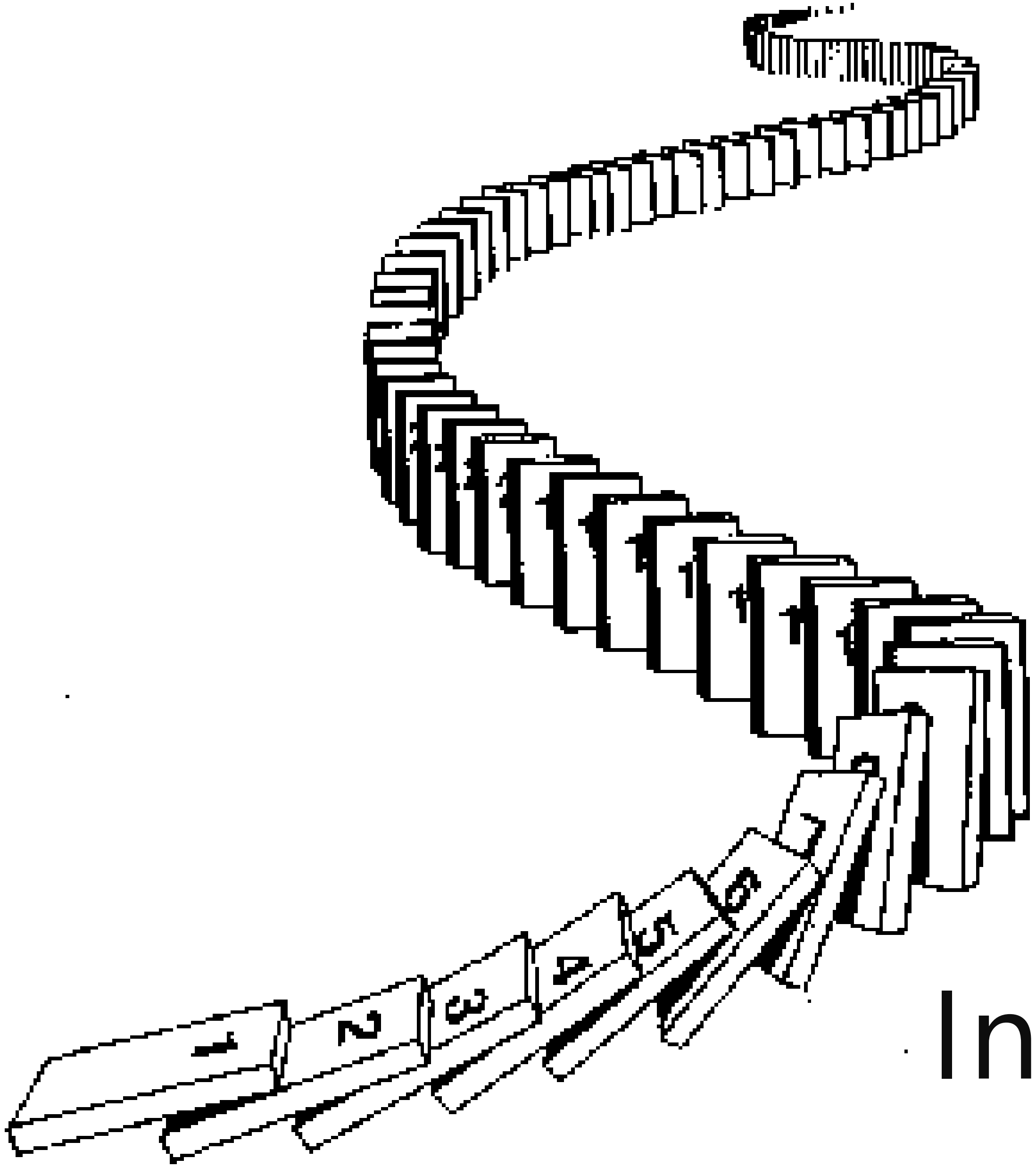


Induction

CSE 311 Summer 2025
Lecture 12

Announcements

- HW4 is released!
  - Start today if you have not already
- Our Midterm is next Friday (8/1) in class!
  - Exam logistics and practice exams are posted on the "Exams" page of the course website
- HW2 solutions are out!

Review

Proof by Contradiction

Proof by contradiction is a strategy for proving statements of any form.

The strategy to prove p is to assume $\neg p$ and derive $\mathbf{False}$. (i.e. (¬claim) → F)

- E.g. the strategy to prove $p \rightarrow q$ is to assume $p \wedge \neg q$ and derive $\mathbf{False}$.
- E.g. the strategy to prove $p \vee q$ is to assume $\neg p \wedge \neg q$ and derive $\mathbf{False}$.

Proof by Contradiction Skeleton

Suppose for the sake of contradiction $\neg p$.

…

Then some statement s must hold.

…

And some statement $\neg s$ must hold.

But s and $\neg s$ is a contradiction. So p must be true.

Claim: No integer is even and odd.

Suppose for the sake of contradiction that there exists an integer x that is both even and odd.

Then $x = 2a$ for some integer a, and $x = 2b + 1$ for some integer b.

Then:

$$2a = 2b + 1$$
$$2a - 2b = 1$$
$$a - b = \frac{1}{2}$$

Since a, b are integers, $a - b$ is an integer. But $\frac{1}{2}$ is not an integer, so $a - b$ cannot equal $\frac{1}{2}$.

This is a contradiction!

Thus no integer can be even and odd.

Proof by Contradiction: Remarks

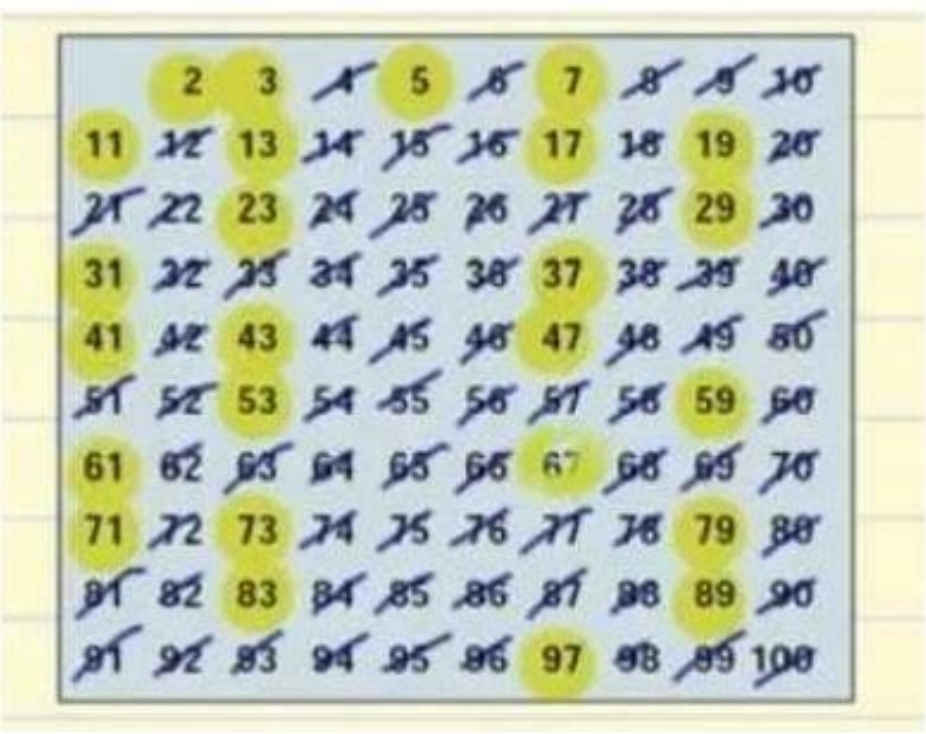
- Unlike other proof techniques, we don't know *where* we're going. We're trying to find any contradiction. That can make it harder.
- Contradiction is a sledge-hammer.
  It can be used to prove many things. But it makes a mess.
- Use contradiction as a last-resort.

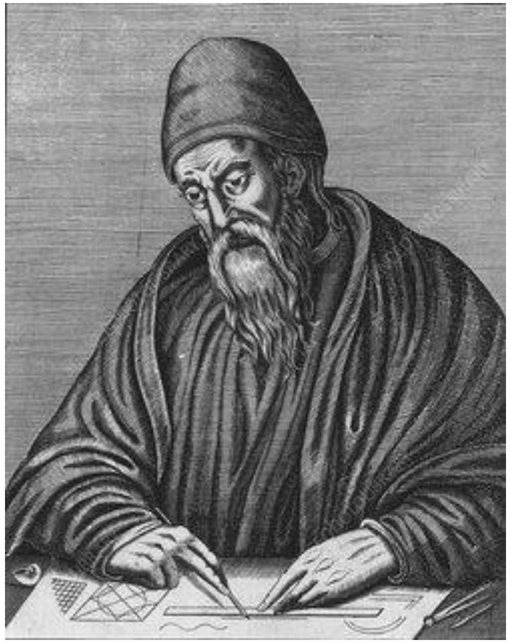
Contradiction is a sledge-hammer

Another Proof by Contradiction

Euclid’s Theorem: There are infinitely many prime numbers.

$2, 3, 5, 7, 11, 13, \ldots$





Euclid ~300 BC

Euclid's Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \ldots, p_k$.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Euclid’s Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \dots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdots p_k + 1$.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Euclid's Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdots p_k + 1$.

Case 1: q is prime.

Case 2: q is composite.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Euclid's Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \dots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdots p_k + 1$.

<u>Case 1</u>: q is prime. Then q is a prime that is larger than p_i for all $i \in \{1, \dots, k\}$. But every prime was supposed to in the list $p_1, \dots, p_k$. This is a contradiction.

<u>Case 2</u>: q is composite.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Euclid's Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdots p_k + 1$.

Case 1: q is prime. Then q is a prime that is larger than p_i for all $i \in \{1, \ldots, k\}$. But every prime was supposed to in the list $p_1, \ldots, p_k$. This is a contradiction.

Case 2: q is composite. Then q must have some prime factor. So there exists some i such that $p_i \mid q$.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Euclid's Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \dots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdots p_k + 1$.

<u>Case 1</u>: q is prime. Then q is a prime that is larger than p_i for all $i \in \{1, \dots, k\}$. But every prime was supposed to in the list $p_1, \dots, p_k$. This is a contradiction.

<u>Case 2</u>: q is composite. Then q must have some prime factor. So there exists some i such that $p_i \mid q$. Then $q \,\%\, p_i = 0$. But

$$q \,\%\, p_i = \quad (p_1 \cdot p_2 \cdots p_k + 1) \,\%\, p_i = 1$$

So we have $q \,\%\, p_i = 0$ and $q \,\%\, p_i = 1$. This is a contradiction.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Proof by Induction

Proof Strategies so Far

- Direct Proof
- Proof by Contrapositive
- Proof of Biconditional
- Proof by Cases
- Existence Proof
- Proof by Contradiction

There are claims we cannot prove using these strategies!

How do we know recursion works?

```
//Assume i is a nonnegative integer
//returns 2^i.
public int CalculatesTwoToTheI(int i){
     if(i == 0)
          return 1;
     else
          return 2*CaclulatesTwoToTheI(i-1);
}
```

Why does `CalculatesTwoToTheI(4)` calculate `2^4?`
Convince the people around you!

How do we know recursion works?

Something like this:

Well, as long as `CalculatesTwoToTheI(3) = 8`, we get 16…

Which happens as long as `CalculatesTwoToTheI(2) = 4`

Which happens as long as `CalculatesTwoToTheI(1) = 2`

Which happens as long as `CalculatesTwoToTheI(0) = 1`

And it is! Because that's what the base case says.

How do we know recursion works?

There are really only two cases.

The Base Case is Correct

`CalculatesTwoToTheI(0) = 1 (which it should!)`

And that means `CalculatesTwoToTheI(1) = 2`, (like it should)

And that means `CalculatesTwoToTheI(2) = 4`, (like it should)

And that means `CalculatesTwoToTheI(3) = 8`, (like it should)

And that means `CalculatesTwoToTheI(4) = 16`, (like it should)

IF the recursive call we make is correct
THEN our value is correct.

How do we know recursion works?

The code has two big cases,

So our proof had two big cases

"The base case of the code produces the correct output"

"IF the calls we rely on produce the correct output THEN the current call produces the right output"

A bit more formally…

"The base case of the code produces the correct output"

"IF the calls we rely on produce the correct output THEN the current call produces the right output"

Let $P(i)$ be "`CalculatesTwoToTheI(i)` returns 2^i."

How do we know $P(4)$?

$P(0)$ is true.

And $P(0) \rightarrow P(1)$, so $P(1)$.

And $P(1) \rightarrow P(2)$, so $P(2)$.

And $P(2) \rightarrow P(3)$, so $P(3)$.

And $P(3) \rightarrow P(4)$, so $P(4)$.

A bit more formally…

This works alright for $P(4)$.

What about $P(1000)$? $P(1000000000)$?

At this point, we'd need to show that implication $P(k) \rightarrow P(k+1)$ for A BUNCH of values of k.

But the code is the same each time.

And so was the argument!

We should instead show $\forall k[P(k) \rightarrow P(k+1)]$.

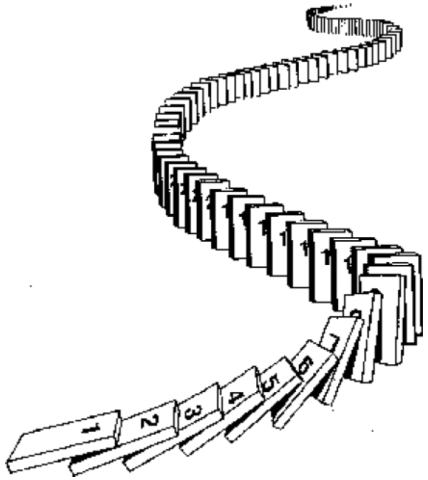
The Principle of Mathematical Induction

$$\mathrm{P}(0) \land \forall k \left(\mathrm{P}(k) \to \mathrm{P}(k+1)\right)$$

<u>Base Case</u>
Prove $\mathrm{P}(0)$ holds.

<u>Inductive Hypothesis</u>
Let $k \geq 0$ be an arbitrary integer.
Suppose $\mathrm{P}(k)$ holds.

<u>Inductive Step</u>
Prove that $\mathrm{P}(k+1)$ holds (using $\mathrm{P}(k)$)



Induction

```
//Assume i is a nonnegative integer
public int CalculatesTwoToTheI(int i){
	if(i == 0)
		return 1;
	else
		return 2*CaclulatesTwoToTheI(i-1);
}
```

Let $P(n)$ be "`CalculatesTwoToTheI(n)` returns 2^n."

Note that if the input n is 0, then the if-statement evaluates to true, and $1 = 2^\wedge 0$ is returned, so $P(0)$ is true.

Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Consider the code run on $k + 1$. Since $k \geq 0$, $k + 1 \geq 1$ and we are in the else branch. By inductive hypothesis, `CalculatesTwoToTheI(`k`)` returns 2^k, so the code run on $k + 1$ returns $2 \cdot 2^k = 2^{k+1}$.

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Making Induction Proofs Pretty

Let $P(n)$ be the predicate "`CalculatesTwoToTheI(n)` returns 2^n." We prove $P(n)$ holds holds for all natural numbers n by induction on n.

Base Case $(\mathrm{n} = 0)$ Note that if the input n is 0, then the if-statement evaluates to true, and $1 = 2\hat{}0$ is returned, so $P(0)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Inductive Step: Since $k \geq 0, k + 1 \geq 1$, so the code goes to the recursive case. We will return $2 \cdot$ `CalculatesTwoToTheI(k)`. By Inductive Hypothesis,

`CalculatesTwoToTheI(k)` = 2^k. Thus we return $2 \cdot 2^k = 2^{k+1}$.

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Making Induction Proofs Pretty

All of our induction proofs will come in 5 easy(?) steps!

1. Define $P(n)$. State that your proof is by induction on n.
2. Show $P(0)$ i.e. show the base case
3. Suppose $P(k)$ for an arbitrary k.
4. Show $P(k + 1)$ (i.e. get $P(k) \rightarrow P(k + 1)$)
5. Conclude by saying $P(n)$ is true for all n by induction.

Induction Template

1. Define $\mathrm{P}(n)$. State that your proof is by induction on n.
2. Base Case: Show $\mathrm{P}(b)$ is true for your base case b.
3. Inductive Hypothesis: Suppose $\mathrm{P}(k)$ holds for an arbitrary integer $k \geq b$.
4. Inductive Step: Prove $\mathrm{P}(k+1)$ (using the Inductive Hypothesis).
5. Conclusion: Conclude by saying $\mathrm{P}(n)$ holds for all integers $n \geq b$ by induction.

Some Other Notes

Always state where you use the inductive hypothesis when you're using it in the inductive step.

It's usually the key step, and the reader really needs to focus on it.

Be careful about what values you're assuming the Inductive Hypothesis for – the smallest possible value of k should assume the base case but nothing more.

More induction!

Find an expression in n for the sum $1 + 2 + 4 + \cdots + 2^n$

- $n = 0$ $\quad 1 \quad = 1$
- $n = 1$ $\quad 1 + 2 \quad = 3$
- $n = 2$ $\quad 1 + 2 + 4 \quad = 7$
- $n = 3$ $\quad 1 + 2 + 4 + 8 \quad = 15$
- $n = 4$ $\quad 1 + 2 + 4 + 8 + 16 \quad = 31$

Find an expression in n for the sum $1 + 2 + 4 + \cdots + 2^n$

- $n = 0$ $\quad 1 = 1$
- $n = 1$ $\quad 1 + 2 = 3$
- $n = 2$ $\quad 1 + 2 + 4 = 7$
- $n = 3$ $\quad 1 + 2 + 4 + 8 = 15$
- $n = 4$ $\quad 1 + 2 + 4 + 8 + 16 = 31$

It *looks* like this sum is $2^{n+1} - 1$.

More Induction

Induction doesn't **only** work for code!

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$." We prove $P(n)$ holds holds for all integers $n \geq 0$ by induction on n.

Base Case $(k = 0)$

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Inductive Step:

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$." We prove $P(n)$ holds holds for all integers $n \geq 0$ by induction on n.

Base Case $(k = 0)$: $\sum_{i=0}^{0} 2^i = 1 = 2 - 1 = 2^{0+1} - 1$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$.

Inductive Step:

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1.$

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$." We prove $P(n)$ holds holds for all integers $n \geq 0$ by induction on n.

Base Case $(k = 0)$: $\sum_{i=0}^{0} 2^i = 1 = 2 - 1 = 2^{0+1} - 1.$

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. i.e. $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

Inductive Step:

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1.$

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$." We prove $P(n)$ holds holds for all integers $n \geq 0$ by induction on n.

Base Case $(k = 0)$: $\sum_{i=0}^{0} 2^i = 1 = 2 - 1 = 2^{0+1} - 1.$

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. i.e. $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

Inductive Step:

[Goal: Show that $\sum_{i=0}^{k+1} 2^i = 2^{k+2} - 1$]

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$." We prove $P(n)$ holds holds for all integers $n \geq 0$ by induction on n.

Base Case $(k = 0)$: $\sum_{i=0}^{0} 2^i = 1 = 2 - 1 = 2^{0+1} - 1$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. i.e. $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

Inductive Step:

$$\sum_{i=0}^{k+1} 2^i = 2^{\mathrm{k}+1} + \sum_{i=0}^{k} 2^i$$

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$." We prove $P(n)$ holds holds for all integers $n \geq 0$ by induction on n.

Base Case $(k = 0)$: $\sum_{i=0}^{0} 2^i = 1 = 2 - 1 = 2^{0+1} - 1$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. i.e. $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

Inductive Step:

$$\sum_{i=0}^{k+1} 2^i = 2^{\mathrm{k+1}} + \sum_{i=0}^{k} 2^i$$

$$= 2^{k+1} + 2^{k+1} - 1 \quad \text{[By IH]}$$

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$." We prove $P(n)$ holds holds for all integers $n \geq 0$ by induction on n.

Base Case $(k = 0)$: $\sum_{i=0}^{0} 2^i = 1 = 2 - 1 = 2^{0+1} - 1$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. i.e. $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

Inductive Step:

$$\sum_{i=0}^{k+1} 2^i = 2^{k+1} + \sum_{i=0}^{k} 2^i$$
$$= 2^{k+1} + 2^{k+1} - 1 \quad \text{[By IH]}$$
$$= 2 \cdot 2^{k+1} - 1$$

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Show that $\sum_{i=0}^{n} 2^i = 1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$.

Let $P(n)$ be "$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$." We prove $P(n)$ holds holds for all integers $n \geq 0$ by induction on n.

Base Case $(k = 0)$: $\sum_{i=0}^{0} 2^i = 1 = 2 - 1 = 2^{0+1} - 1$.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. i.e. $\sum_{i=0}^{k} 2^i = 2^{k+1} - 1$

Inductive Step:

$$\sum_{i=0}^{k+1} 2^i = 2^{k+1} + \sum_{i=0}^{k} 2^i$$

$$= 2^{k+1} + 2^{k+1} - 1 \quad \text{[By IH]}$$

$$= 2 \cdot 2^{k+1} - 1$$

$$= 2^{k+2} - 1$$

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Even More Induction Examples

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

<u>Examples</u>

$n = 3$ Sum: $1 + 2 + 3 = 6$

Formula: $\frac{3(3+1)}{2} = \frac{3\cdot 4}{2} = 6$

$n = 5$ Sum: $1 + 2 + 3 + 4 + 5 = 15$

Formula: $\frac{5(5+1)}{2} = \frac{5\cdot 6}{2} = 15$

Carl Friedrich Gauss
(1777-1855)

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Let $P(n)$ be "." We prove $P(n)$ holds holds for all integers $n \geq 1$ by induction on n.

Base Case ( )

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq$ [ ].

Inductive Step:

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 1$ by the principle of induction.

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Let $P(n)$ be "$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$." We prove $P(n)$ holds holds for all integers $n \geq 1$ by induction on n.

Base Case (   )

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq$ [ ].

Inductive Step:

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 1$ by the principle of induction.

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Let $P(n)$ be "$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$." We prove $P(n)$ holds holds for all integers $n \geq 1$ by induction on n.

Base Case $(\mathrm{n} = 1)$: The LHS evaluates to 1. The RHS evaluates to $\frac{1(1+1)}{2} = 1$. Since $1 = 1$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 1$.

Inductive Step:

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 1$ by the principle of induction.

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Let $P(n)$ be "$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$." We prove $P(n)$ holds holds for all integers $n \geq 1$ by induction on n.

Base Case $(\mathrm{n} = 1)$**:** The LHS evaluates to 1. The RHS evaluates to $\frac{1(1+1)}{2} = 1$. Since $1 = 1$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 1$. That is,

$1 + 2 + \ldots + k = \frac{k(k+1)}{2}$.

Inductive Step:

[Goal: Show $1 + 2 + \ldots + (k + 1) = \frac{(k+1)(k+2)}{2}$]

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 1$ by the principle of induction.

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Let $P(n)$ be "$1 + 2 + 3 + \ ... + n = \frac{n(n+1)}{2}$." We prove $P(n)$ holds holds for all integers $n \geq 1$ by induction on n.

Base Case $(\mathrm{n} = 1\ )$**:** The LHS evaluates to 1. The RHS evaluates to $\frac{1(1+1)}{2} = 1$. Since $1 = 1$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 1$**.** That is,

$1 + 2 + \ ... + k = \frac{k(k+1)}{2}$.

Inductive Step:

$$1 + 2 + 3 + \ ... + (k + 1) = 1 + 2 + \ ... + k + (k + 1)$$

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 1$ by the principle of induction.

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Let $P(n)$ be "$1 + 2 + 3 + \ ... + n = \frac{n(n+1)}{2}$." We prove $P(n)$ holds holds for all integers $n \geq 1$ by induction on n.

Base Case ($\mathrm{n} = 1$): The LHS evaluates to 1. The RHS evaluates to $\frac{1(1+1)}{2} = 1$. Since $1 = 1$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 1$. That is,

$1 + 2 + \ ... + k = \frac{k(k+1)}{2}$.

Inductive Step:

$$
\begin{aligned}
1 + 2 + 3 + \ ... + (k+1) &= 1 + 2 + \ ... + k + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \qquad \text{[By IH]}
\end{aligned}
$$

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 1$ by the principle of induction.

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Let $P(n)$ be "$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$." We prove $P(n)$ holds holds for all integers $n \geq 1$ by induction on n.

Base Case ($\mathrm{n} = 1$): The LHS evaluates to 1. The RHS evaluates to $\frac{1(1+1)}{2} = 1$. Since $1 = 1$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 1$. That is,

$1 + 2 + \ldots + k = \frac{k(k+1)}{2}$.

Inductive Step:

$$
\begin{aligned}
1 + 2 + 3 + \ldots + (k+1) &= 1 + 2 + \ldots + k + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \qquad \text{[By IH]} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2}
\end{aligned}
$$

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 1$ by the principle of induction.

Prove that the sum of the first n positive integers is $\frac{n(n+1)}{2}$.

Let $P(n)$ be "$1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$." We prove $P(n)$ holds holds for all integers $n \geq 1$ by induction on n.

Base Case ($\mathrm{n} = 1$): The LHS evaluates to 1. The RHS evaluates to $\frac{1(1+1)}{2} = 1$. Since $1 = 1$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 1$. That is,

$1 + 2 + \ldots + k = \frac{k(k+1)}{2}$.

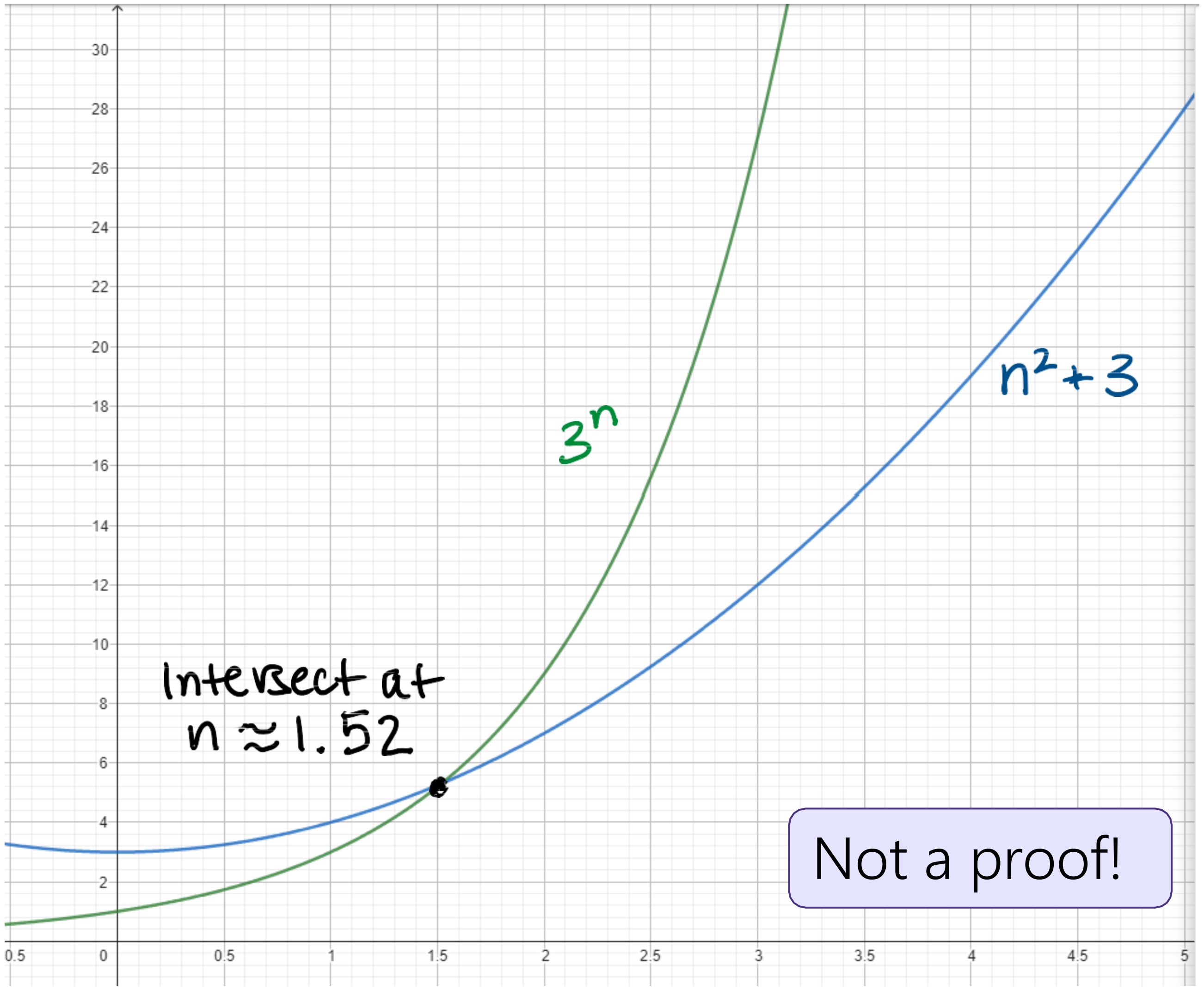
Inductive Step:

$$
\begin{aligned}
1 + 2 + 3 + \ldots + (k+1) &= 1 + 2 + \ldots + k + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) \qquad \text{[By IH]} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{(k+2)(k+1)}{2}
\end{aligned}
$$

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 1$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.



Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case ( )

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq$ [ ].

Inductive Step:

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "$3^n \geq n^2 + 3$." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case ( )

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq$ [ ].

Inductive Step:

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "$3^n \geq n^2 + 3$." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case $(n = 2)$

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step:

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "$3^n \geq n^2 + 3$." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case $(n = 2)$: The LHS evaluates to $3^2 = 9$. The RHS evaluates to $2^2 + 3 = 7$. Since $9 \geq 7$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$.

Inductive Step:

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "$3^n \geq n^2 + 3$." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case $(n = 2)$: The LHS evaluates to $3^2 = 9$. The RHS evaluates to $2^2 + 3 = 7$. Since $9 \geq 7$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$. In other words, $3^k \geq k^2 + 3$

Inductive Step:

[Goal: Show $3^{k+1} \geq (k+1)^2 + 3$]

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "$3^n \geq n^2 + 3$." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case $(n = 2)$: The LHS evaluates to $3^2 = 9$. The RHS evaluates to $2^2 + 3 = 7$. Since $9 \geq 7$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$. In other words, $3^k \geq k^2 + 3$

Inductive Step:

$$3^{k+1} = 3 \cdot 3^k$$

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "$3^n \geq n^2 + 3$." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case ($n = 2$): The LHS evaluates to $3^2 = 9$. The RHS evaluates to $2^2 + 3 = 7$. Since $9 \geq 7$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$. In other words, $3^k \geq k^2 + 3$

Inductive Step:

$$\begin{aligned} 3^{k+1} &= 3 \cdot 3^k \\ &\geq 3 \cdot (k^2 + 3) \quad \text{[By IH]} \end{aligned}$$

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "$3^n \geq n^2 + 3$." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case ($n = 2$): The LHS evaluates to $3^2 = 9$. The RHS evaluates to $2^2 + 3 = 7$. Since $9 \geq 7$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$. In other words, $3^k \geq k^2 + 3$

Inductive Step:

$$3^{k+1} = 3 \cdot 3^k$$

$$\geq 3 \cdot (k^2 + 3) \qquad \text{[By IH]}$$

$$= 3k^2 + 9$$

$$= k^2 + 2k^2 + 9$$

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "$3^n \geq n^2 + 3$." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case ($n = 2$): The LHS evaluates to $3^2 = 9$. The RHS evaluates to $2^2 + 3 = 7$. Since $9 \geq 7$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$. In other words, $3^k \geq k^2 + 3$

Inductive Step:

$$
\begin{aligned}
3^{k+1} &= 3 \cdot 3^k \\
&\geq 3 \cdot (k^2 + 3) && \text{[By IH]} \\
&= 3k^2 + 9 \\
&= k^2 + 2k^2 + 9 \\
&\geq k^2 + 2k + 9 && \text{[Since } k \geq 1 \text{, so } 2k^2 \geq 2k\text{]}
\end{aligned}
$$

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "$3^n \geq n^2 + 3$." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case $(n = 2)$: The LHS evaluates to $3^2 = 9$. The RHS evaluates to $2^2 + 3 = 7$. Since $9 \geq 7$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$. In other words, $3^k \geq k^2 + 3$

Inductive Step:

$$
\begin{aligned}
3^{k+1} &= 3 \cdot 3^k \\
&\geq 3 \cdot (k^2 + 3) && \text{[By IH]} \\
&= 3k^2 + 9 \\
&= k^2 + 2k^2 + 9 \\
&\geq k^2 + 2k + 9 && \text{[Since } k \geq 1 \text{, so } 2k^2 \geq 2k\text{]} \\
&\geq k^2 + 2k + 4
\end{aligned}
$$

So $P(k + 1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Prove that $3^n \geq n^2 + 3$ for all integers $n \geq 2$.

Let $P(n)$ be "$3^n \geq n^2 + 3$." We prove $P(n)$ holds holds for all integers $n \geq 2$ by induction on n.

Base Case ($n = 2$): The LHS evaluates to $3^2 = 9$. The RHS evaluates to $2^2 + 3 = 7$. Since $9 \geq 7$, the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 2$. In other words, $3^k \geq k^2 + 3$

Inductive Step:

$$
\begin{aligned}
3^{k+1} &= 3 \cdot 3^k \\
&\geq 3 \cdot (k^2 + 3) \text{ [By IH]} \\
&= 3k^2 + 9 \\
&= k^2 + 2k^2 + 9 \\
&\geq k^2 + 2k + 9 \qquad \text{[Since } k \geq 1\text{, so } 2k^2 \geq 2k\text{]} \\
&\geq k^2 + 2k + 4 \\
&= (k+1)^2 + 3
\end{aligned}
$$

So $P(k+1)$ holds.

Therefore $P(n)$ holds for all $n \geq 2$ by the principle of induction.

Todo

Tonight:

- CC 12 is out and due Wednesday at noon
- Start HW4 if you have not already!
- Read the midterm logistics on the Exams page of the course website and post on the Ed board if you have any questions