Structural Induction Continued

CSE 311 Summer 2025
Lecture 16

Announcements

- HW5 is released and due Saturday, 8/2!
  - You *cannot* use any late days to submit HW5 after Saturday
- HW4 resubmission is due Friday, 8/1!
- Our Midterm is Friday (8/1) in class!
  - Exam logistics and practice exams are posted on the “Exams” page of the course website
  - The problem topics for the midterm are listed in the slides from last lecture

Review

Recursive Definition of Sets

Define a set S as follows:

Basis Step: $0 \in S$

Recursive Step: If $x \in S$ then $x + 2 \in S$.

Exclusion Rule: Every element of S is in S from the basis step (alone) or a finite number of recursive steps starting from a basis step.

What is S?

Recursive Definitions of Sets

All Natural Numbers

Basis Step: $0 \in S$

Recursive Step: If $x \in S$ then $x + 1 \in S$.

All Integers

Basis Step: $0 \in S$

Recursive Step: If $x \in S$ then $x + 1 \in S$ and $x - 1 \in S$.

Integer coordinates in the line $y = x$

Basis Step: $(0,0) \in S$

Recursive Step: If $(x, y) \in S$ then $(x + 1, y + 1) \in S$ and $(x - 1, y - 1) \in S$.

Structural Induction Template

1. Define $P()$ State that you will show $P(x)$ holds for all $x \in S$ and that your proof is by structural induction.

2. Base Case: Show $P(b)$

[Do that for every b in the basis step of defining S]

3. Inductive Hypothesis: Suppose $P(x)$

[Do that for every x listed as already in S in the recursive rules].

4. Inductive Step: Show $P()$ holds for the "new elements."

[You will need a separate step for every element created by the recursive rules].

5. Therefore $P(x)$ holds for all $x \in S$ by the principle of induction.

Structural Induction

Let $P(x)$ be "x is divisible by 3."

We show $P(x)$ holds for all $x \in S$ by structural induction.

Base Cases:
$6 = 2 \cdot 3$ so $3|6$, and $P(6)$ holds. $15 = 5 \cdot 3$, so$3|15$ and $P(15)$ holds.

Inductive Hypothesis: Suppose $P(x)$ and $P(y)$ for arbitrary $x, y \in S$.

Inductive Step: By IH $3|\mathrm{x}$ and $3|\mathrm{y}$. So $x = 3n$ and $y = 3m$ for integers m, n.

Adding the equations, $x + y = 3(n + m)$. Since n, m are integers, we have $3|(x + y)$ by definition of divides. This gives $P(x + y)$.

We conclude $P(x) \forall x \in \mathrm{S}$ by the principle of induction.

Basis: $6 \in S, 15 \in S$
Recursive: if $x, y \in S$ then $x + y \in S$.

Strings

ε is "the empty string"

The string with 0 characters – `""` in Java (not `null!`)

Σ^*:

Basis: $\varepsilon \in \Sigma^*$.

Recursive: If $w \in \Sigma^*$ and $a \in \Sigma$ then $wa \in \Sigma^*$

wa means the string of w with the character a appended.

You'll also see $w \cdot a$ (a $\cdot$ to mean "concatenate" i.e. + in Java)

Functions on Strings

Since strings are defined recursively, most functions on strings are as well.

Length:

len(ε)=0;

len(wa)=len(w)+1 for $w \in \Sigma^*$, $a \in \Sigma$

Reversal:

$\varepsilon^R = \varepsilon$;
$(wa)^R = aw^R$ for $w \in \Sigma^*$, $a \in \Sigma$

Concatenation

$x \cdot \varepsilon = x$ for all $x \in \Sigma^*$;
$x \cdot (wa) = (x \cdot w)a$ for $w \in \Sigma^*, a \in \Sigma$

Number of c's in a string

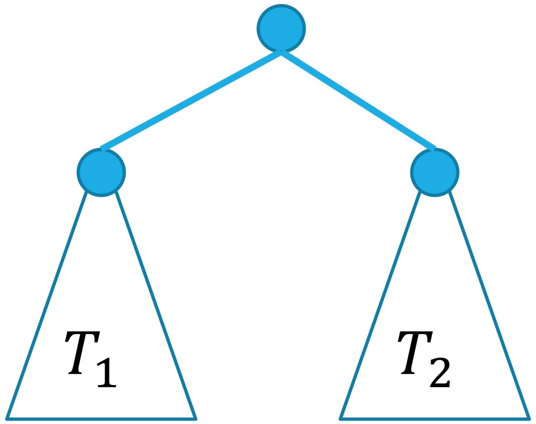
$\#_c(\varepsilon) = 0$
$\#_c(wc) = \#_c(w) + 1$ for $w \in \Sigma^*$;
$\#_c(wa) = \#_c(w)$ for $w \in \Sigma^*, a \in \Sigma \setminus \{c\}$.

More Structural Sets

Binary Trees are another common source of structural induction.

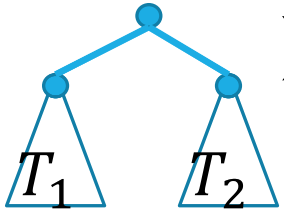
Basis: A single node is a rooted binary tree. 

Recursive Step: If T_1 and T_2 are rooted binary trees with roots r_1 and r_2, then a tree rooted at a new node, with children r_1, r_2 is a binary tree.

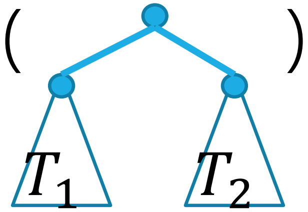


Functions on Binary Trees

size()=1

size() = size(T_1) + size(T_2) + 1

height() = 0

height() = 1+max(height(T_1),height(T_2))

Warm Up: Structural Induction

What does the inductive step look like?

Here's a recursively-defined set:

Basis: $0 \in T$ and $5 \in T$

Recursive: If $x, y \in T$ then $x + y \in T$ and $x - y \in T$.

Let's prove $5|x$ for all $x \in T$.

Basis: $0 \in T,\ 5 \in T$
Recursive: If $x, y \in T$ then
$x + y \in T$ and $x - y \in T$.

Claim: $5|x$ for all $x \in T$.

Let $P(x)$ be "$5|x$." We prove $P(x)$ holds for all $x \in T$ by structural induction.

Base Case ( ):

Inductive Hypothesis:

Inductive Step:

Thus $\mathrm{P}(\mathrm{x})$ holds for all $x \in T$ by structural induction

Basis: $0 \in T,\ 5 \in T$
Recursive: If $x, y \in T$ then $x + y \in T$ and $x - y \in T$.

Claim: $5|x$ for all $x \in T$.

Let $P(x)$ be "$5|x$." We prove $P(x)$ holds for all $x \in T$ by structural induction.
Base Cases:
$(\boldsymbol{x = 0})$:

$(\boldsymbol{x = 5})$:

Inductive Hypothesis:
Inductive Step:

Thus $\mathrm{P(x)}$ holds for all $x \in T$ by structural induction

Basis: $0 \in T,\ 5 \in T$
Recursive: If $x, y \in T$ then
$x + y \in T$ and $x - y \in T$.

Claim: $5|x$ for all $x \in T$.

Let $P(x)$ be "$5|x$." We prove $P(x)$ holds for all $x \in T$ by structural induction.
Base Cases:
$(\boldsymbol{x = 0})$: $5(0) = 0$, so by definition of divides $\mathbf{5|0}$, and our base case holds.

$(\boldsymbol{x = 5})$: $5(1) = 5$, so by definition of divides $5|5$, and our base case holds.

Inductive Hypothesis:
Inductive Step:

Thus $\mathrm{P(x)}$ holds for all $x \in T$ by structural induction

Basis: $0 \in T,\ 5 \in T$
Recursive: If $x, y \in T$ then
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($\boldsymbol{x = 5}$): $5(1) = 5$, so by definition of divides $5|5$, and our base case holds.

Inductive Hypothesis: Assume $P(x)$ and $P(y)$ hold for arbitrary $x, y \in T$.
Inductive Step:

Thus P(x) holds for all $x \in T$ by structural induction

Claim: $5|x$ for all $x \in T$.

Basis: $0 \in T,\ 5 \in T$
Recursive: If $x, y \in T$ then
$x + y \in T$ and $x - y \in T$.

Let $P(x)$ be "$5|x$." We prove $P(x)$ holds for all $x \in T$ by structural induction.
Base Cases:
($\boldsymbol{x = 0}$): $5(0) = 0$, so by definition of divides $5|0$, and our base case holds.

($\boldsymbol{x = 5}$): $5(1) = 5$, so by definition of divides $5|5$, and our base case holds.

Inductive Hypothesis: Assume $P(x)$ and $P(y)$ hold for arbitrary $x, y \in T$. i.e., suppose that $5|x$ and $5|y$

Inductive Step:

Thus $\mathrm{P(x)}$ holds for all $x \in T$ by structural induction

What does the inductive step look like?

Basis: $0 \in T$ and $5 \in T$

Recursive: If $x, y \in T$ then $x + y \in T$ and $x - y \in T$.

Let $P(x)$ be "$5|x$"

What does the inductive step look like?

Well there's two recursive rules, so we have two things to show

Claim: $5|x$ for all $x \in T$.

Basis: $0 \in T,\ 5 \in T$
Recursive: If $x, y \in T$ then
$x + y \in T$ and $x - y \in T$.

Let $P(x)$ be "$5|x$." We prove $P(x)$ holds for all $x \in T$ by structural induction.
Base Cases:
$(\boldsymbol{x = 0})$: $5(0) = 0$, so by definition of divides $5|0$, and our base case holds.

$(\boldsymbol{x = 5})$: $5(1) = 5$, so by definition of divides $5|5$, and our base case holds.

Inductive Hypothesis: Assume $P(x)$ and $P(y)$ hold for arbitrary $x, y \in T$. i.e., suppose that $5|x$ and $5|y$.

Inductive Step: Now, consider a new element $t \in T$ made from x and y.

Case 1: t $= x + y$: [Show $P(x + y)$ holds]

Case 2: t $= x - y$: [Show $P(x - y)$ holds]

Thus $\mathrm{P(x)}$ holds for all $x \in T$ by structural induction

Claim: $5|x$ for all $x \in T$.

Basis: $0 \in T,\ 5 \in T$
Recursive: If $x, y \in T$ then
$x + y \in T$ and $x - y \in T$.

Let $P(x)$ be "$5|x$." We prove $P(x)$ holds for all $x \in T$ by structural induction.
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Inductive Hypothesis: Assume $P(x)$ and $P(y)$ hold for arbitrary $x, y \in T$. i.e., suppose that $5|x$ and $5|y$.

Inductive Step: Now, consider a new element $t \in T$ made from x and y.

Case 1: $\mathrm{t} = x + y$: By IH $5|x$ and $5|y$ so $5a = x$ and $5b = y$ for integers a, b.
Adding, we get $x + y = 5a + 5b = 5(a + b)$. Since a, b are integers, so is $a + b$, and $P(x + y)$, i.e. $P(t)$, holds.
Case 2: $\mathrm{t} = x - y$: By IH $5|x$ and $5|y$ so $5a = x$ and $5b = y$ for integers a, b.
Subtracting, we get $x - y = 5a - 5b = 5(a - b)$. Since a, b are integers, so is $a - b$, and $P(x - y)$, i.e., $P(t)$, holds.

In all cases, we have $\mathrm{P(t)}$. Thus $\mathrm{P(x)}$ holds for all $x \in T$ by structural induction

Practice: Structural Induction on Strings

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $\mathrm{y} \in \Sigma^*$ by structural induction.

Base Case:

Inductive Hypothesis

Inductive Step:

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $y \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step:

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

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Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

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Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

len(xy)=len(xwa) =len(xw)+1 (by definition of len)

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

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len(xy)=len(xwa) =len(xw)+1 (by definition of len)

=len(x) + len(w) + 1 (by IH)

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Claim for all $x, y \in \Sigma^*$ len(x·y)=len(x) + len(y).

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Inductive Hypothesis: Suppose $P(w)$ for an arbitrary string w.

Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

len(xy)=len(xwa) =len(xw)+1 (by definition of len)

=len(x) + len(w) + 1 (by IH)

=len(x) + len(wa) (by definition of len)

Therefore, len(xy)=len(x) + len(y), as required.

We conclude that $P(y)$ holds for all string y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

Why all those arbitraries?

Let $P(y)$ be "len(x·y)=len(x) + len(y) for all $x \in \Sigma^*$."

We prove $P(y)$ for all $y \in \Sigma^*$ by structural induction.

Base Case: Let x be an arbitrary string, len($x \cdot \epsilon$)=len(x) =len(x)+0=len(x)+len(ε)

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Inductive Step: Let $y = wa$ for an arbitrary $a \in \Sigma$. We show $P(y)$. Let x be an arbitrary string.

len(xy)=len(xwa) =len(xw)+1 (by definition of len)

=len(x) + len(w) + 1 (by IH)

=len(x) + len(wa) (by definition of len)

Therefore, len(xy)=len(x) + len(y), as required.

We conclude that $P(y)$ holds for all strings y by the principle of induction. Unwrapping the definition of P, we get $\forall x \forall y \in \Sigma^*$ len(xy)=len(x)+len(y), as required.

$P(\varepsilon)$ is a for-all statement, introduce arbitrary variable to show for-all.

Needs to be arbitrary because it's in the IH (induction wouldn't show "all strings" otherwise)

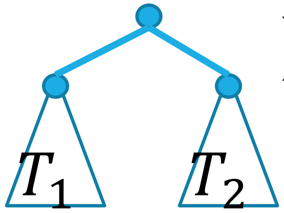
Recursive rule says "every $a \in \Sigma$" so we need to argue for every a.

$P(y)$ is a for-all statement, introduce arbitrary variable to show for-all.

Practice: Structural Induction on Trees

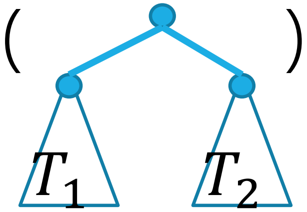
Functions on Binary Trees

size()=1

size(

) = size(T_1) + size(T_2) + 1

leaves() = 1

leaves(

) = leaves(T_1) + leaves(T_2)

Claim

For all trees T, $\mathbf{leaves}(T) \geq \frac{\mathrm{size}(T)}{2} + \frac{1}{2}$

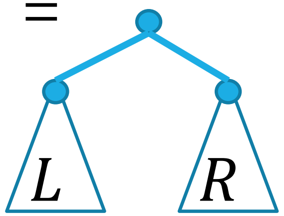
Take a moment to absorb this formula, then we'll do induction!

Practice: Structural Induction on Trees

Let $P(T)$ be "$\text{leaves}(T) \geq \frac{\text{size}(T)}{2} + \frac{1}{2}$". We show $P(T)$ for all binary trees T by structural induction.

Base Case: Let $T =$  .

Inductive Hypothesis: Suppose $\text{P}(L)$ and $\text{P}(R)$ hold for arbitrary trees L, R. Let $T =$



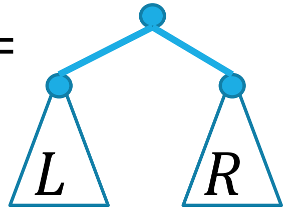
So $P(T)$ holds, and we have $P(T)$ for all binary trees T by the principle of induction.

Practice: Structural Induction on Trees

Let $P(T)$ be "$\text{leaves}(T) \geq \frac{\text{size}(T)}{2} + \frac{1}{2}$ ". We show $P(T)$ for all binary trees T by structural induction.

Base Case: Let $T =$  . $\text{leaves}(T) = 1$ and $\text{size}(T) = 1$, so $\frac{\text{size}(T)}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$. Thus, $\text{leaves}(T) = 1 \geq 1 = \frac{\text{size}(T)}{2} + \frac{1}{2}$, so $P(T)$ holds for the base case.

Inductive Hypothesis: Suppose $\text{P}(L)$ and $\text{P}(R)$ hold for arbitrary trees L, R. Let $T =$



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Practice: Structural Induction on Trees

Let $P(T)$ be "$\mathrm{leaves}(T) \geq \frac{\mathrm{size}(T)}{2} + \frac{1}{2}$ ". We show $P(T)$ for all binary trees T by structural induction.

Base Case: Let $T =$  . $\mathrm{leaves}(T) = 1$ and $\mathrm{size}(T) = 1$, so $\frac{\mathrm{size}(T)}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$. Thus, $\mathrm{leaves}(T) = 1 \geq 1 = \frac{\mathrm{size}(T)}{2} + \frac{1}{2}$, so $P(T)$ holds for the base case.

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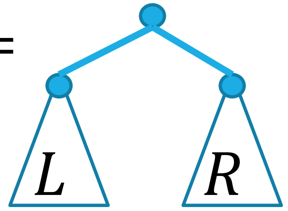

$\mathrm{leaves}(T) = \mathrm{leaves}(L) + \mathrm{leaves}(R)$ [By Def. of leaves]

So $P(T)$ holds, and we have $P(T)$ for all binary trees T by the principle of induction.

Practice: Structural Induction on Trees

Let $P(T)$ be "$\text{leaves}(T) \geq \frac{\text{size}(T)}{2} + \frac{1}{2}$". We show $P(T)$ for all binary trees T by structural induction.

Base Case: Let $T =$  . $\text{leaves}(T) = 1$ and $\text{size}(T) = 1$, so $\frac{\text{size}(T)}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$. Thus, $\text{leaves}(T) = 1 \geq 1 = \frac{\text{size}(T)}{2} + \frac{1}{2}$, so $P(T)$ holds for the base case.

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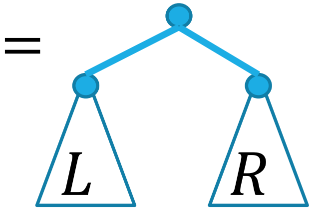

$\text{leaves}(T) = \text{leaves}(L) + \text{leaves}(R)$ [By Def. of leaves]

$\geq \left(\frac{\text{size}(L)}{2} + \frac{1}{2}\right) + \left(\frac{\text{size}(R)}{2} + \frac{1}{2}\right)$ [By IH]

So $P(T)$ holds, and we have $P(T)$ for all binary trees T by the principle of induction.

Practice: Structural Induction on Trees

Let $P(T)$ be "$\text{leaves}(T) \geq \frac{\text{size}(T)}{2} + \frac{1}{2}$ ". We show $P(T)$ for all binary trees T by structural induction.

Base Case: Let $T =$  . $\text{leaves}(T) = 1$ and $\text{size}(T) = 1$, so $\frac{\text{size}(T)}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$. Thus, $\text{leaves}(T) = 1 \geq 1 = \frac{\text{size}(T)}{2} + \frac{1}{2}$, so $P(T)$ holds for the base case.

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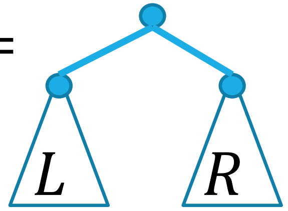


$$\text{leaves}(T) = \text{leaves}(L) + \text{leaves}(R)$$ [By Def. of leaves]

$$\geq \left(\frac{\text{size}(L)}{2} + \frac{1}{2}\right) + \left(\frac{\text{size}(R)}{2} + \frac{1}{2}\right)$$ [By IH]

$$= \frac{1 + \text{size}(L) + \text{size}(R)}{2} + \frac{1}{2}$$ [By Algebra]

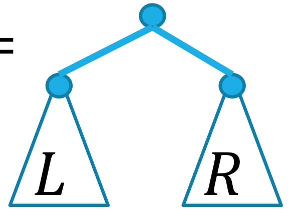
So $P(T)$ holds, and we have $P(T)$ for all binary trees T by the principle of induction.

Practice: Structural Induction on Trees

Let $P(T)$ be "$\text{leaves}(T) \geq \frac{\text{size}(T)}{2} + \frac{1}{2}$". We show $P(T)$ for all binary trees T by structural induction.

Base Case: Let $T =$  . $\text{leaves}(T) = 1$ and $\text{size}(T) = 1$, so $\frac{\text{size}(T)}{2} + \frac{1}{2} = \frac{1}{2} + \frac{1}{2} = 1$. Thus, $\text{leaves}(T) = 1 \geq 1 = \frac{\text{size}(T)}{2} + \frac{1}{2}$, so $P(T)$ holds for the base case.

Inductive Hypothesis: Suppose $\mathrm{P}(L)$ and $\mathrm{P}(R)$ hold for arbitrary trees L, R. Let $T =$



$$\begin{aligned} \text{leaves}(T) &= \text{leaves}(L) + \text{leaves}(R) && \text{[By Def. of leaves]} \\ &\geq \left(\frac{\text{size}(L)}{2} + \frac{1}{2}\right) + \left(\frac{\text{size}(R)}{2} + \frac{1}{2}\right) && \text{[By IH]} \\ &= \frac{1 + \text{size}(L) + \text{size}(R)}{2} + \frac{1}{2} && \text{[By Algebra]} \\ &= \frac{\text{size}(T)}{2} + \frac{1}{2} && \text{[By Def. of size]} \end{aligned}$$

So $P(T)$ holds, and we have $P(T)$ for all binary trees T by the principle of induction.

A few last comments

If you don't have a recursively-defined set

You won't do structural induction.

You can do weak or strong induction though.

For example, Let $P(n)$ be "for all elements of S of "size" n <something> is true"

To prove "for all $x \in S$ of size n…" you need to start with "let x be an arbitrary element of size $k + 1$ in your IS.

You CAN'T start with size k and "build up" to an arbitrary element of size $k + 1$ it isn't arbitrary.

Extra Induction Practice

Induction: Hats!

You have n people in a line ($n \geq 2$). Each of them wears either a purple hat or a gold hat. The person at the front of the line wears a purple hat. The person at the back of the line wears a gold hat.

Show that for every arrangement of the line satisfying the rule above, there is a person with a purple hat next to someone with a gold hat.

Yes, this is kinda obvious. I promise this is good induction practice.

Yes, you could argue this by contradiction. I promise this is good induction practice.

Induction: Hats!

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$

Inductive Hypothesis:

Inductive Step:

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats!

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Target: there is someone in a purple hat next to someone in a gold hat.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Induction: Hats!

Define $P(n)$ to be "in every line of n people with gold and purple hats, with a purple hat at one end and a gold hat at the other, there is a person with a purple hat next to someone with a gold hat"

We show $P(n)$ for all integers $n \geq 2$ by induction on n.

Base Case: $n = 2$ The line must be just a person with a purple hat and a person with a gold hat, who are next to each other.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step: Consider an arbitrary line with $k + 1$ people in purple and gold hats, with a gold hat at one end and a purple hat at the other.

Case 1: There is someone with a purple hat next to the person in the gold hat at one end. Then those people are the required adjacent opposite hats.

Case 2:. There is a person with a gold hat next to the person in the gold hat at the end. Then the line from the second person to the end is length k, has a gold hat at one end and a purple hat at the other. Applying the inductive hypothesis, there is an adjacent, opposite-hat wearing pair.

In either case we have $P(k + 1)$.

By the principle of induction, we have $P(n)$ for all $n \geq 2$

Todo

Tonight:

- CC 16 is out and due **Monday** at noon
- Start reviewing for the midterm if you haven’t already!