Functions And Graphs

CSE 311 Summer 2025
Lecture 17

Announcements

- HW6 is released and due Wednesday, 8/6!
- Come to our next lecture on Wednesday for HW3, HW4, and midterm solutions

Announcements

- Midterm Retake
  - Midterm retake sessions will happen next Monday (8/11) and Tuesday (8/12) outside of our normal class time
  - The exam structure will be identical to the midterm, and we aim to make the difficulty about the same
  - Your final midterm score will be computed by taking the max score per question between the midterm and the midterm retake
  - You'll indicate *all* times you would be able to take the midterm retake through a scheduling form that will be posted on the Ed board (due Thursday, 8/7)
  - On Friday, Parker will post the midterm retake schedule

Review

Recursive Definition of Sets

Define a set S as follows:

Basis Step: $0 \in S$

Recursive Step: If $x \in S$ then $x + 2 \in S$.

Exclusion Rule: Every element of S is in S from the basis step (alone) or a finite number of recursive steps starting from a basis step.

What is S?

Structural Induction Template

1. Define $P()$ State that you will show $P(x)$ holds for all $x \in S$ and that your proof is by structural induction.

2. Base Case: Show $P(b)$
[Do that for every b in the basis step of defining S]

3. Inductive Hypothesis: Suppose $P(x)$
[Do that for every x listed as already in S in the recursive rules].

4. Inductive Step: Show $P()$ holds for the "new elements."
[You will need a separate step for every element created by the recursive rules].

5. Therefore $P(x)$ holds for all $x \in S$ by the principle of induction.

Strings

ε is "the empty string"

The string with 0 characters – `""` in Java (not `null`!)

Σ^*:

Basis: $\varepsilon \in \Sigma^*$.

Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$

wa means the string of w with the character a appended.

You'll also see $w \cdot a$ (a $\cdot$ to mean "concatenate" i.e. + in Java)

Functions on Strings

Since strings are defined recursively, most functions on strings are as well.

Length:

len(ε)=0;

len(wa)=len(w)+1 for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$;
$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation

$x \cdot \varepsilon = x$ for all $x \in \Sigma^*$;
$x \cdot (wa) = (x \cdot w)a$ for $w \in \Sigma^*, a \in \Sigma$

Number of c's in a string

$\#_c(\varepsilon) = 0$
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$;
$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*, a \in \Sigma \setminus \{c\}$.

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $\mathrm{y} \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

len(xy)=len(xwa) =len(xw)+1 (by definition of len)

=len(x) + len(w) + 1 (by IH)

=len(x) + len(wa) (by definition of len)

Therefore, len(xy)=len(x) + len(y), as required.

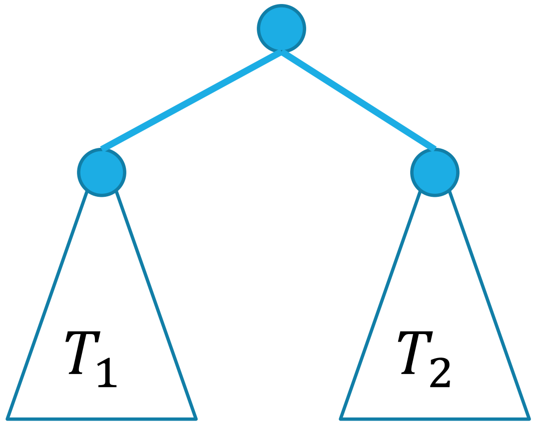
We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

More Structural Sets

Binary Trees are another common source of structural induction.

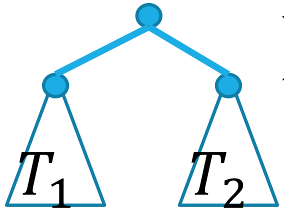
Basis: A single node is a rooted binary tree. 

Recursive Step: If T_1 and T_2 are rooted binary trees with roots r_1 and r_2, then a tree rooted at a new node, with children r_1, r_2 is a binary tree.



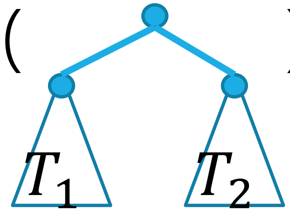
Functions on Binary Trees

size()=1

size(

) = size(T_1) + size(T_2) + 1

height() = 0

height(

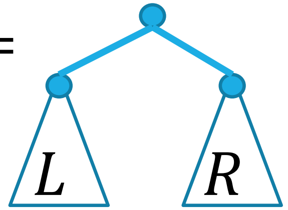
) = 1+max(height(T_1),height(T_2))

Practice: Structural Induction on Trees

Let $P(T)$ be "$\text{leaves}(T) \geq \frac{\text{size}(T)}{2} + \frac{1}{2}$ ". We show $P(T)$ for all binary trees T by structural induction.

Base Case: Let $T =$  . $\text{leaves}(T) = 1$ and $\text{size}(T) = 1$, so $\frac{\text{size}(T)}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$. Thus, $\text{leaves}(T) = 1 \geq 1 = \frac{\text{size}(T)}{2} + \frac{1}{2}$, so $P(T)$ holds for the base case.

Inductive Hypothesis: Suppose $\text{P}(L)$ and $\text{P}(R)$ hold for arbitrary trees L, R. Let $T =$



$$\text{leaves}(T) = \text{leaves}(L) + \text{leaves}(R) \quad \text{[By Def. of leaves]}$$

$$\geq \left(\frac{\text{size}(L)}{2} + \frac{1}{2}\right) + \left(\frac{\text{size}(R)}{2} + \frac{1}{2}\right) \quad \text{[By IH]}$$

$$= \frac{1 + \text{size}(L) + \text{size}(R)}{2} + \frac{1}{2} \quad \text{[By Algebra]}$$

$$= \frac{\text{size}(T)}{2} + \frac{1}{2} \quad \text{[By Def. of size]}$$

So $P(T)$ holds, and we have $P(T)$ for all binary trees T by the principle of induction.

Proof Strategies Complete List

- Direct Proof
- Proof by Contrapositive
- Proof of Biconditional
- Proof by Cases
- Proof of Existence
- Proof by Counterexample
- Proof by Contradiction
- Proof by Weak/Strong/Structural Induction

Functions

Some types of functions

Why?

We'll want to talk about sizes of infinite sets during the last week of classes. It'll help us find problems our computers can't solve.

Ok, but why now?

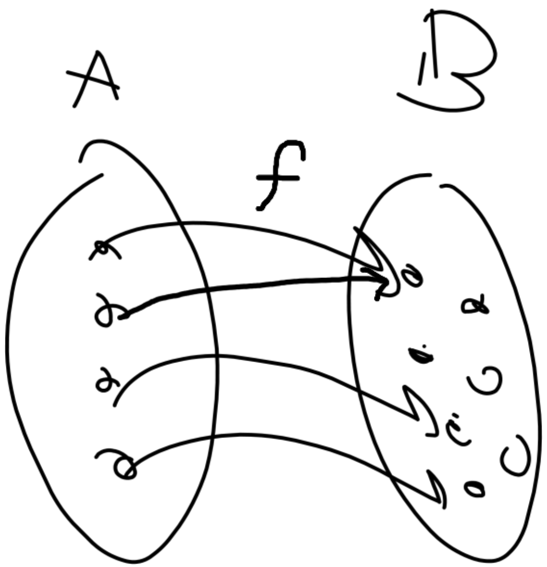
It'll let us practice set proofs a bit more over the next few weeks!

Functions!

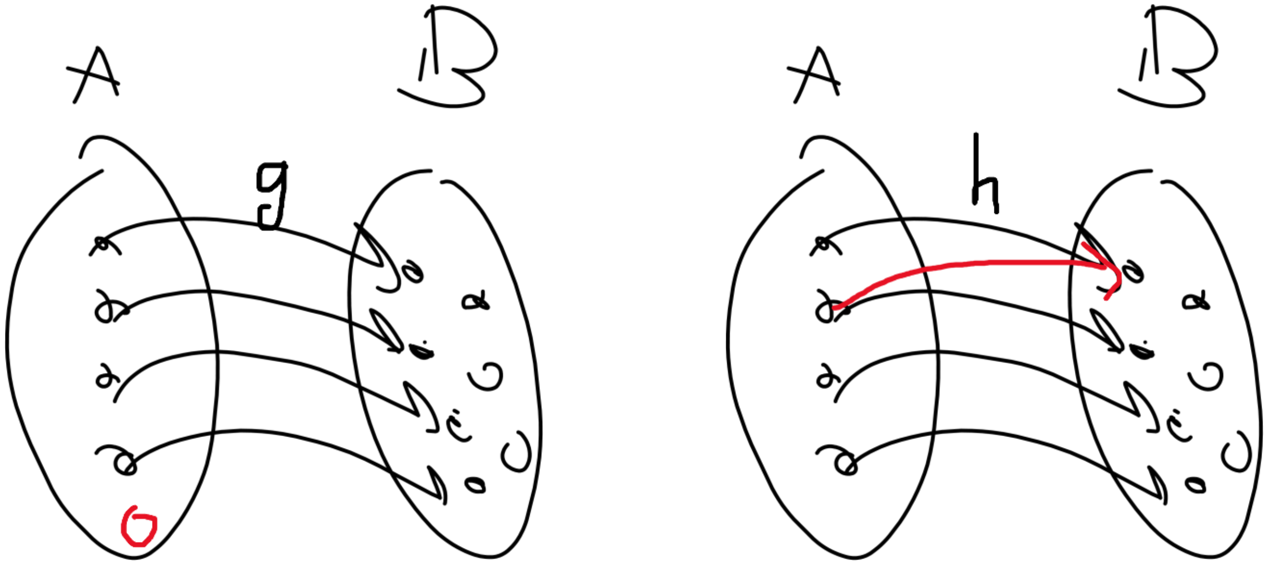
A function $f: A \to B$ maps every element of A to one element of B

A is the "domain", B is the "co-domain" (also called the "image" or "range")

Good function



Not a function



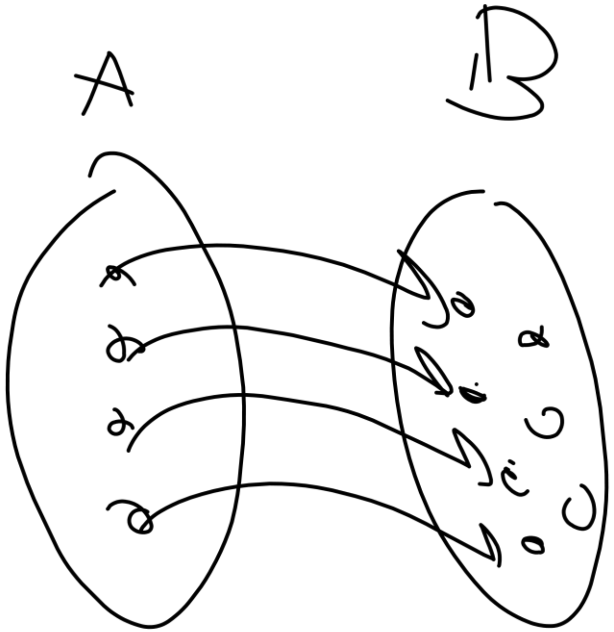
Two Requirements for a Bijection

A function $f: A \to B$ maps every element of A to one element of B

A is the "domain", B is the "co-domain"

One-to-one (aka injection)

A function f is one-to-one iff
$$\forall a \forall b (f(a) = f(b) \to a = b)$$



That is, every output has at most one possible input.

One-to-one (injection)

What did that definition say?

$\forall a \forall b (f(a) = f(b) \rightarrow a = b)$

In contrapositive that looks like

$\forall a \forall b \left(a \neq b \rightarrow f(a) \neq f(b)\right)$

So, if you get two different inputs, then you get two different outputs.

One-to-one proofs

One-to-one (aka injection)

A function f is one-to-one iff
$\forall a \forall b(f(a) = f(b) \rightarrow a = b)$

It's a for-all statement! We know how to prove it.

Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be the function given by $f(x) = x + 5$.

Claim: f is one-to-one

Proof:

What's the outline? What do we introduce, what do we assume, what's our target?

One-to-one proofs

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Proof: Let a, b be arbitrary elements of our domain, and suppose $f(a) = f(b)$.

…

$a = b$

One-to-one proofs

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Let $f: \mathbb{Z} \rightarrow \mathbb{Z}$ be the function given by $f(x) = x + 5$.

Claim: f is one-to-one

Proof: Let a, b be arbitrary elements of our domain, and suppose $f(a) = f(b)$.

By definition of the function, we have $a + 5 = b + 5$

Subtracting 5 from each side, we have $a = b$, meeting the definition of one-to-one.

Two Requirements for a Bijection

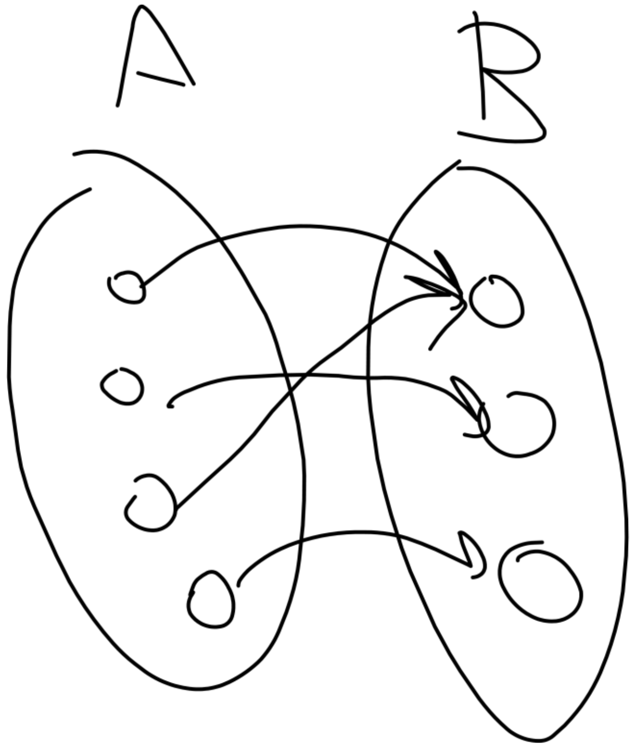
A function $f: A \to B$ maps every element of A to one element of B

A is the "domain", B is the "co-domain"

Onto (aka surjection)

A function $f: A \to B$ is onto iff
$\forall b \in B \exists a \in A(b = f(a))$

Every output has at least one input that maps to it.



Onto proofs

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Onto proofs

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A function $f: A \to B$ is onto iff $\forall b \in B \exists a \in A(b = f(a))$

It's a for-all statement, with an exists inside! We know how to prove it.

Let $f: \mathbb{Z} \to \mathbb{Z}$ be the function given by $f(x) = x + 5$.

Claim: f is onto

Proof: Let b be an arbitrary element of the codomain.

Consider $a =$…

…

So $f(a) = b$

Onto proofs

Onto (aka surjection)

A function $f: A \to B$ is onto iff
$\forall b \in B \exists a \in A(b = f(a))$

It's a for-all statement, with an exists inside! We know how to prove it.

Let $f: \mathbb{Z} \to \mathbb{Z}$ be the function given by $f(x) = x + 5$.

Claim: f is onto

Proof: Let b be arbitrary element of the codomain.

Consider $a = b - 5$

Observe that $f(a) = a + 5 = b - 5 + 5 = b$.

Since $b \in \mathbb{Z}$, a is also an integer so it is in the domain. Thus f meets the definition of onto.

Bijection

One-to-one (aka injection)

A function f is one-to-one iff
$\forall a \forall b(f(a) = f(b) \rightarrow a = b)$

Onto (aka surjection)

A function $f: A \rightarrow B$ is onto iff
$\forall b \in B \exists a \in A(b = f(a))$

Bijection

A function $f: A \rightarrow B$ is a bijection iff
f is one-to-one and onto

A bijection maps every element of the domain to **exactly** one element of the co-domain, and every element of the codomain to **exactly** one element of the domain.

Sizes of sets

How do we know two sets are the same size?

Easy. Count the number of elements in both.

That works great for finite sets, but ∞ isn't really a number we get to count to…

More Practical

What does it mean that two sets have the same size?

More Practical

What does it mean that two sets have the same size?

Why do we care about bijections?

Bijections create a (confusingly-named) one-to-one correspondence between sets.

There is a bijection $f: A \to B$ if and only if A and B are the same size.

A bijection “matches the elements up”

For finite sets we usually tell which of two sets is bigger by counting the number of elements in each and comparing the numbers.

These functions let you compare set sizes even if you can’t count the elements. We’ll use that idea for infinite sets in a few weeks.

Definition

Two sets A, B have the same size (same cardinality) if and only if there is a bijection $f: A \to B$

This matches our intuition on finite sets.

But it also works for infinite sets!

Let's see just how infinite these sets are.

Some infinite sets

Two sets A, B have the same size (same cardinality) if and only if there is a bijection $f: A \to B$

Let's compare the sizes of: $\mathbb{N}$, $\mathbb{Z}$, $\{x : x \text{ is an even integer}\}$

Some infinite sets

Two sets A, B have the same size (same cardinality) if and only if there is a bijection $f: A \to B$

Let's compare the sizes of: $\mathbb{N}$, $\mathbb{Z}$, $\{x : x \text{ is an even integer}\}$

$\mathbb{N}$ $0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ \ldots$

$\mathbb{Z}$

Some infinite sets

Two sets A, B have the same size (same cardinality) if and only if there is a bijection $f: A \to B$

Let's compare the sizes of: $\mathbb{N}$, $\mathbb{Z}$, $\{x : x$ is an even integer$\}$

$\mathbb{Z}$ $\quad \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$

$Even$

They're all the same size.

$\mathbb{Z}$ and even integers?

$f(x) = 2x$ Is it a bijection?

One-to-one? Let $a, b \in \mathbb{Z}$ be arbitrary. Suppose $f(a) = f(b)$. By definition of f, $2a = 2b$. Dividing by 2, $a = b$.

Onto? Let b be an arbitrary even integer. Since b is even, there must be some a $\in \mathbb{Z}$ such that $b = 2a$. By definition of f, $f(a) = b$.

They're all the same size.

$\mathbb{Z}$ and even integers?

$f(x) = 2x$ Is it a bijection?

YES

$\mathbb{N}$ and $\mathbb{Z}$?

They're all the same size.

$\mathbb{Z}$ and even integers?

$f(x) = 2x$ Is it a bijection?

YES

$\mathbb{N}$ and $\mathbb{Z}$

$$\mathrm{g}(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

They’re all the same size…

$\mathbb{N}$ and even integers?

$f(g(x))$ will work nicely. You can also build one explicitly.

Good exercise: show that if f and g are bijections then $f \circ g$ is also a bijection.

They're all the same size…

$\mathbb{N}$ and even integers?

$f(g(x))$ will work nicely. You can also build one explicitly.

$$\mathbb{N} \xrightarrow{g} \mathbb{Z} \xrightarrow{f} \text{evens}$$

Good exercise: show that if f and g are bijections then $f \circ g$ is also a bijection.

Composition of 2 bijections is a bijection.

Let $f\colon B \to C, g\colon A \to B$ be arbitrary bijections.

Consider the composition $f \circ g\colon\ A \to C$. We will use the alternative notation $f\big(g(x)\big)$ for clarity.

(1) We show $f\big(g(x)\big)$ is **one-to-one**. Let $a, b \in A$ be arbitrary. Suppose $f\big(g(a)\big) = f(g(b))$. Since f is a bijection, it's also one-to-one so $g(a) = g(b)$. Since g is a bijection, it's also one-to-one so $a = b$. Since a, b were arbitrary, $f\big(g(x)\big)$ is one-to-one.

…

Thus, $f \circ g$ is a bijection because it's both one-to-one and onto.

Composition of 2 bijections is a bijection.

Let $f: B \to C, g: A \to B$ be arbitrary bijections.

Consider the composition $f \circ g$: $A \to C$. We will use the alternative notation $f(g(x))$ for clarity.

…

(2) We show $f(g(x))$ is **onto**. Let $c \in C$ be arbitrary. Since f is onto, there is some $b \in B$ such that $f(b) = c$. Since g is onto, there is some $a \in A$ such that $g(a) = b$. Combining the facts that $f(b) = c$ and $g(a) = b$, we have $f(g(a)) = c$. Since c was arbitrary, $f(g(x))$ is onto.

Thus, $f \circ g$ is a bijection because it's both one-to-one and onto.

Countable

Countable

The set A is countable iff there's a one-to-one function from $\boldsymbol{A}$ to $\mathbb{N}$,
Equivalently, $\boldsymbol{A}$ is countable iff it is finite or there is a bijection from $\boldsymbol{A}$ to $\mathbb{N}$

$\mathbb{N}, \mathbb{Z}, \{x: x$ is an even integer$\}$ are all countable.

To build a bijection from $\boldsymbol{A}$ to $\mathbb{N}$, just list all the elements!

Let’s Try one that’s a little harder

What about $\mathbb{Q}$. There’s gotta be more of those right?

It’s pretty intuitive to think there are more rationals than integers.

Between every two rationals, there’s another rational number.

Or said in more intimidating fashion: between every two rationals there are infinitely many others!

The set of positive rational numbers

1/1	1/2	1/3	1/4	1/5	1/6	1/7	1/8	…
2/1	2/2	2/3	2/4	2/5	2/6	2/7	2/8	…
3/1	3/2	3/3	3/4	3/5	3/6	3/7	3/8	…
4/1	4/2	4/3	4/4	4/5	4/6	4/7	4/8	…
5/1	5/2	5/3	5/4	5/5	5/6	5/7	…	
6/1	6/2	6/3	6/4	6/5	6/6	…		
7/1	7/2	7/3	7/4	7/5	….			
…	…	…	…	…				

In bijection with the natural numbers

Order the rationals by their denominator (increasing), breaking ties by numerator.

$1/1, 1/2, 1/3, 2/3, 1/4, 3/4, 1/5, 2/5, 3/5, 4/5, 1/6, \ldots$

$f(x) =$the x^{th} number in that list (indexed from 0)

That's a bijection from $\mathbb{N}$ to $\mathbb{Q}^+$(it's not a nice clean formula, but it's definitely a function)

Are all infinite sets countable?

No. We will prove this in a few weeks.

$\mathbb{R}$ is uncountable

Graphs

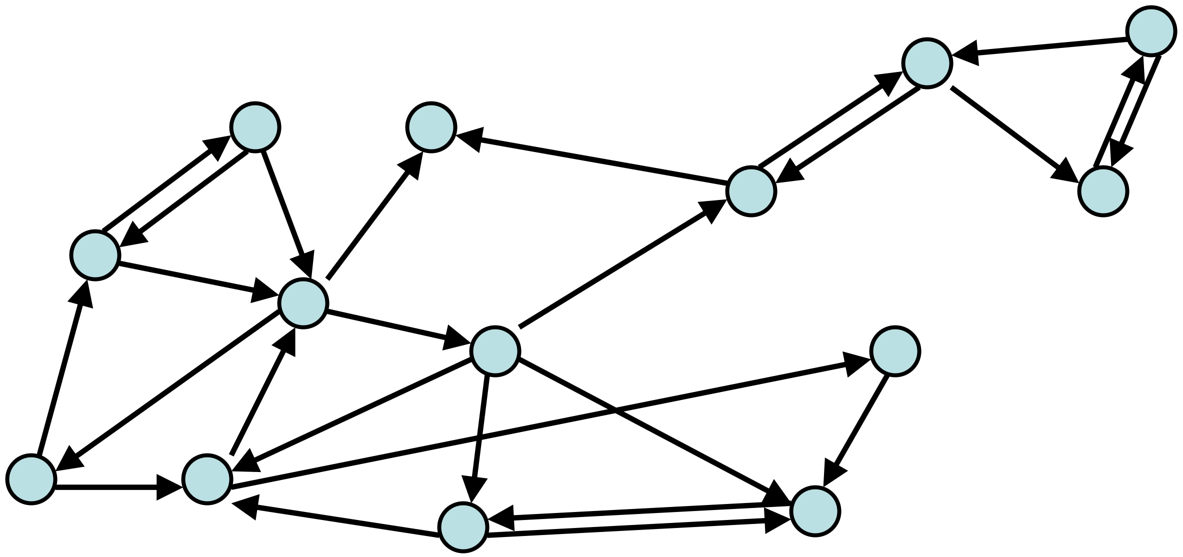
Directed Graphs

$G = (V, E)$

V is a set of vertices (an underlying set of elements)

E is a set of edges (ordered pairs of vertices; i.e. connections from one to the next).

Path $v_0, v_1, \ldots, v_k$ such that $(v_i, v_{i+1}) \in E$
Simple Path: path with all v_i distinct
Cycle: path with $v_0 = v_k$ (and $k > 0$)
Simple Cycle: simple path plus edge (v_k, v_0) with $k > 0$



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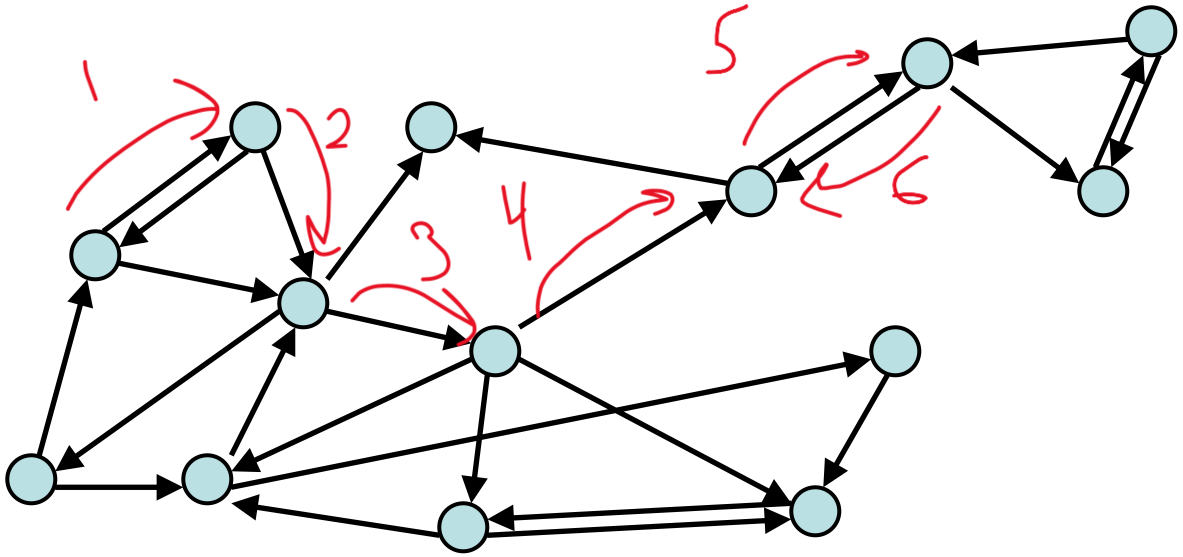

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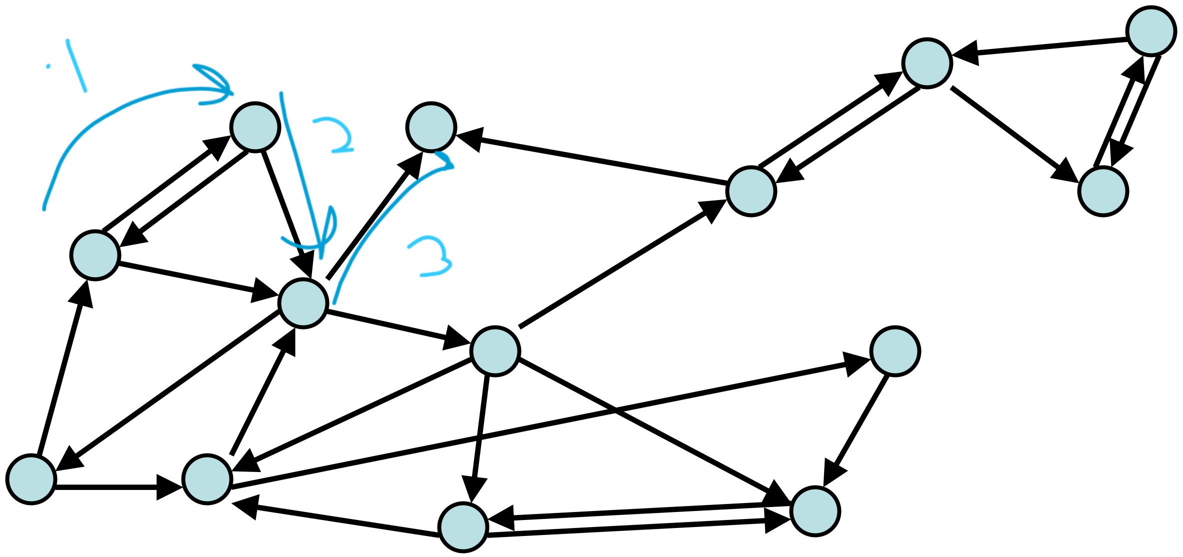

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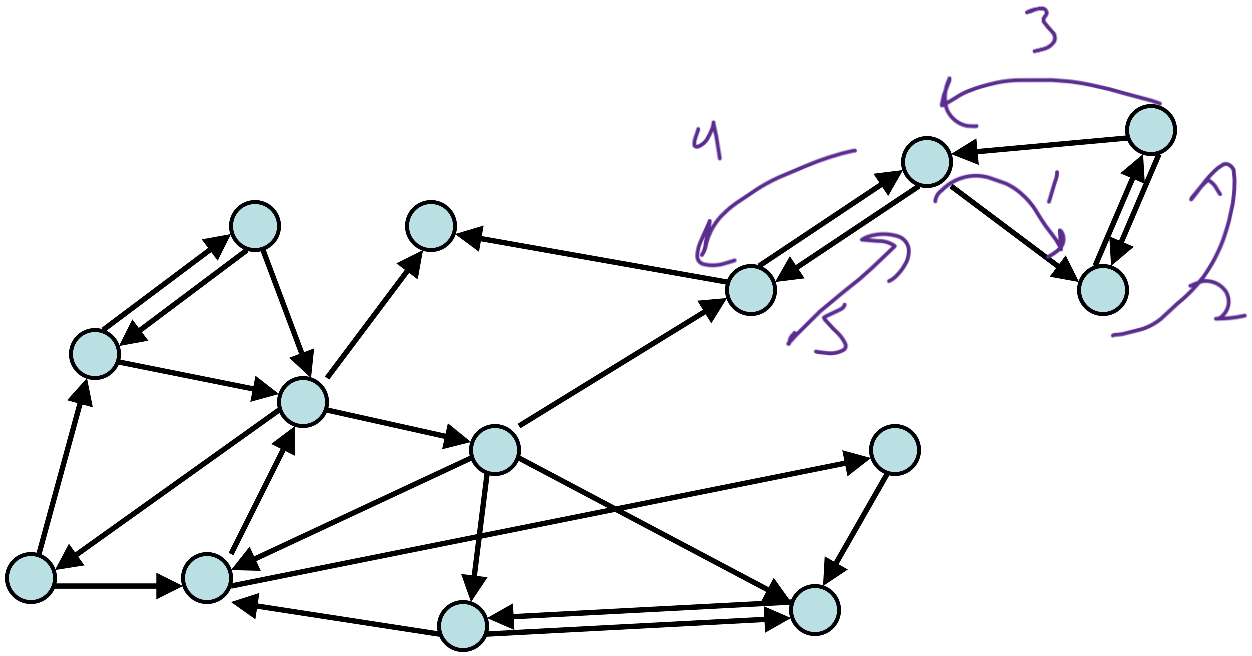

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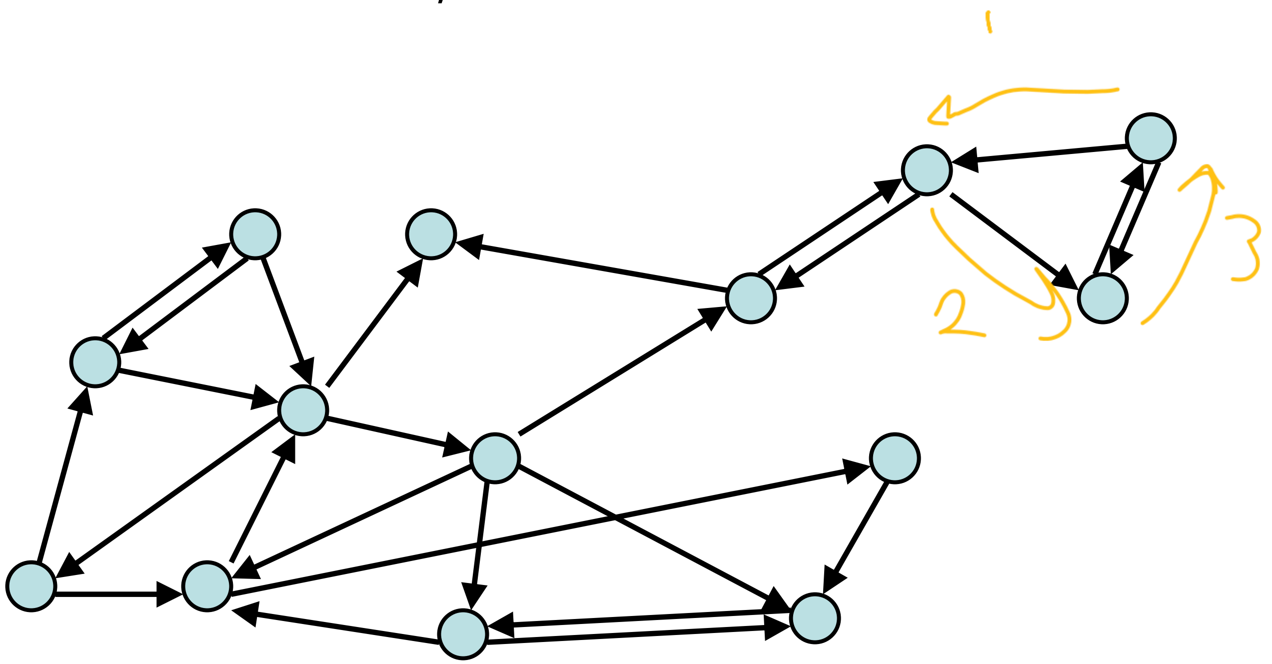

Extra examples

Between every two distinct rationals, there's another rational number.

Let $a, b \in \mathbb{Q}$ be arbitrary and suppose $a \neq b$. By the definition of rational, there's some integers p, r and non-zero integers q, s such that $a = \frac{p}{q},\ b = \frac{r}{s}$. Without loss of generality, assume $a < b$.

Consider $c = \frac{a+b}{2}$ (the average of a, b).

Since $a < b, a + b < 2b$ holds and we have $\mathrm{c} = \frac{a+b}{2} < b$. Similarly, $a < b$ gives $2a < a + b$ and we have $a < \frac{a+b}{2} = c$. Thus, c is "between" a, b.

$c = \frac{a+b}{2} = \frac{\frac{p}{q}+\frac{r}{s}}{2} = \frac{ps+rq}{2qs}$. As p, q, r, s are integers and q, s non-zero, $ps + rq$ is an integer and $2qs$ is a non-zero integer. So c is rational by definition of rational.

So c is a rational number between a, b. Since a, b were arbitrary, we can find a rational number between any two distinct rational numbers.

There are infinitely many rational numbers between any two distinct rational numbers.

Let $a, b \in \mathbb{Q}$ be arbitrary and suppose $a \neq b$.

For the sake of contradiction, suppose there are finitely many rational numbers between a, b. Then we can list all of them (rational numbers have ordering so we can list them from least to greatest):

$$a = p_1 < \cdots < p_n = b \text{ where each } p_i \in \mathbb{Q}$$

Notice that p_1, p_2 are distinct rational numbers. By the previous proof, there's a rational number q between these 2 distinct rational numbers. But $a < q < b$ and q isn't in our list, so we have a contradiction.

Since a, b were arbitrary, between 2 distinct rationals there are infinitely many rationals!

Show that the following function is a bijection.

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

(1) Let $a, b \in \mathbb{N}$ be arbitrary. Suppose that $g(a) = g(b)$. We go by cases:

$g(a) \geq 0$: So a, b even because $a, b \geq 0$

(non-negative outputs have even inputs, since $\frac{a}{2}, \frac{b}{2} > 0$):

$\frac{a}{2} = \frac{b}{2} \Rightarrow a = b$

$g(a) < 0$: So a, b odd because $a, b \geq 0$

(negative outputs have odd inputs, since $\frac{a+1}{2}, \frac{b+1}{2} > 0$):

$-\frac{a+1}{2} = -\frac{b+1}{2} \Rightarrow -a - 1 = -b - 1 \Rightarrow a = b$

These cases are exhaustive so $a = b$. Since $a, b \in \mathbb{N}$ were arbitrary, the function one-to-one.

Show that the following function is a bijection.

$$g(x) = \begin{cases} \frac{x}{2} & \text{if } x \text{ is even} \\ -\frac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

(2) Let $b \in \mathbb{Z}$ be arbitrary. We go by cases:

$b < 0$: Consider $a = 2(-b) - 1$. $a > 0$ because $-b > 0$ and multiplying and adding positive integers results in a positive integer. So $a \in \mathbb{N}$ (the domain). a is odd, so rearranging with algebra (and using the appropriate definition for g in the last step):

$$a = -2b - 1 \Rightarrow a + 1 = -2b \Rightarrow -\frac{(a+1)}{2} = b \Rightarrow g(a) = b$$

$b \geq 0$: Left as an exercise to the reader.

Since b was arbitrary, the function is onto.

Since the function is one-to-one and onto, it's a bijection.

Todo

Tonight:

- CC 17 is out and due Wednesday at noon
- Look out for the midterm retake sign-up form on the Ed board and fill it out
- Start working on HW6 after lecture if you haven’t already