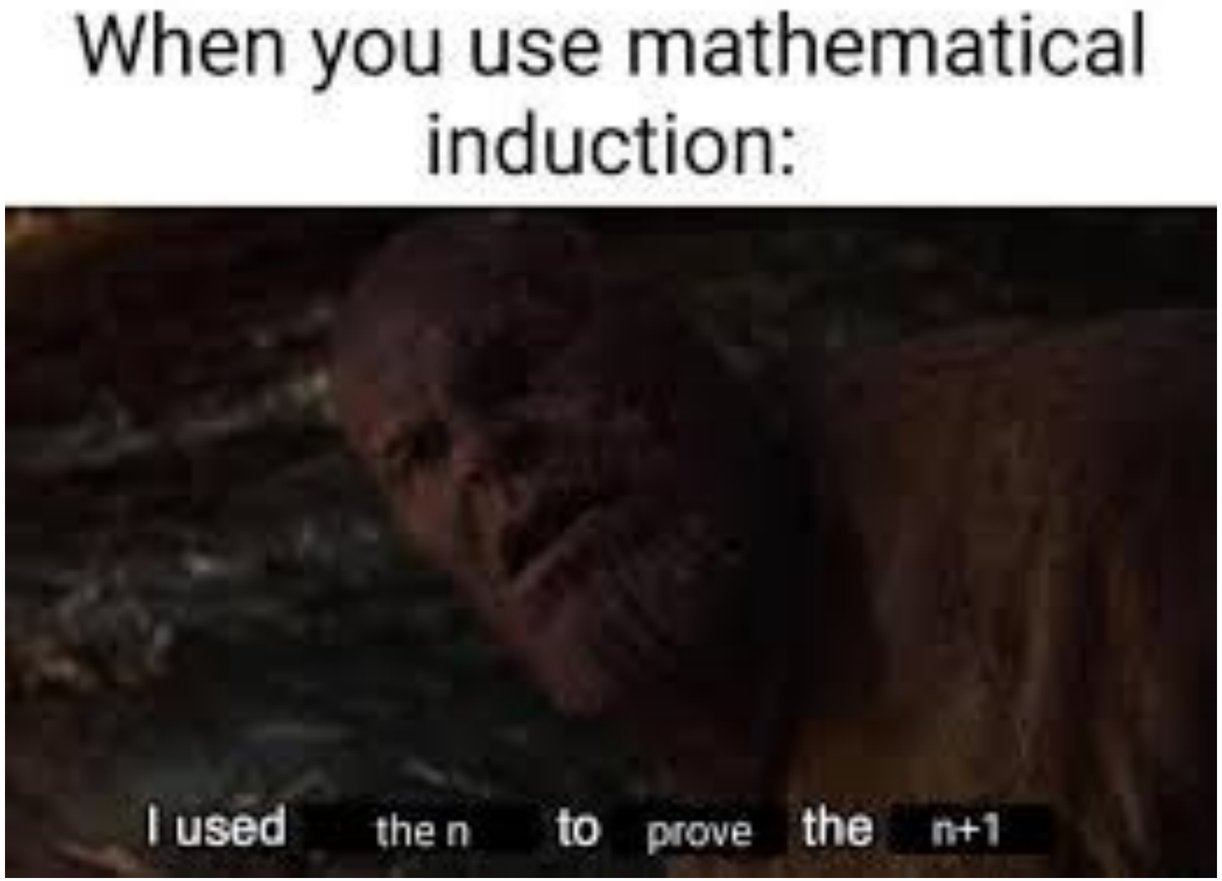

Induction practice

CSE 311 Summer 2025
Lecture 14

Announcements

- HW5 has been released!
- HW4 was due on Wednesday (7/23)
  - Late submissions open until Saturday 7/25
- HW3 Feedback and resubmission is out
  - Due **today at 11:59pm—no late submissions!**

- Our Midterm is next Friday (8/1) in class!
  - Exam logistics and practice exams are posted on the "Exams" page of the course website

Practice

Induction Template

1. Define $\mathrm{P}(n)$. State that your proof is by induction on n.
2. Base Case: Show $\mathrm{P}(b)$ is true for your base case b.
3. Inductive Hypothesis: Suppose $\mathrm{P}(k)$ holds for an arbitrary integer $k \geq b$.
4. Inductive Step: Prove $\mathrm{P}(k+1)$ (using the Inductive Hypothesis).
5. Conclusion: Conclude by saying $\mathrm{P}(n)$ holds for all integers $n \geq b$ by induction.

Problem 1: Induction with Sums

Let a be an integer with $a > 1$. Prove, by induction, that

$$\sum_{i=0}^{n} a^i < \frac{a^{n+1}}{a-1}$$

for all integers $n \geq 0$.

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> for all integers $n \geq 0$.

Let a be an arbitrary integer. Suppose that $a > 1$.

Let $P(n)$ be the claim: "$\sum_{i=0}^{n} a^i < \frac{a^{n+1}}{a-1}$". We will prove $P(n)$ by induction on all integers $n \geq 0$.

Base Case ($n = 0$):

$$\begin{aligned}
\sum_{i=0}^{0} a^i &= a^0 \\
&= 1 \\
&< 1 + \frac{1}{a-1} && \text{[since a > 1]} \\
&= \frac{a-1}{a-1} + \frac{1}{a-1} \\
&= \frac{a^{0+1}}{a-1}
\end{aligned}$$

Thus $P(0)$ holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary integer $k \geq 0$.

> Let a be an integer with $a > 1$. Prove, by induction, that
>
> $$\sum_{i=0}^{n} a^i < \frac{a^{n+1}}{a-1}$$
>
> for all integers $n \geq 0$.

Inductive Step: $\boxed{\text{Goal: } P(k+1)}$

$$\begin{aligned}
\sum_{i=0}^{k+1} a^i &= \sum_{i=0}^{k} a_i + a^{k+1} \\
&< \frac{a^{k+1}}{a-1} + a^{k+1} && \text{[Inductive Hypothesis]} \\
&= a^{k+1}(\frac{1}{a-1} + 1)]] \\
&= a^{k+1}(\frac{1}{a-1} + \frac{a-1}{a-1}) \\
&= a^{k+1}(\frac{a}{a-1}) \\
&= \frac{a^{k+2}}{a-1}
\end{aligned}$$

Thus $P(k+1)$ holds. Since a was arbitrary the claim holds by the principle of induction for all integers $n \geq 0$ and for all $a > 1$.

Problem 2: Fibonacci Inequality

Fibonacci Inequality

Define the Fibonacci numbers as follows:

$f(0) = 1$

$f(1) = 1$

$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

*This is a somewhat unusual definition, $f(0) = 0, f(1) = 1$ is more common.

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

$f(0) = 1; \qquad f(1) = 1$
$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2.$

[Define $P(n)$]

Base Cases:

Inductive Hypothesis:

Inductive step:

Therefore, we have $P(n)$ for all $n \geq 2$ by the principle of induction.

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

$$f(0) = 1; \qquad f(1) = 1$$
$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases:

Inductive Hypothesis:

Inductive step:

Therefore, we have $P(n)$ for all $n \geq 2$ by the principle of induction.

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

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Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$

Inductive Hypothesis:

Inductive step:

Therefore, we have $P(n)$ for all $n \geq 2$ by the principle of induction.

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

$$f(0) = 1; \qquad f(1) = 1$$
$$f(n) = f(n-1) + f(n-2) \text{ for all } n \in \mathbb{N}, n \geq 2.$$

Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

$$f(3) = f(2) + f(1) = 2 + 1 = 3 = 2 \cdot \frac{3}{2} \geq 2\sqrt{2} = 2^{1.5} = 2^{3/2}$$

Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

Therefore, we have $P(n)$ for all $n \geq 2$ by the principle of induction.

Show that $f(n) \geq 2^{n/2}$ for all $n \geq 2$ by induction.

$$f(0) = 1; \qquad f(1) = 1$$
$f(n) = f(n-1) + f(n-2)$ for all $n \in \mathbb{N}, n \geq 2$.

Define $P(n)$ to be "$f(n) \geq 2^{n/2}$" We show $P(n)$ is true for all $n \geq 2$ by induction on n.

Base Cases: $f(2) = f(1) + f(0) = 2 \geq 2 = 2^1 = 2^{2/2}$

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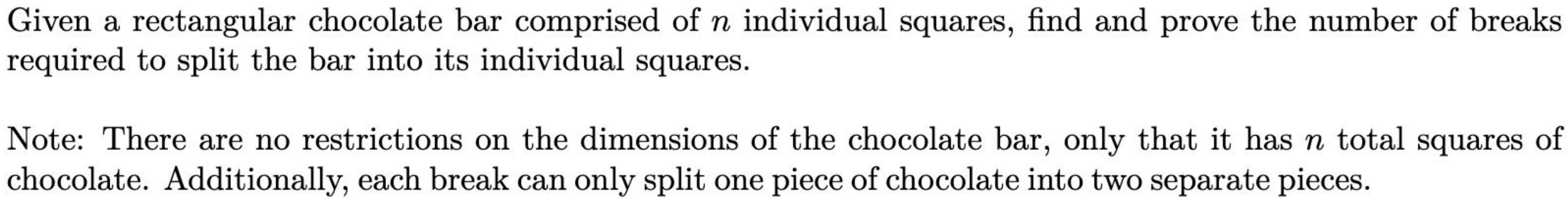
Inductive Hypothesis: Suppose $P(2) \wedge P(3) \wedge \cdots \wedge P(k)$ for an arbitrary $k \geq 3$.

Inductive step: $f(k+1) = f(k) + f(k-1)$ by the definition of the Fibonacci numbers. Applying IH twice, we have

$$\begin{aligned} f(k+1) &\geq 2^{k/2} + 2^{(k-1)/2} \\ &= 2^{(k-1)/2}(\sqrt{2} + 1) \\ &\geq 2^{(k-1)/2} \cdot 2 \\ &\geq 2^{(k+1)/2} \end{aligned}$$

Therefore, we have $P(n)$ for all $n \geq 2$ by the principle of induction.

Problem 3: Sharing Chocolate



> Given a rectangular chocolate bar comprised of n individual squares, find and prove the number of breaks required to split the bar into its individual squares.
>
> Note: There are no restrictions on the dimensions of the chocolate bar, only that it has n total squares of chocolate. Additionally, each break can only split one piece of chocolate into two separate pieces.

[Define $P(n)$]

Base Cases:

Inductive Hypothesis:

Inductive step:

Therefore, we have $P(n)$ for all $n \geq 0$ by the principle of induction.

> Given a rectangular chocolate bar comprised of n individual squares, find and prove the number of breaks required to split the bar into its individual squares.

Let $P(n)$ be the claim: "It takes $n-1$ breaks to split a rectangular chocolate bar into its n individual pieces". We will prove $P(n)$ for all integers $n \geq 1$.

Base Case: $(n = 1)$:
Since there are no breaks necessary to split this bar into its individual squares of chocolate, the base case holds.

Inductive Hypothesis:
Assume $P(1) \wedge \cdots \wedge P(k)$ is true for some integer $k \geq 1$.

Inductive Step:
Goal: $P(k{+}1)$ is true, i.e., it takes k splits to break a $k{+}1$-squared chocolate bar into its individual pieces.

First, note that any break must split the bar into two rectangular chocolate bars with a and b number of squares, where $a, b \geq 1$ and $a + b = k + 1$. From the Inductive Hypothesis, we know that $P(a)$ and $P(b)$ are true, meaning to split the two parts into individual squares takes $a-1$ and $b-1$ splits, respectively. Adding to this the break to create the two pieces, and we have $(a{-}1){+}(b{-}1){+}1 = (k{+}1){-}1 = k$ breaks.

Therefore it takes k breaks to split the bare into $k+1$ pieces, and $P(k+1)$ holds.
Thus our claims hold by the principle of induction for all integers $n \geq 1$.

Todo

Tonight:

- HW3 resubmissions are due tonight!
- Submit HW4 late if needed
- CC 17 is out and due Friday at noon
- Read the midterm logistics on the Exams page of the course website and post on the Ed board if you have any questions