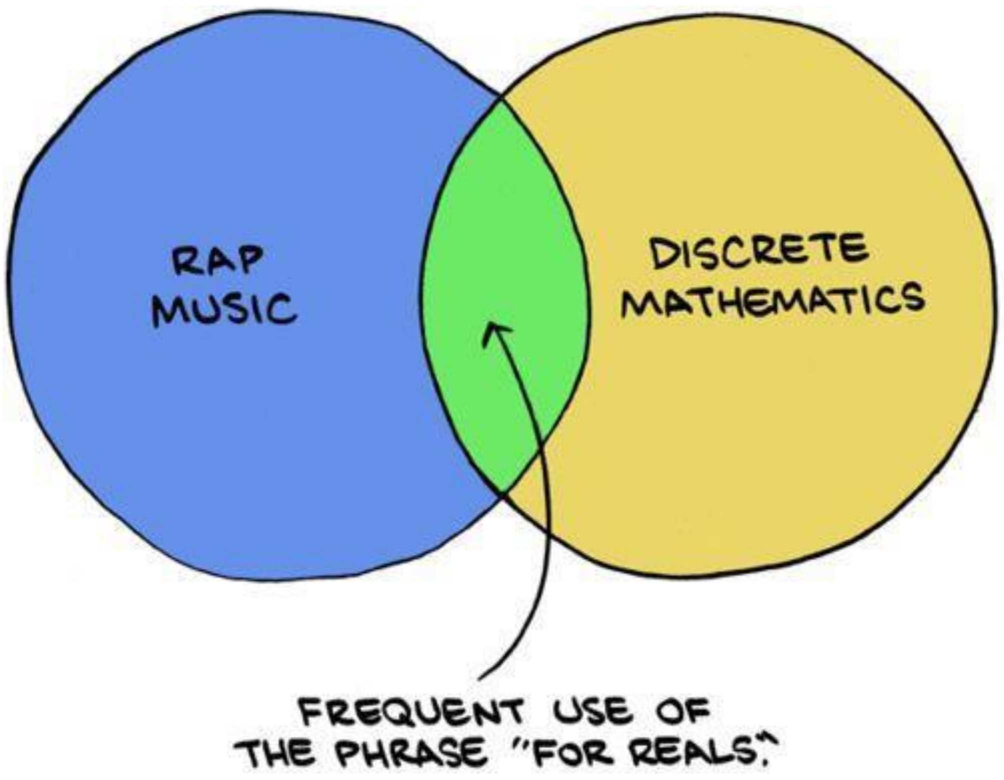


Set Theory Continued

CSE 311 Summer 25
Lecture 11

Announcements

- HW4 is released!
  - Make sure to start this one early. You'll be writing more complex set and number theory proofs.

- HW2 Feedback is out

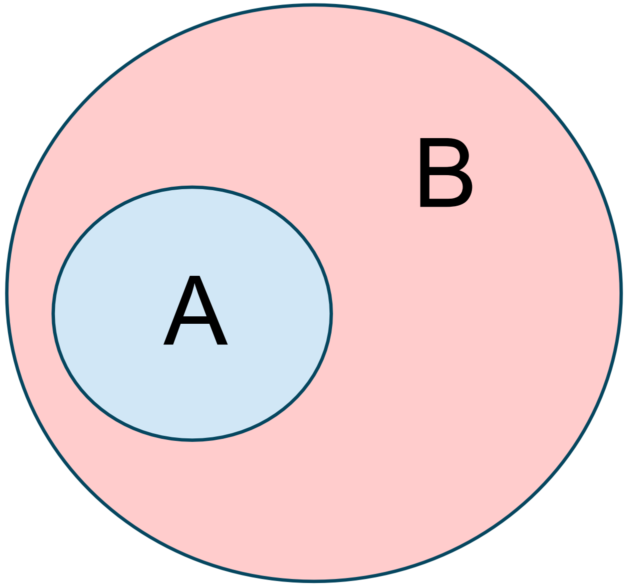
- HW2 Resubmission is due tonight!

Review

Sets

- A set is an unordered collection of distinct objects, called elements.
- The cardinality of a set is the number of elements in the set, denote $|S|$ for a set S.
- Sets A and B are equal if they have the same elements.
- Set A is a subset of B if every element of A is also in B.

A is a subset of B



Common Sets

$\mathbb{R}$ is the set of Real Numbers. E.g. $1, -17, \pi, \sqrt{2}$

$\mathbb{Z}$ is the set of Integers. $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$

$\mathbb{N}$ is the set of Natural Numbers. $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$

$\mathbb{Q}$ is the set of Rational Numbers (fractions) E.g. $\frac{1}{2}, -\frac{11}{3}, 17$

$\emptyset = \{\}$ is the Empty Set. $\emptyset$ has no elements

Sets Builder Notation

Another way to describe a set is using set-builder notation.

$S = \{x : \mathrm{P}(x)\}$ means S is the set of all x for which $\mathrm{P}(x)$ is true.

For example:

- $\{x \in \mathbb{Z} : x > 0\}$ is the set of all positive integers.
- $\{x \in \mathbb{N} : x \equiv_3 2\}$ is the set $\{2, 5, 8, 11, 14, \dots\}$.
- $\{\frac{a}{b} : a, b \in \mathbb{Z}, \mathrm{b} \neq 0\}$ is the set of rational numbers

Set Equality

Sets A and B are equal iff they have the same elements.

In predicate logic, $A = B$ is defined as:

$$\forall x\ (x \in A \leftrightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

Which sets are equal?

$$C = D = E$$

Subset

Set A is a subset of B if every element of A is also in B.

In predicate logic, $A \subseteq B$ is defined as:

$$\forall x(x \in A \rightarrow x \in B)$$

A = {1, 2, 3}
B = {3, 4, 5}
C = {3, 4}
D = {4, 3, 3}
E = {3, 4, 3}
F = {4, {3}}

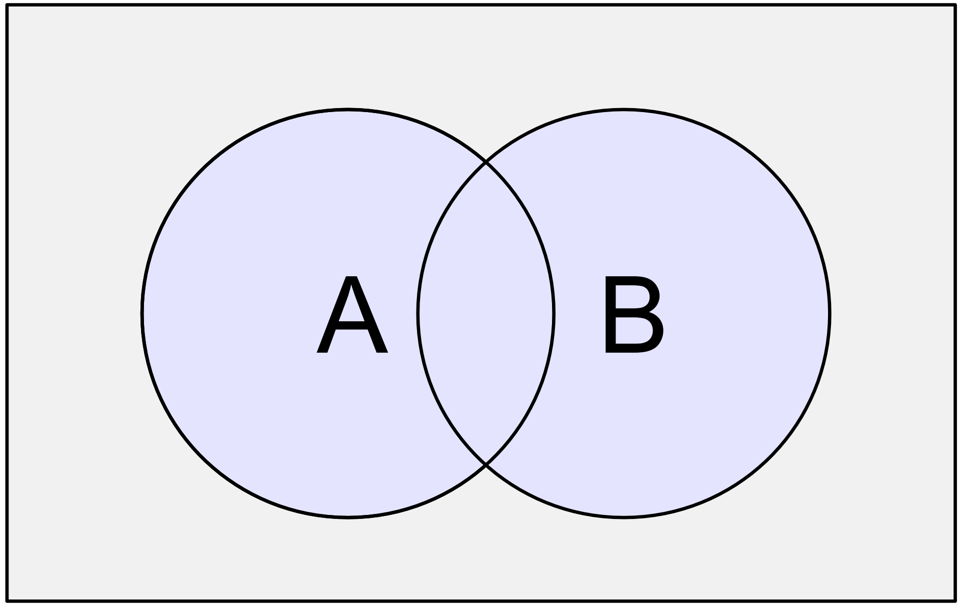
Which sets are subsets?

$C \subseteq B$, $D \subseteq E$, $E \subseteq D$, etc.

Set Operations

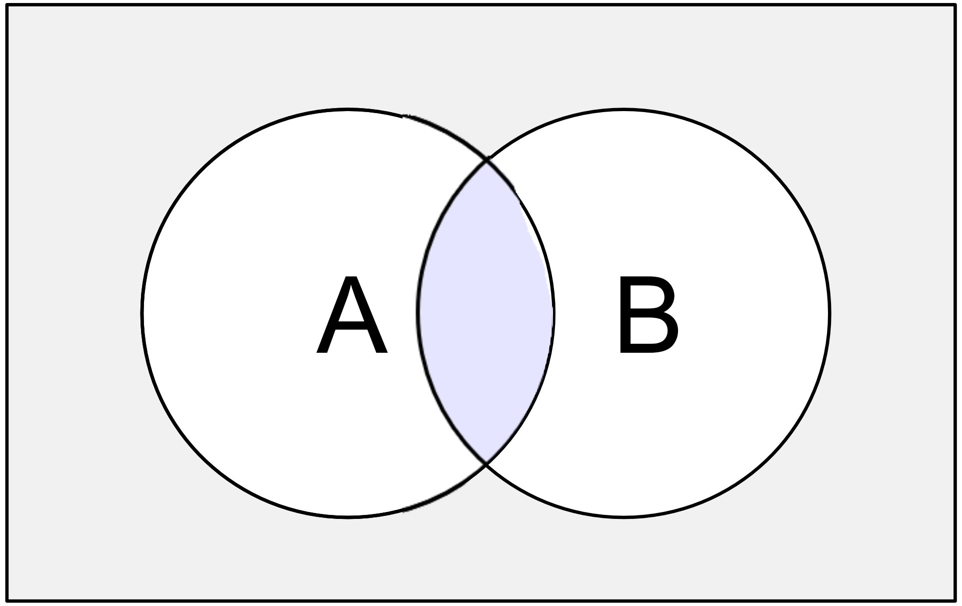
Union: $A \cup B$

$A \cup B = \{x : x \in A \vee x \in B\}$



Intersection: $A \cap B$

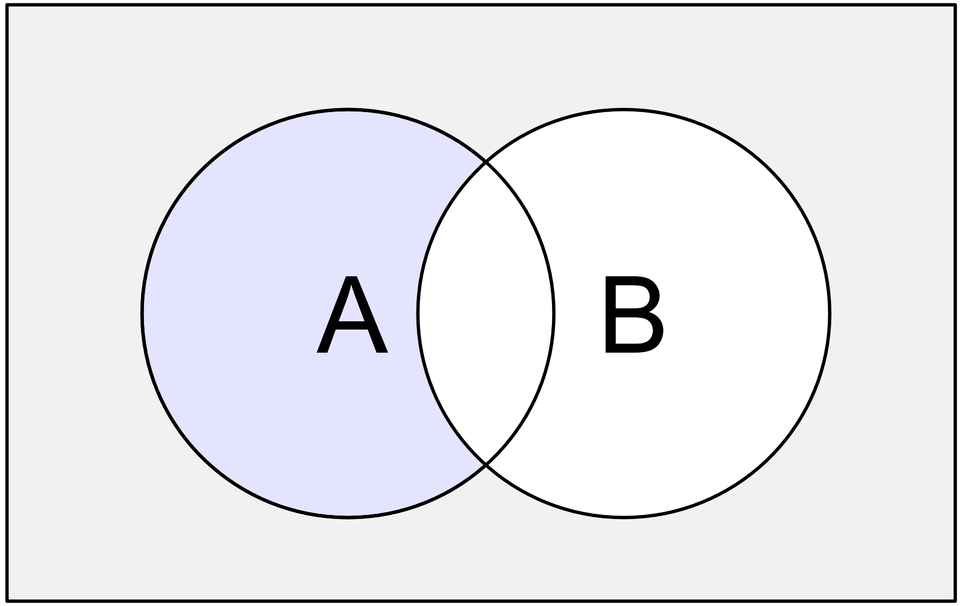
$A \cap B = \{x : x \in A \wedge x \in B\}$



Set Operations

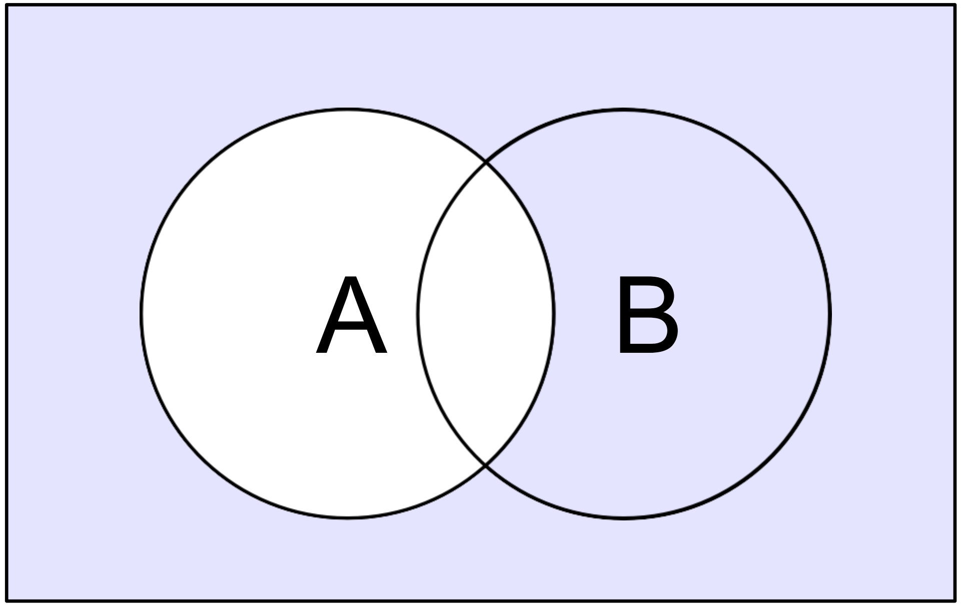
Set Difference: $A \setminus B$

$A \setminus B = \{x : x \in A \wedge x \notin B\}$



Set Complement: $\bar{A} = A^c$
(with respect to the universe $\mathcal{U}$)

$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$



Set Operations

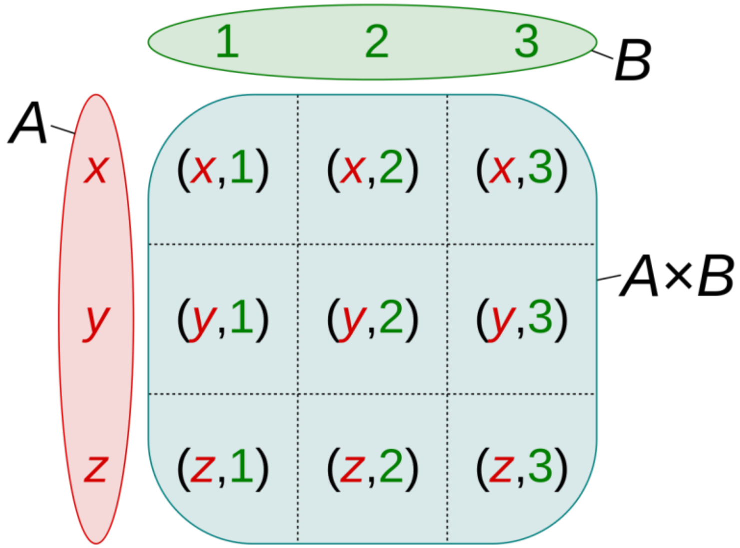
Powerset: $\mathcal{P}(A)$

$$\mathcal{P}(A) = \{B : B \subseteq A\}$$

$$\mathcal{P}(\{1,2,3\}) = \left\{ \begin{array}{c} \{\ \} \\ \{1\} \quad \{2\} \quad \{3\} \\ \{1,2\} \ \{1,3\} \ \{2,3\} \\ \{1,2,3\} \end{array} \right\}$$

Cartesian Product: $A \times B$

$$A \times B = \{(a, b) : a \in A, b \in B\}$$



Warm Up: Two Claims

Two Claims

Determine if the following claims are true or false.

Claim 1: For all sets A, B, C, if $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$.

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Two Claims

Determine if the following claims are true or false.

Claim 1: For all sets A, B, C, if $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$.

False.

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

True.

Claim 1

Claim 1: For all sets A, B, C, if $A \subseteq (B \cup C)$ then $A \subseteq B$ or $A \subseteq C$.

We disprove this claim with a counterexample.

Let $A = \{1,2\}$, let $B = \{1\}$ and $C = \{2\}$. Then $A \subseteq (B \cup C)$, but $A \nsubseteq B$ and $A \nsubseteq C$.

Claim 2

Definition
$A \subseteq B \equiv \forall x(x \in A \to x \in B)$

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Proof Strategy

- Let A, B, C be arbitrary sets.
- Let $x \in A \cap B \cap C$ be arbitrary.
- Prove that $x \in A \cup B$.

Claim 2

Definition
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Proof

Let sets A, B, C be arbitrary. Let $x \in A \cap B \cap C$ be an arbitrary element.

So $x \in A \cup B$. Since x was arbitrary, $A \cap B \cap C \subseteq A \cup B$.

Claim 2

Definition
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Proof

Let sets A, B, C be arbitrary. Let $x \in A \cap B \cap C$ be an arbitrary element.

[Unroll definitions]

[Use facts to move towards goal]

[Reroll definitions]

So $x \in A \cup B$. Since x was arbitrary, $A \cap B \cap C \subseteq A \cup B$.

Claim 2

Definition
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Proof

Let sets A, B, C be arbitrary. Let $x \in A \cap B \cap C$ be an arbitrary element.

Then by definition of intersection, $x \in A$ and $x \in B$ and $x \in C$.

[Use facts to move towards goal]

[Reroll definitions]

So $x \in A \cup B$. Since x was arbitrary, $A \cap B \cap C \subseteq A \cup B$.

Claim 2

Definition
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Proof

Let sets A, B, C be arbitrary. Let $x \in A \cap B \cap C$ be an arbitrary element.

Then by definition of intersection, $x \in A$ and $x \in B$ and $x \in C$.

Then certainly $x \in A$. So $x \in A$ or $x \in B$.

[Reroll definitions]

So $x \in A \cup B$. Since x was arbitrary, $A \cap B \cap C \subseteq A \cup B$.

Claim 2

Definition
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$

Claim 2: For all sets A, B, C it holds that $A \cap B \cap C \subseteq A \cup B$.

Proof

Let sets A, B, C be arbitrary. Let $x \in A \cap B \cap C$ be an arbitrary element.

Then by definition of intersection, $x \in A$ and $x \in B$ and $x \in C$.

Then certainly $x \in A$. So $x \in A$ or $x \in B$.

By definition of union, $x \in A \cup B$.

Since x was arbitrary, $A \cap B \cap C \subseteq A \cup B$.

Subset Proofs

Claim 1

Definitions

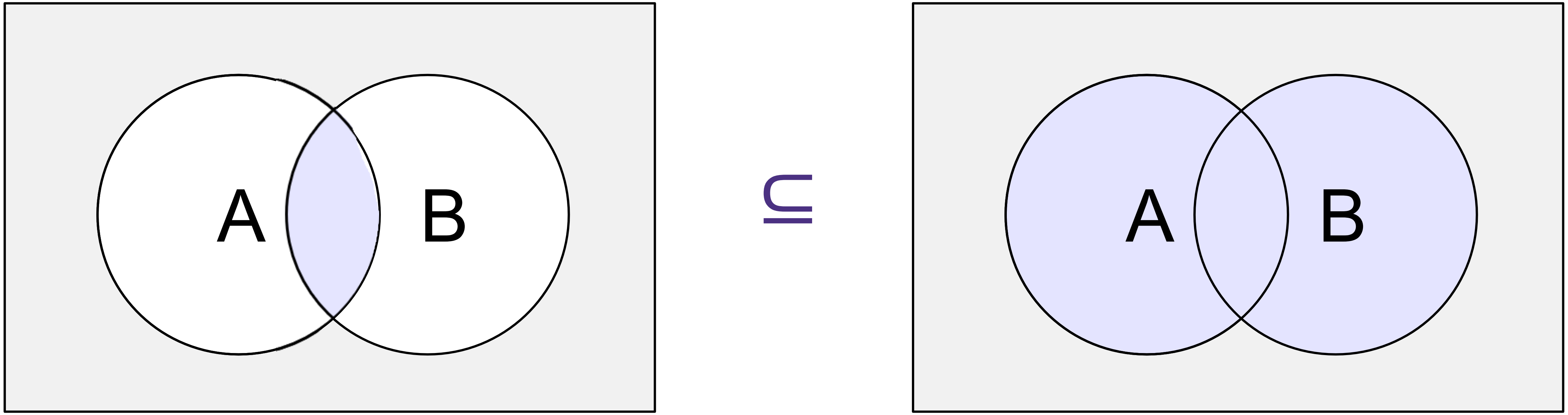
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$

$A \cup B = \{x : x \in A \vee x \in B\}$

$A \cap B = \{x : x \in A \wedge x \in B\}$

Claim 1: For all sets A, B, we have $A \cap B \subseteq A \cup B$.

Intuition (Venn Diagram)



Claim 1

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$

Claim 1: For all sets A, B, we have $A \cap B \subseteq A \cup B$.

Claim 1

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$

Claim 1: For all sets A, B, we have $A \cap B \subseteq A \cup B$.

Proof:

Let A, B be arbitrary sets. Let $x \in A \cap B$ be arbitrary.

[Unroll definitions]

[Apply facts to move towards our goal]

[Reroll definitions]

Since x was arbitrary, $A \cap B \subseteq A \cup B$.
Since A, B were arbitrary sets, the claim holds for all sets A, B.

Claim 1

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$

Claim 1: For all sets A, B, we have $A \cap B \subseteq A \cup B$.

Proof:

Let A, B be arbitrary sets. Let $x \in A \cap B$ be arbitrary.

Then by definition of intersection, $x \in A$ and $x \in B$.

Since x was arbitrary, $A \cap B \subseteq A \cup B$.
Since A, B were arbitrary sets, the claim holds for all sets A, B.

Claim 1

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$

Claim 1: For all sets A, B, we have $A \cap B \subseteq A \cup B$.

Proof:

Let A, B be arbitrary sets. Let $x \in A \cap B$ be arbitrary.

Then by definition of intersection, $x \in A$ and $x \in B$.

Since x was arbitrary, $A \cap B \subseteq A \cup B$.
Since A, B were arbitrary sets, the claim holds for all sets A, B.

Claim 1

Definitions
$A \subseteq B \equiv \forall x (x \in A \rightarrow x \in B)$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$

Claim 1: For all sets A, B, we have $A \cap B \subseteq A \cup B$.

Proof:

Let A, B be arbitrary sets. Let $x \in A \cap B$ be arbitrary.

Then by definition of intersection, $x \in A$ and $x \in B$.

Thus by definition of union, $x \in A \cup B$.

Since x was arbitrary, $A \cap B \subseteq A \cup B$.
Since A, B were arbitrary sets, the claim holds for all sets A, B.

Claim 1

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$

Claim 1: For all sets A, B, we have $A \cap B \subseteq A \cup B$.

Proof:

Let A, B be arbitrary sets. Let $x \in A \cap B$ be arbitrary.

Then by definition of intersection, $x \in A$ and $x \in B$.

So certainly $x \in A$ or $x \in B$.

Thus by definition of union, $x \in A \cup B$.

Since x was arbitrary, $A \cap B \subseteq A \cup B$.
Since A, B were arbitrary sets, the claim holds for all sets A, B.

Proving Subsets

To prove that $X \subseteq Y$, we let $x \in X$ be arbitrary and prove that $x \in Y$.

Claim 2

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}\,(A) \subseteq \mathcal{P}(B)$.

<u>Intuition (Example)</u>

$A = \{1,2\} \quad \mathcal{B} = \{1,2,3\}$

$\mathcal{P}(A) = \big\{\emptyset, \{1\}, \{2\}, \{1,2\}\big\}$

$\mathcal{P}(B) = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}\big\}$

Claim 2

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$\mathcal{P}(A) = \{B : B \subseteq A\}$

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof

Let A, B be arbitrary sets. Suppose $A \subseteq B$. Let $X \in \mathcal{P}(A)$ be arbitrary.

Thus, $X \in \mathcal{P}(B)$. Since X was arbitrary, we have shown that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Claim 2

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$\mathcal{P}(A) = \{B : B \subseteq A\}$

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof

Let A, B be arbitrary sets. Suppose $A \subseteq B$. Let $X \in \mathcal{P}(A)$ be arbitrary.

Then by definition of powerset, $X \subseteq A$.

Thus, $X \in \mathcal{P}(B)$. Since X was arbitrary, we have shown that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Claim 2

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$\mathcal{P}(A) = \{B : B \subseteq A\}$

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof

Let A, B be arbitrary sets. Suppose $A \subseteq B$. Let $X \in \mathcal{P}(A)$ be arbitrary.

Then by definition of powerset, $X \subseteq A$.

Starting Point: $X \subseteq A$

Goal: Show $X \in \mathcal{P}(B)$. i.e. show that $X \subseteq B$

Available Facts: $X \subseteq A$ and $A \subseteq B$

Thus, $X \in \mathcal{P}(B)$. Since X was arbitrary, we have shown that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Claim 2

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$\mathcal{P}(A) = \{B : B \subseteq A\}$

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof

Let A, B be arbitrary sets. Suppose $A \subseteq B$. Let $X \in \mathcal{P}(A)$ be arbitrary.

Then by definition of powerset, $X \subseteq A$.

Let $x \in X$ be arbitrary. Since $x \in X$ and $X \subseteq A$, then $x \in A$.

Available Facts: $X \subseteq A$ and $A \subseteq B$

Goal: Show $X \in \mathcal{P}(B)$. . i.e. show that $X \subseteq B$

Thus, $X \in \mathcal{P}(B)$. Since X was arbitrary, we have shown that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Claim 2

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$\mathcal{P}(A) = \{B : B \subseteq A\}$

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof

Let A, B be arbitrary sets. Suppose $A \subseteq B$. Let $X \in \mathcal{P}(A)$ be arbitrary.

Then by definition of powerset, $X \subseteq A$.

Let $x \in X$ be arbitrary. Since $x \in X$ and $X \subseteq A$, then $x \in A$.

Since $x \in A$ and $A \subseteq B$, then $x \in B$. Since x was arbitrary, $X \subseteq B$.

Available Facts: $X \subseteq A$ and $A \subseteq B$

Goal: Show $X \in \mathcal{P}(B)$. i.e. show that $X \subseteq B$

Thus, $X \in \mathcal{P}(B)$. Since X was arbitrary, we have shown that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Claim 2

Definitions
$A \subseteq B \equiv \forall x(x \in A \rightarrow x \in B)$
$\mathcal{P}(A) = \{B : B \subseteq A\}$

Claim 2: For all sets A, B if $A \subseteq B$ then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

Proof

Let A, B be arbitrary sets. Suppose $A \subseteq B$. Let $X \in \mathcal{P}(A)$ be arbitrary.

Then by definition of powerset, $X \subseteq A$.

Let $x \in X$ be arbitrary. Since $x \in X$ and $X \subseteq A$, then $x \in A$.

Since $x \in A$ and $A \subseteq B$, then $x \in B$. Since x was arbitrary, $X \subseteq B$.

Now since we have that $X \subseteq B$, by definition of powerset, $X \in \mathcal{P}(B)$

Since X was arbitrary, we have shown that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$

Symbols and Sets

Note that when writing set proofs, we follow various conventions.

We DO tend use symbols like $\in$, $\subseteq$, $\cup$, $\cap$, $\times$ etc. (instead of writing out the symbol in English).

E.g. "Let $x \in A$ be arbitrary"

We DO NOT tend to use symbols like $\wedge$, $\vee$, $\neg$ (but rather write them out in English).**

E.g. "Then $x \in A$ and $x \in B$"

**There are exceptions to this if logical symbols provide clarity when applying equivalence rules (Absorption, DeMorgan's Laws, etc.). The proof of Claim 3 will be an example of that.

Exercises

Which of the following statements are true?

If $x \in A \cap B$ then $(x \in A) \cap (x \in B)$.

If $x \in C \backslash D$ then $x \in C \wedge \neg(x \in D)$.

If $X \subseteq \mathcal{P}(A)$ then $X \in A$.

If $(a, b) \in E \times F$ then $a \in E$ and $b \in F$.

Exercises

Which of the following statements are true?

If $x \in A \cap B$ then $(x \in A) \cap (x \in B)$.

False. This should be $(x \in A) \wedge (x \in B)$.

If $x \in C \backslash D$ then $x \in C \wedge \neg(x \in D)$.

True.

If $X \subseteq \mathcal{P}(A)$ then $X \in A$.

False. This should be if $X \in \mathcal{P}(A)$ then $X \subseteq A$.

If $(a, b) \in E \times F$ then $a \in E$ and $b \in F$.

True.

Set Equality Proofs

Claim 3 (DeMorgan's Law for Sets)

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Definitions

$A = B \equiv A \subseteq B \land B \subseteq A$

$A \cup B = \{x : x \in A \lor x \in B\}$

$A \cap B = \{x : x \in A \land x \in B\}$

$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

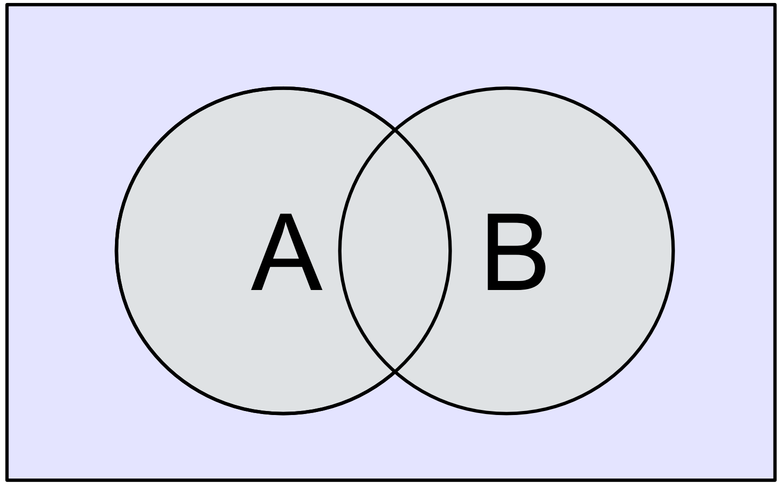
Claim 3 (DeMorgan's Law for Sets)

Definitions
$A = B \equiv A \subseteq B \wedge B \subseteq A$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$
$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

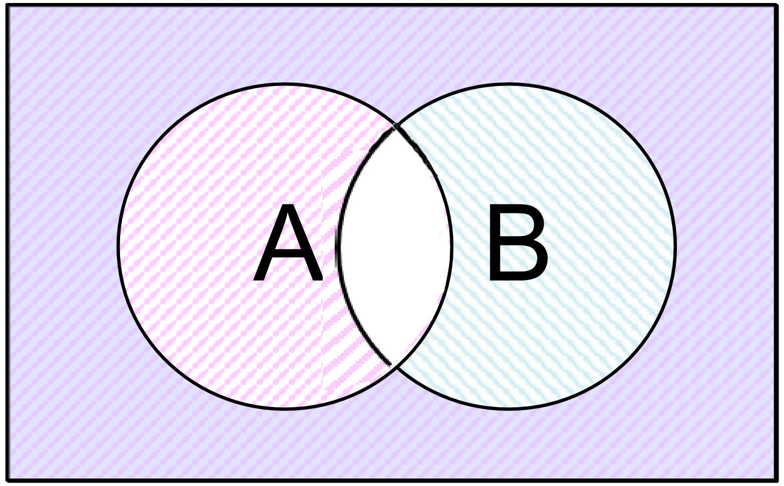
Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Intuition (Venn Diagram)

$\overline{A \cup B}$



$\bar{A} \cap \bar{B}$



Claim 3 (DeMorgan's Law for Sets)

Definitions
$A = B \equiv A \subseteq B \land B \subseteq A$
$A \cup B = \{x : x \in A \lor x \in B\}$
$A \cap B = \{x : x \in A \land x \in B\}$
$\bar{A} = \{x \in \mathcal{U}: x \notin A\}$

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof Strategy

- Let A, B be arbitrary sets.
- Prove that $\overline{A \cup B} \subseteq \bar{A} \cup \bar{B}$
- Prove that $\bar{A} \cup \bar{B} \subseteq \overline{A \cup B}$

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof (Method 1)

Let A, B be arbitrary sets.

Definitions

$A = B \equiv A \subseteq B \land B \subseteq A$

$A \cup B = \{x : x \in A \lor x \in B\}$

$A \cap B = \{x : x \in A \land x \in B\}$

$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Definitions
$A = B \equiv A \subseteq B \land B \subseteq A$
$A \cup B = \{x : x \in A \lor x \in B\}$
$A \cap B = \{x : x \in A \land x \in B\}$
$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

Proof (Method 1)

Let A, B be arbitrary sets.

$\Rightarrow$ First we show that $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Let $x \in \overline{A \cup B}$ be arbitrary.

By definition of intersection, $x \in \bar{A} \cap \bar{B}$. Since x was arbitrary, $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Definitions
$A = B \equiv A \subseteq B \wedge B \subseteq A$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$
$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

Proof (Method 1)

Let A, B be arbitrary sets.

⇒ First we show that $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Let $x \in \overline{A \cup B}$ be arbitrary.

By definition of complement, we have that $\neg(x \in A \cup B)$.

By definition of intersection, $x \in \bar{A} \cap \bar{B}$. Since x was arbitrary, $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Definitions
$A = B \equiv A \subseteq B \land B \subseteq A$
$A \cup B = \{x : x \in A \lor x \in B\}$
$A \cap B = \{x : x \in A \land x \in B\}$
$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

Proof (Method 1)

Let A, B be arbitrary sets.

⇒ First we show that $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Let $x \in \overline{A \cup B}$ be arbitrary.

By definition of complement, we have that $\neg(x \in A \cup B)$.

Then by definition of union, $\neg(x \in A \lor x \in B)$.

By definition of intersection, $x \in \bar{A} \cap \bar{B}$. Since x was arbitrary, $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Claim 3

Definitions
$A = B \equiv A \subseteq B \land B \subseteq A$
$A \cup B = \{x : x \in A \lor x \in B\}$
$A \cap B = \{x : x \in A \land x \in B\}$
$\bar{A} = \{x \in \mathcal{U}: x \notin A\}$

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof (Method 1)

Let A, B be arbitrary sets.

⇒ First we show that $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Let $x \in \overline{A \cup B}$ be arbitrary.

By definition of complement, we have that $\neg(x \in A \cup B)$.

Then by definition of union, $\neg(x \in A \lor x \in B)$.

So by DeMorgan's Law, $x \notin A \land x \notin B$.

By definition of intersection, $x \in \bar{A} \cap \bar{B}$. Since x was arbitrary, $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Claim 3

Definitions
$A = B \equiv A \subseteq B \wedge B \subseteq A$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$
$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof (Method 1)

Let A, B be arbitrary sets.

⇒ First we show that $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Let $x \in \overline{A \cup B}$ be arbitrary.

By definition of complement, we have that $\neg(x \in A \cup B)$.

Then by definition of union, $\neg(x \in A \vee x \in B)$.

So by DeMorgan's Law, $x \notin A \wedge x \notin B$.

Then by definition of complement, $x \in \bar{A}$ and $x \in \bar{B}$

By definition of intersection, $x \in \bar{A} \cap \bar{B}$. Since x was arbitrary, $\overline{A \cup B} \subseteq \bar{A} \cap \bar{B}$.

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof (Method 1)

Let A, B be arbitrary sets.

$\Leftarrow$ Now we show $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

Definitions

$A = B \equiv A \subseteq B \wedge B \subseteq A$

$A \cup B = \{x : x \in A \vee x \in B\}$

$A \cap B = \{x : x \in A \wedge x \in B\}$

$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Definitions

$A = B \equiv A \subseteq B \land B \subseteq A$

$A \cup B = \{x : x \in A \lor x \in B\}$

$A \cap B = \{x : x \in A \land x \in B\}$

$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

Proof (Method 1)

Let A, B be arbitrary sets.

$\Leftarrow$ Now we show $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.

Since x was arbitrary, $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

<u>Proof (Method 1)</u>

Let A, B be arbitrary sets.

$\Leftarrow$ Now we show $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.

By definition of intersection, $x \in \bar{A} \wedge x \in \bar{B}$.

Since x was arbitrary, $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

<u>Definitions</u>

$A = B \equiv A \subseteq B \wedge B \subseteq A$

$A \cup B = \{x : x \in A \vee x \in B\}$

$A \cap B = \{x : x \in A \wedge x \in B\}$

$\bar{A} = \{x \in \mathcal{U}: x \notin A\}$

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Definitions
$A = B \equiv A \subseteq B \land B \subseteq A$
$A \cup B = \{x : x \in A \lor x \in B\}$
$A \cap B = \{x : x \in A \land x \in B\}$
$\bar{A} = \{x \in \mathcal{U}: x \notin A\}$

Proof (Method 1)

Let A, B be arbitrary sets.

$\Leftarrow$ Now we show $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.

By definition of intersection, $x \in \bar{A} \land x \in \bar{B}$.

By definition of complement, $x \notin A$ and x $\notin B$.

Since x was arbitrary, $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Definitions
$A = B \equiv A \subseteq B \land B \subseteq A$
$A \cup B = \{x : x \in A \lor x \in B\}$
$A \cap B = \{x : x \in A \land x \in B\}$
$\bar{A} = \{x \in \mathcal{U}: x \notin A\}$

Proof (Method 1)

Let A, B be arbitrary sets.

$\Leftarrow$ Now we show $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.

By definition of intersection, $x \in \bar{A} \land x \in \bar{B}$.

By definition of complement, $x \notin A$ and x $\notin B$.

Applying DeMorgan's Law, we get $\neg(x \in A \lor x \in B)$.

Since x was arbitrary, $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

Claim 3

Definitions
$A = B \equiv A \subseteq B \wedge B \subseteq A$
$A \cup B = \{x : x \in A \vee x \in B\}$
$A \cap B = \{x : x \in A \wedge x \in B\}$
$\bar{A} = \{x \in \mathcal{U} : x \notin A\}$

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof (Method 1)

Let A, B be arbitrary sets.

$\Leftarrow$ Now we show $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.

By definition of intersection, $x \in \bar{A} \wedge x \in \bar{B}$.

By definition of complement, $x \notin A$ and $\mathrm{x} \notin B$.

Applying DeMorgan's Law, we get $\neg(x \in A \vee x \in B)$.

By the definition of union, we get: $\neg(x \in A \cup B)$, so $x \notin A \cup B$.

Since x was arbitrary, $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Definitions
$A = B \equiv A \subseteq B \land B \subseteq A$
$A \cup B = \{x : x \in A \lor x \in B\}$
$A \cap B = \{x : x \in A \land x \in B\}$
$\bar{A} = \{x \in \mathcal{U}: x \notin A\}$

Proof (Method 1)

Let A, B be arbitrary sets.

$\Leftarrow$ Now we show $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$

Let x be an arbitrary element of $\bar{A} \cap \bar{B}$.

By definition of intersection, $x \in \bar{A} \land x \in \bar{B}$.

By definition of complement, $x \notin A$ and $\mathrm{x} \notin B$.

Applying DeMorgan's Law, we get $\neg(x \in A \lor x \in B)$.

By the definition of union, we get: $\neg(x \in A \cup B)$, so $x \notin A \cup B$.

From the definition of complement, we get $\mathrm{x} \in \overline{A \cup B}$, as required.

Since x was arbitrary, $\bar{A} \cap \bar{B} \subseteq \overline{A \cup B}$.

Claim 3

Claim 3: For all sets A, B, $\overline{A \cup B} = \bar{A} \cap \bar{B}$

Proof (Method 1)

Since the subset relation holds in both directions, we have $\bar{A} \cap \bar{B} = \overline{A \cup B}$.

Definitions

$A = B \equiv A \subseteq B \land B \subseteq A$

$A \cup B = \{x : x \in A \lor x \in B\}$

$A \cap B = \{x : x \in A \land x \in B\}$

$\bar{A} = \{x \in \mathcal{U}: x \notin A\}$

Proving Set Equality: Method 1

One way to prove that $X = Y$ is to show that $X \subseteq Y$ by one subset proof, and $Y \subseteq X$ by another subset proof.

Proving Set Equality: Method 2

Another way to prove that $X = Y$ is to show that for arbitrary element x, $x \in X \equiv \cdots \equiv x \in Y$ by a chain of equivalences.

Claim 4

Definitions

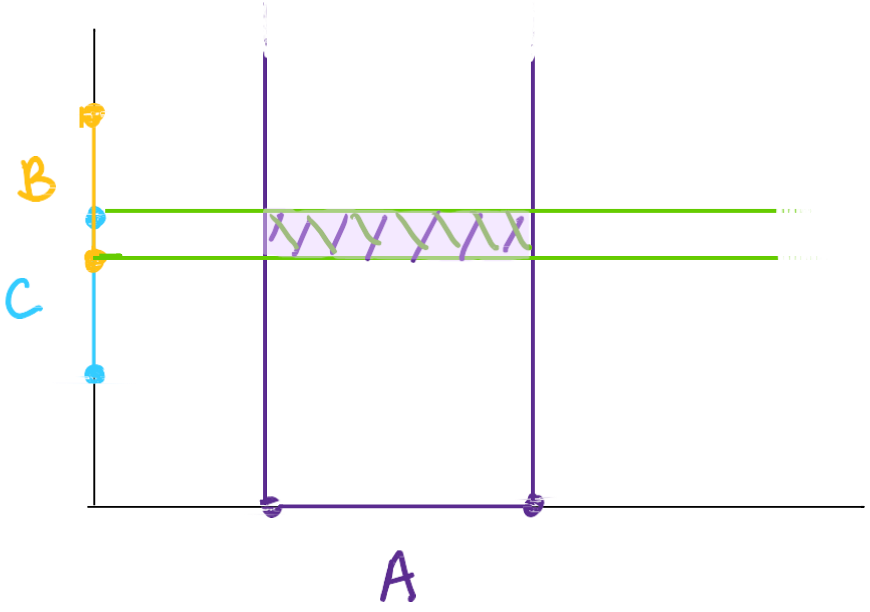
$A \times B = \{(a, b) : a \in A, b \in B\}$

$A \cup B = \{x : x \in A \vee x \in B\}$

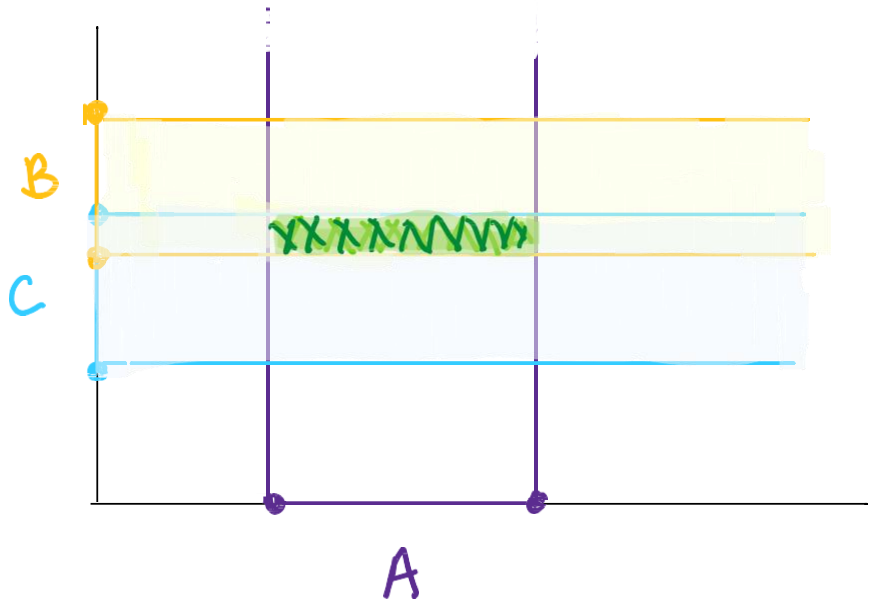
Claim 4: For all sets A, B, C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Intuition (Diagram)

$A \times (B \cap C)$



$(A \times B) \cap (A \times C)$



Claim 4

Definitions
$A \times B = \{(a,b): a \in A, b \in B\}$
$A \cup B = \{x : x \in A \vee x \in B\}$

Claim 4: For all sets A, B, C, $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof (Method 2)

Let A, B, C be arbitrary sets. Let (y, z) be an arbitrary pair. Then:

$(y,z) \in A \times (B \cap C)$	$\equiv (y \in A) \wedge (z \in B \cap C)$	Def. of $\times$
	$\equiv (y \in A) \wedge \big((z \in B) \wedge (z \in C)\big)$	Def. of $\cap$
	$\equiv (y \in A) \wedge (y \in A) \wedge (z \in B) \wedge (z \in C)$	Idempotency
	$\equiv (y \in A) \wedge (z \in B) \wedge (y \in A) \wedge (z \in C)$	Assoc. and Commutativity
	$\equiv (y,z) \in A \times B \wedge (y,z) \in A \times C$	Def of $\times$
	$\equiv (y,z) \in (A \times B) \cap (A \times C)$	Def of $\cap$

Since (y, z) was arbitrary, we have shown that $A \times (B \cap C) = (A \times B) \cap (A \times C)$. Since A, B, C were arbitrary sets, the claim holds.

Proof by Contradiction

Proof by Contradiction

Proof by contradiction is a strategy for proving statements of any form.

The strategy to prove p is to assume $\neg p$ and derive $\mathbf{False}$. (i.e. (¬claim) → F)

- E.g. the strategy to prove $p \rightarrow q$ is to assume $p \wedge \neg q$ and derive $\mathbf{False}$.
- E.g. the strategy to prove $p \vee q$ is to assume $\neg p \wedge \neg q$ and derive $\mathbf{False}$.

Why does that work?

Observe from our logical equivalences:

$\neg p \rightarrow \mathrm{F} \equiv \neg\neg p \vee \mathrm{F}$	Law of Implication
$\equiv p \vee \mathrm{F}$	Double Negation
$\equiv p$	Identity

Proof by Contradiction Skeleton

Suppose for the sake of contradiction $\neg p$.

…

Then some statement s must hold.

…

And some statement $\neg s$ must hold.

But s and $\neg s$ is a contradiction. So p must be true.

What’s the difference?

What’s the difference between proof by contrapositive and proof by contradiction?

Show $p \to q$	**Proof by contradiction**	**Proof by contrapositive**
Starting Point	$\neg(p \to q) \equiv (p \wedge \neg q)$	$\neg q$
Target	Something false	$\neg p$

Show p	**Proof by contradiction**	**Proof by contrapositive**
Starting Point	$\neg p$	---
Target	Something false	---

Proof by Contradiction: Remarks

- Unlike other proof techniques, we don't know *where* we're going. We're trying to find any contradiction. That can make it harder.
- Contradiction is a sledge-hammer.
  It can be used to prove many things. But it makes a mess.
- Use contradiction as a last-resort.

Contradiction is a sledge-hammer

Claim: No integer is even and odd.

Suppose for the sake of contradiction that there exists an integer x that is both even and odd.

This is a contradiction!

Thus no integer can be even and odd.

Claim: No integer is even and odd.

Suppose for the sake of contradiction that there exists an integer x that is both even and odd.

[Show there exists some contradiction if we make this assumption]

This is a contradiction!

Thus no integer can be even and odd.

Claim: No integer is even and odd.

Suppose for the sake of contradiction that there exists an integer x that is both even and odd.

Then $x = 2a$ for some integer a, and $x = 2b + 1$ for some integer b.

This is a contradiction!

Thus no integer can be even and odd.

Claim: No integer is even and odd.

Suppose for the sake of contradiction that there exists an integer x that is both even and odd.

Then $x = 2a$ for some integer a, and $x = 2b + 1$ for some integer b.

Then:

$$2a = 2b + 1$$
$$2a - 2b = 1$$
$$a - b = \frac{1}{2}$$

This is a contradiction!

Thus no integer can be even and odd.

Claim: No integer is even and odd.

Suppose for the sake of contradiction that there exists an integer x that is both even and odd.

Then $x = 2a$ for some integer a, and $x = 2b + 1$ for some integer b.

Then:

$$2a = 2b + 1$$
$$2a - 2b = 1$$
$$a - b = \frac{1}{2}$$

Since a, b are integers, $a - b$ is an integer. But $\frac{1}{2}$ is not an integer, so $a - b$ cannot equal $\frac{1}{2}$.

This is a contradiction!

Thus no integer can be even and odd.

Proof By Contradiction

How in the world did we know how to do that?

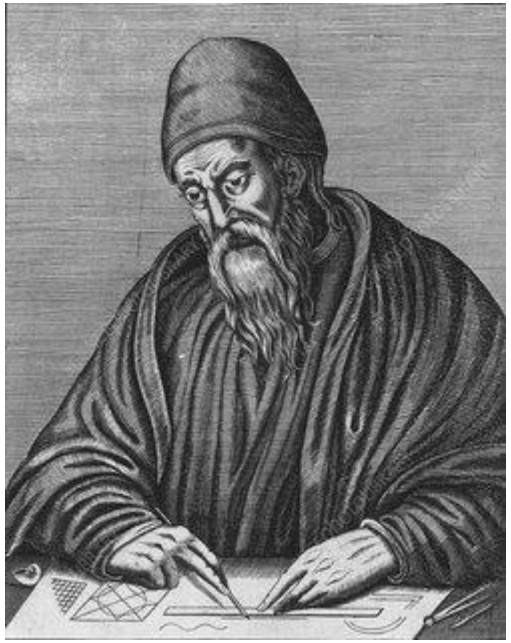
In real life…lots of attempts that didn't work.

Be very careful with proof by contradiction – without a clear target, you can easily end up in a loop of trying random things and getting nowhere.

Another Proof by Contradiction

Euclid’s Theorem: There are infinitely many prime numbers.

$2, 3, 5, 7, 11, 13, \ldots$



Euclid ~300 BC

Euclid’s Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \ldots, p_k$.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Euclid’s Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdots p_k + 1$.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Euclid's Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdots p_k + 1$.

<u>Case 1</u>: q is prime.

<u>Case 2</u>: q is composite.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Euclid’s Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdots p_k + 1$.

Case 1: q is prime. Then q is a prime that is larger than p_i for all $i \in \{1, \ldots, k\}$. But every prime was supposed to in the list $p_1, \ldots, p_k$. This is a contradiction.

Case 2: q is composite.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Euclid's Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdots p_k + 1$.

Case 1: q is prime. Then q is a prime that is larger than p_i for all $i \in \{1, \ldots, k\}$. But every prime was supposed to in the list $p_1, \ldots, p_k$. This is a contradiction.

Case 2: q is composite. Then q must have some prime factor. So there exists some i such that $p_i \mid q$.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Euclid's Theorem: There are infinitely many prime numbers.

Assume for the sake of contradiction that there are finitely many prime numbers. Call them $p_1, p_2, \ldots, p_k$.

Consider the number $q = p_1 \cdot p_2 \cdots p_k + 1$.

Case 1: q is prime. Then q is a prime that is larger than p_i for all $i \in \{1, \ldots, k\}$. But every prime was supposed to in the list $p_1, \ldots, p_k$. This is a contradiction.

Case 2: q is composite. Then q must have some prime factor. So there exists some i such that $p_i \mid q$. Then $q \,\%\, p_i = 0$. But

$$q \,\%\, p_i = \ (p_1 \cdot p_2 \cdots p_k + 1) \,\%\, p_i = 1$$

So we have $q \,\%\, p_i = 0$ and $q \,\%\, p_i = 1$. This is a contradiction.

In either case, we have a contradiction. Thus there must be infinitely many primes.

Extra Practice: Proof by Contradiction

Just the Skeleton

"For all integers x, if x^2 is even, then x is even."

Just the Skeleton

"For all integers x, if x^2 is even, then x is even."

Suppose for the sake of contradiction, there is an integer x, such that x^2 is even and x is odd.

…

[] is a contradiction, so for all integers x, if x^2 is even, then x is even.

Just the Skeleton

“There is not an integer k such that for all integers n, $k \geq n$.

Just the Skeleton

"There is not an integer k such that for all integers n, $k \geq n$.

Suppose, for the sake of contradiction, that there is an integer k such that for all integers n, $k \geq n$.

…

[] is a contradiction! So there is not an integer k such that for all integers $n, k \geq n$.

Todo

Tonight:

- HW2 Resubmissions due tonight!
- CC 11 is out and due Monday at noon
- Take a look at HW4 today or over the weekend!