases} 1 & \text{if } f(x) \text{ outputs 0 on input } x \\ 0 & \text{if } f(x) \text{ outputs 1 on input } x \end{cases}$$

# Our Second big takeaway

There are more functions  $g: \mathbb{N} \rightarrow B$  than there are Java programs to compute them.

Some function must be **uncomputable**.

That is there is no piece of code which tells you the output of the function when you give it the appropriate input.

# Not just Java

This isn't just about java programs. (all we used about java was that its programs are strings)...that's...well every programming language.

There are functions that simply cannot be computed.

Doesn't matter how clever you are. How fancy your new programming language is. Just doesn't work.\*

\*there's a difference between `int` and  $\mathbb{N}$  here, for the proof to work you really need all integers to be valid inputs, not just integers in a certain range.

# Does this matter?

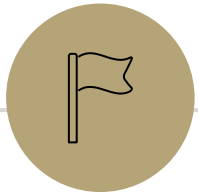
It's even worse than that – almost all functions are not computable.

So...how come this has never happened to you?

This might not be meaningful yet. Almost all functions are also inexpressible in a finite amount of English (English is a language too!)

You've probably never decided to write a program that computes a function you couldn't describe in English...

Are there any problems anyone is **interested** in solving that aren't computable?



**What's a computer?**

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# Formal definition of computer

To prove a theorem, we'll need some kind of definition.

The first theoretical description of a ("normal") computer is a **Turing Machine**

**Turing Machines** have:

1. A finite control
  - think: a DFA-like object to represent program state, what is the program I'm executing, what line number am I on, etc.
2. As much memory as we need
  - Stored on an infinite "tape"
3. A "focus-of-attention"
  - The bit of memory on the tape we're paying attention to right now. Can read and write at this spot. The finite control can move the focus.



# Does that sound like a computer?

Maybe not the first analogy you would have come up with.

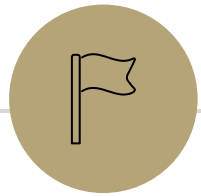
But computer scientists still love talking about Turing Machines---take 431 to learn more

But an idea called "Turing-completeness" says we can think about Java programs instead

A computational device is Turing-complete if it can do everything a Turing-machine can do.

Java is Turing-complete, as are C++, Python, and any other general-purpose programming language you know. And moreover, none can really solve problems the others can't

See 431 for more on these.



# The Halting Problem

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# A Practical Uncomputable Problem

Ever pressed the run button on your code and have it take a long time?

Like an infinitely long time?

What didn't your compiler...like, tell you **not** to push the button yet.

It tells you when your code doesn't compile before it runs it...why doesn't it check for infinite loops?

# The Halting Problem

## The Halting Problem

Given: source code for a program  $P$  and  $x$  an input we could give to  $P$   
Return: True if  $P$  will halt on  $x$ , False if it runs forever (e.g. goes in an infinite loop or infinitely recurses)

This would be super useful to solve!

We can't solve it...let's find out why.

# A Proof By Contradiction

Suppose, for the sake of contradiction, there is a program  $H$ , which given input `P.java, x` will accurately report

" $P$  would halt when run with input  $x$ " or

" $P$  will run forever on input  $x$ ."

**Important:**  $H$  does not just compile `P.java` and run it. To count,  $H$  needs to return "halt" or "doesn't" in a finite amount of time.

And remember, it's not a good idea to say "but  $H$  has to run `P.java` to tell if it'll go into an infinite loop" that's what we're trying to prove!!

# A Very Tricky Program.

```
Diagonal.java (String x) {  
    Run H.exe on input <x, x>  
    if (H.exe says "x halts on x")  
        while (true) { // Go into an infinite loop  
            int x=2+2;  
        }  
    else // H.exe says "x doesn't halt on x"  
        return; // halt.  
}
```

So, uhh that's a weird program.

What do we do with it?

USE IT TO BREAK STUFF

Does `Diagonal.java` halt when its input is `Diagonal.java`?

Let's assume it does and see what happens...

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Imagine Diagonal.java halts on  
Diagonal.java.

Then H better say it halts.  
So it goes into an infinite loop.

Wait shoot.

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That didn't work.

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```

Imagine Diagonal.java doesn't halt on Diagonal.java. Then H better say it doesn't halt. So we go into the else branch. And it halts. Wait shoot.

# So, uhh that's a weird program.

What do we do with it?

USE IT TO BREAK STUFF

Does `Diagonal.java` halt when its input is `Diagonal.java`?

Let's assume it does and see what happens...

That didn't work.

Let's assume it doesn't and see what happens...

That didn't work either.

There's no third option. It either halts or it doesn't. And it doesn't do either. That's a contradiction! `H.exe` can't exist.

So...

So there is no general-purpose algorithm that decides whether any input program halts (on any input string).

The Halting Problem is undecidable (i.e. uncomputable) there is no algorithm that solves every instance of the problem correctly.

# What this does and doesn't mean

Does this code halt?

```
public static void main(String[] args)
{
    int i=0;
    while (i<100) {
        System.out.println("*");
        i++;
    }
}
```

# What this does and doesn't mean

Does this code halt?

```
public static void main(String[] args)
{
    int i=0;
    while (i<100) {
        System.out.println("*");
        i++;
    }
}
```

It does halt!

# What this does and doesn't mean

Does this code halt?

```
public static void main(String[] args)
{
    int i=0;
    while (i<100) {
        System.out.println("*");
        //i++; //doesn't seem important.
    }
}
```

# What this does and doesn't mean

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```
public static void main(String[] args)
{
    int i=0;
    while (i<100) {
        System.out.println("*");
        //i++; //doesn't seem important.
    }
}
```

It doesn't halt!

# What that does and doesn't mean

That doesn't mean that there aren't algorithms that often get the answer right

For example, if there's no loops, no recursion, and no method calls, it definitely halts. No problem with a program that checks that there's no loops, recursion, or method calls and tells you it's going to halt if it doesn't find any.

This isn't just a failure of computers – if you think **you** can do this by hand, well...

...you can't either.

# Takeaways

Don't expect that there's a better IDE/better compiler/better programming language coming that will make it possible to tell if your code is going to hit an infinite loop.

It's not coming.

# More Uncomputable problems

Imagine we gave the following task to 121 students:

Write a program that prints "Hello World"

Can you make an autograder?

Technically...NO!

# More Uncomputable problems

Imagine we gave the following task to 121 students:

Write a program that prints "Hello World"

Can you make an autograder?

Technically...NO!

In practice, we declare the program wrong if it runs for 1 minute or so. That's not right 100% of the time, but it's good enough for your programming classes.

# How Would we prove that?

With a **reduction**

Suppose, for the sake of contradiction, I can solve the HelloWorld problem. (i.e. on input P.java I can tell whether it eventually prints HelloWorld)

Let W.exe solve that problem.

Consider this program...

# A Reduction

```
Trick(P, x) {  
    Run P on x, //(but only simulate printing if P prints things)  
    Print "Hello World"  
}
```

This actually prints "hello world" iff P halts on x.

Plug Trick into W and....we solved the Halting Problem!

# Reductions in General

The big idea for reductions is “reusing code”

Just like calling a library

But doing it in contrapositive form.

Instead of

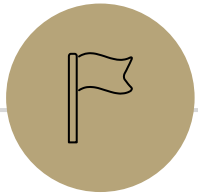
“If I have a library, then I can solve a new problem” reductions can do the contrapositive:

“If I can solve a problem I know I shouldn’t be able to, then that library function can’t exist”

# Fun (Scary?) Fact

## Rice's Theorem

Says any "non-trivial" input-output behavior of programs cannot be computed (in finite time).



**Victory Lap!**

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# We've Covered A LOT

Propositional Logic.

Boolean logic and circuits.

Boolean algebra. You'll use quantifiers in 332 to define big-O

Predicates, quantifiers and predicate logic.

Inference rules and formal proofs for propositional and predicate logic.

English proofs.

Set theory. 431 is basically 10 weeks of fun set proofs.

Modular arithmetic.

Prime numbers. Interested in crypto? They'll come back.

GCD, Euclid's algorithm and modular inverse

# No really. A lot

Induction and Strong Induction.

Lots of induction proof [sketches] in 332

Recursively defined functions and sets.

Structural induction.

Regular expressions.

You'll see these in compilers

Context-free grammars and languages.

One-to-one and Onto

Graphs

You'll use graphs at least once a week for the rest of your CS career.

# Like A lot a lot.

DFAs, NFAs and language recognition.

Cross Product construction for DFAs.

Finite state machines with outputs at states.

Conversion of regular expressions to NFAs.

Powerset construction to convert NFAs to DFAs.

Equivalence of DFAs, NFAs, Regular Expressions

Method to prove languages not accepted by DFAs.

Cardinality, countability and diagonalization

Undecidability: [Halting problem](#) and evaluating properties of programs.

Promise you won't ever try to solve the Halting Problem? It's tempting to try to sometimes if you don't remember it's undecidable

# What Comes next?

CSE 312 (foundations II)

Fewer proofs ☹️

Basics of probability theory (super useful in algorithms, ML, and just everyday life).  
Fundamental statistics.

CSE 332 (data structures and parallelism)

Data structures, a few fundamental algorithms, parallelism.

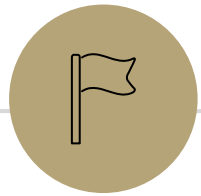
Graphs. Graphs everywhere.

Also, induction. [same for 417/421, 422 the algorithms courses]

CSE 431 (complexity theory)

What can't you do with computers **in a reasonable amount of time.**

Beautiful theorems – more on CFGs, DFAs/NFAs as well.



**[If time] Q&A**

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