

# More Proof Strategies

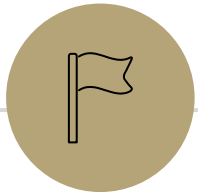
CSE 311 Summer 25  
Lecture 7

# Announcements

- HW2 is due tonight!
  - Stop by OH for support while finishing the assignment!
- HW3 is released!
  - Please start early- this HW is a significant step up from HW1 and HW2
- HW1 feedback is out!
  - Make sure to read *all* of your feedback!
  - HW1 Regrade requests are open for one week
  - Submit your regrade request(s) by Thursday EOD if you want your request to be responded to before the resubmission window closes

# Announcements

- Homework Resubmissions
  - For each homework, you may resubmit up to 2 problems for an improved grade
  - Three steps to resubmit your work
    - 1) Revise your work using the feedback on your original submission
    - 2) Submit your revised work using the "Homework 1 Resubmission" assignment on Gradescope
    - 3) Fill out the HW1 Resubmission form on the Ed board
  - Both steps 2 and 3 must be completed for your resubmission to be graded
  - Resubmissions are open from Wednesday at noon until 11:59 pm on Friday
  - Late days cannot be used on resubmissions (unless there are extenuating circumstances)



**Review**

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# Theorems and Proofs

**Theorem:** A statement that has been proven to be true.

**Proof:** A valid argument that establishes a statement to be true.

You'll also see

"claim" (the statement we're about to prove)

"lemma" (small theorem, used to prove a bigger theorem)

"corollary" (small theorem, proven using a bigger theorem)

# Integer

We need a basic starting point to be able to prove things.

Objects to work with.

An integer: is any real number with no fractional part.

Some **definitions** to analyze

## Even

**Even** ( $x$ ) := An integer,  $x$ , is even if and only if there is an integer  $k$  such that  $x = 2k$ .

## Odd

**Odd** ( $x$ ) := An integer,  $x$ , is odd if and only if there is an integer  $k$  such that  $x = 2k + 1$ .

# Direct Proof Template

Declare an arbitrary variable for each  $\forall$ .

Assume the left side of the implication.

Unroll the predicate definitions.

Manipulate towards the goal.

Reroll definitions into the right side of the implication.

Conclude that you have proved the claim.

Prove:  $\forall x(\text{Even}(x) \rightarrow \text{Even}(x^2))$

Let  $x$  be an arbitrary integer.

Suppose that  $x$  is even.

Then by definition of even, there exists some integer  $k$  such that  $x = 2k$ .

Squaring both sides, we see that:

$$x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$$

Because  $k$  is an integer, then  $2k^2$  is also an integer. So  $x^2$  is two times an integer.

So by definition of even,  $x^2$  is even.

Since  $x$  was an arbitrary integer, we can conclude that for all integers  $x$ , if  $x$  is even then  $x^2$  is even.

# Direct Proof Steps

These are the usual steps. We'll see different outlines in the future!!

- Introduction
  - Declare an arbitrary variable for each  $\forall$  quantifier
  - Assume the left side of the implication
- Core of the proof
  - Unroll the predicate definitions
  - Manipulate towards the goal (using creativity, algebra, etc.)
  - Reroll definitions into the right side of the implication
- Conclude that you have proved the claim

# Proof by Contrapositive

Proof by contrapositive is another strategy for proving statements of the form  $\forall x(P(x) \rightarrow Q(x))$ .

The strategy is to prove the contrapositive, i.e. prove  $\forall x (\neg Q(x) \rightarrow \neg P(x))$

Remember, an implication is equivalent to its contrapositive!

# Proof by Contrapositive

Definitions

$\text{Odd}(x) := \exists k(x = 2k + 1)$

Prove: For an integer  $x$ , if  $3x + 2$  is odd, then  $x$  is odd.

$$\forall x (\text{Odd}(3x + 2) \rightarrow \text{Odd}(x)) \equiv \forall x (\text{Even}(x) \rightarrow \text{Even}(3x + 2))$$

We prove by contrapositive.

Let  $x$  be an arbitrary integer. Suppose that  $x$  is even.

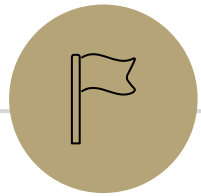
Then by definition of even,  $x = 2k$  for some integer  $k$ .

Consider  $3x + 2$ :

$$3x + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$$

Since  $k$  is an integer,  $3k + 1$  is an integer. So by definition of even,  $3x + 2$  is even.

Since  $x$  was arbitrary, we have shown that for all integers  $x$ , if  $x$  is even then  $3x + 2$  is even. Thus the contrapositive also holds: for all integers  $x$ , if  $3x + 2$  is odd, then  $x$  is odd.



**Warm Up**

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# Another Proof by Contrapositive

Definitions

$\text{Even}(x) := \exists k(x = 2k)$

Prove by Contrapositive: For an integer  $n$ , if  $n^3$  is even, then  $n$  is even.

# Another Proof by Contrapositive

Definitions

$\text{Even}(x) := \exists k(x = 2k)$

Prove by Contrapositive: For an integer  $n$ , if  $n^3$  is even, then  $n$  is even.

$$\forall n (\text{Even}(n^3) \rightarrow \text{Even}(n)) \equiv \forall n (\text{Odd}(n) \rightarrow \text{Odd}(n^3))$$

# Another Proof by Contrapositive

Definitions

$\text{Even}(x) := \exists k(x = 2k)$

Prove by Contrapositive: For an integer  $n$ , if  $n^3$  is even, then  $n$  is even.

$$\forall n (\text{Even}(n^3) \rightarrow \text{Even}(n)) \equiv \forall n (\text{Odd}(n) \rightarrow \text{Odd}(n^3))$$

We prove by contrapositive.

Let  $n$  be an arbitrary integer. Suppose that  $n$  is odd.

Thus by definition of odd,  $n^3$  is odd. Since  $n$  was arbitrary, we have shown that for all integers  $n$ , if  $n$  is odd then  $n^3$  is odd. Thus the contrapositive also holds: for all integers  $n$ , if  $n^3$  is even, then  $n$  is even.

# Another Proof by Contrapositive

Definitions

$\text{Even}(x) := \exists k(x = 2k)$

Prove by Contrapositive: For an integer  $n$ , if  $n^3$  is even, then  $n$  is even.

$$\forall n (\text{Even}(n^3) \rightarrow \text{Even}(n)) \equiv \forall n (\text{Odd}(n) \rightarrow \text{Odd}(n^3))$$

We prove by contrapositive.

Let  $n$  be an arbitrary integer. Suppose that  $n$  is odd.

[Unroll predicate definitions]

[Manipulate towards goal]

Thus by definition of odd,  $n^3$  is odd. Since  $n$  was arbitrary, we have shown that for all integers  $n$ , if  $n$  is odd then  $n^3$  is odd. Thus the contrapositive also holds: for all integers  $n$ , if  $n^3$  is even, then  $n$  is even.

# Another Proof by Contrapositive

Definitions

$\text{Even}(x) := \exists k(x = 2k)$

Prove by Contrapositive: For an integer  $n$ , if  $n^3$  is even, then  $n$  is even.

$$\forall n (\text{Even}(n^3) \rightarrow \text{Even}(n)) \equiv \forall n (\text{Odd}(n) \rightarrow \text{Odd}(n^3))$$

We prove by contrapositive.

Let  $n$  be an arbitrary integer. Suppose that  $n$  is odd.

Then by definition of odd,  $n = 2k + 1$  for some integer  $k$ .

[Manipulate towards goal]

Thus by definition of odd,  $n^3$  is odd. Since  $n$  was arbitrary, we have shown that for all integers  $n$ , if  $n$  is odd then  $n^3$  is odd. Thus the contrapositive also holds: for all integers  $n$ , if  $n^3$  is even, then  $n$  is even.

# Another Proof by Contrapositive

Definitions

$\text{Even}(x) := \exists k(x = 2k)$

Prove by Contrapositive: For an integer  $n$ , if  $n^3$  is even, then  $n$  is even.

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We prove by contrapositive.

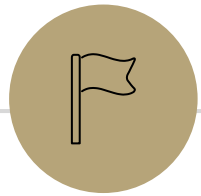
Let  $n$  be an arbitrary integer. Suppose that  $n$  is odd.

Then by definition of odd,  $n = 2k + 1$  for some integer  $k$ .

Consider  $n^3$ :

$$n^3 = (2k + 1)^3 = 8k^3 + 8k^2 + 4k + 1 = 2(4k^3 + 4k^2 + 2k) + 1$$

Since  $k$  is an integer,  $4k^3 + 4k^2 + 2k$  is an integer. Thus by definition of odd,  $n^3$  is odd. Since  $n$  was arbitrary, we have shown that for all integers  $n$ , if  $n$  is odd then  $n^3$  is odd. Thus the contrapositive also holds: for all integers  $n$ , if  $n^3$  is even, then  $n$  is even.



## **Proof Strategy: Biconditional**

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# Proof of a Biconditional

Recall that biconditionals are statements of the form:

$$\forall x (P(x) \leftrightarrow Q(x))$$

The strategy is to prove such statements is to prove an implication in both directions. I.e. prove  $\forall x (P(x) \rightarrow Q(x)) \wedge \forall x (Q(x) \rightarrow P(x))$ .

$$\begin{aligned} \forall x (P(x) \leftrightarrow Q(x)) &\equiv \forall x (P(x) \rightarrow Q(x) \wedge Q(x) \rightarrow P(x)) \\ &\equiv \forall x (P(x) \rightarrow Q(x)) \wedge \forall x (Q(x) \rightarrow P(x)) \end{aligned}$$

# Proof of a Biconditional

**Prove:** “For an integer  $x$ ,  $2x + 3 = 15$  if and only if  $x = 6$ .”

What’s the claim in logic?

$$\forall x(2x + 3 = 15 \leftrightarrow x = 6) \equiv \forall x(2x + 3 \rightarrow x = 6) \wedge \forall x(x = 6 \rightarrow 2x + 3)$$

How would we prove this claim?

Write two sub-proofs, one for each implication:

$$\forall x(2x + 3 = 15 \rightarrow x = 6) \text{ and } \forall x(x = 6 \rightarrow 2x + 3 = 15)$$

# Proof of a Biconditional

Prove: For an integer  $x$ ,  $2x + 3 = 15$  if and only if  $x = 6$ .

# Proof of a Biconditional

Prove: For an integer  $x$ ,  $2x + 3 = 15$  if and only if  $x = 6$ .

[Proof that  $\forall x(2x + 3 = 15 \rightarrow x = 6)$ , called the  $\Rightarrow$  direction ]

[Proof that  $\forall x(x = 6 \rightarrow 2x + 3 = 15)$ , called the  $\Leftarrow$  direction ]

# Proof of a Biconditional

Prove: For an integer  $x$ ,  $2x + 3 = 15$  if and only if  $x = 6$ .

$\Rightarrow$  Let  $x$  be an arbitrary integer. Suppose  $2x + 3 = 15$ .

[Proof that  $x = 6$ ]

Since  $x$  was arbitrary, for all integers  $x$  if  $2x + 3 = 15$  then  $x = 6$ .

[Proof that  $\forall x(x = 6 \rightarrow 2x + 3 = 15)$ , called the  $\Leftarrow$  direction ]

# Proof of a Biconditional

Prove: For an integer  $x$ ,  $2x + 3 = 15$  if and only if  $x = 6$ .

$\Rightarrow$  Let  $x$  be an arbitrary integer. Suppose  $2x + 3 = 15$ .

Then  $2x = 12$ . Thus,  $x = 6$

Since  $x$  was arbitrary, for all integers  $x$  if  $2x + 3 = 15$  then  $x = 6$ .

[Proof that  $\forall x(x = 6 \rightarrow 2x + 3 = 15)$ , called the  $\Leftarrow$  direction ]

# Proof of a Biconditional

Prove: For an integer  $x$ ,  $2x + 3 = 15$  if and only if  $x = 6$ .

$\Rightarrow$  Let  $x$  be an arbitrary integer. Suppose  $2x + 3 = 15$ .

Then  $2x = 12$ . Thus,  $x = 6$

Since  $x$  was arbitrary, for all integers  $x$  if  $2x + 3 = 15$  then  $x = 6$ .

$\Leftarrow$  Let  $x$  be an arbitrary integer. Suppose  $x = 6$ .

[Proof that  $2x + 3 = 15$ ]

Since  $x$  was arbitrary, for all integers  $x$  if  $x = 6$  then  $2x + 3 = 15$ .

# Proof of a Biconditional

Prove: For an integer  $x$ ,  $2x + 3 = 15$  if and only if  $x = 6$ .

$\Rightarrow$  Let  $x$  be an arbitrary integer. Suppose  $2x + 3 = 15$ .

Then  $2x = 12$ . Thus,  $x = 6$

Since  $x$  was arbitrary, for all integers  $x$  if  $2x + 3 = 15$  then  $x = 6$ .

$\Leftarrow$  Let  $x$  be an arbitrary integer. Suppose  $x = 6$ .

Consider  $2x + 3$ :

$$2x + 3 = 2(6) + 3 = 12 + 3 = 15$$

Since  $x$  was arbitrary, for all integers  $x$  if  $x = 6$  then  $2x + 3 = 15$ .

## Remark: Biconditional Proofs

Each direction of the biconditional proof can use whichever proof type fits best (direct, contrapositive, etc.).

Consider the claim: For an integer  $n$ ,  $3n + 3$  is odd iff  $n$  is even.

$\Leftarrow$  Prove that  $\forall n(\text{Even}(n) \rightarrow \text{Odd}(3n + 3))$ . Use a direct proof.

$\Rightarrow$  Prove that  $\forall n(\text{Odd}(3n + 3) \rightarrow \text{Even}(n))$ . Use contrapositive. I.e. prove that  $\forall n(\text{Odd}(n) \rightarrow \text{Even}(3n + 3))$

# More Practice: Another Proof of a Biconditional

## Definitions

$$\text{Even}(x) := \exists k(x = 2k)$$

$$\text{Odd}(x) := \exists k(x = 2k + 1)$$

Prove: For an integer  $n$ ,  $3n + 3$  is odd iff  $n$  is even.

# More Practice: Another Proof of a Biconditional

## Definitions

$$\text{Even}(x) := \exists k(x = 2k)$$

$$\text{Odd}(x) := \exists k(x = 2k + 1)$$

Prove: For an integer  $n$ ,  $3n + 3$  is odd iff  $n$  is even.

[Proof that  $\forall x(\text{Odd}(3n + 3) \rightarrow \text{Even}(n))$ ]

[Proof that  $\forall x(\text{Even}(n) \rightarrow \text{Odd}(3n + 3))$ ]

# More Practice: Another Proof of a Biconditional

## Definitions

$$\text{Even}(x) := \exists k(x = 2k)$$

$$\text{Odd}(x) := \exists k(x = 2k + 1)$$

Prove: For an integer  $n$ ,  $3n + 3$  is odd iff  $n$  is even.

$\Leftarrow$  Let  $n$  be an arbitrary integer. Suppose  $n$  is even. Then by definition of even,  $n = 2k$  for some integer  $k$ . Then consider  $3n + 3$ :

$$3n + 3 = 3(2k) + 3 = 6k + 3 = 2(3k + 1) + 1$$

Since  $k$  is an integer,  $3k + 1$  is an integer. So  $3n + 3$  is 2 times an integer plus 1. So by definition of odd,  $3n + 3$  is odd. Since  $n$  was arbitrary, this shows that for all integers  $n$  if  $n$  is even then  $3n + 3$  is odd.

$\Rightarrow$  We prove by contrapositive. Let  $n$  be an arbitrary integer. Suppose that  $n$  is odd. Then by definition of odd,  $n = 2k + 1$  for some integer  $k$ . Then consider  $3n + 3$ :

$$3n + 3 = 3(2k + 1) + 3 = 6k + 3 + 3 = 6k + 6 = 2(3k + 3)$$

Since  $k$  is an integer,  $3k + 3$  is an integer. So  $3n + 3$  is 2 times an integer. So by definition of even,  $3n + 3$  is even. Since  $n$  was arbitrary, this shows that for all integers  $n$ , if  $n$  is odd then  $3n + 3$  is even. Then the contrapositive also holds: for all integers  $n$ , if  $3n + 3$  is odd then  $n$  is even.

## Remark: Multiple Biconditionals

Suppose you wanted to prove  $p \leftrightarrow q \leftrightarrow r$ .

How many sub-proofs would you need?

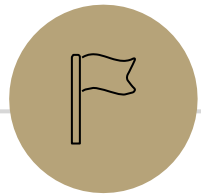
We could do every pair:  $(p \rightarrow q) \wedge (q \rightarrow p) \wedge (q \rightarrow r) \wedge (r \rightarrow q) \wedge (p \rightarrow r) \wedge (r \rightarrow p)$

But it turns out we only need 3. For instance,  $(p \rightarrow q) \wedge (q \rightarrow r) \wedge (r \rightarrow p)$

Any chain of conditional statements work so long as you can follow the chain of implications to get from any statement to any other.

# Proof Strategies So Far

- Direct Proof
- Proof by Contrapositive
- Proof of Biconditional
- Proof by Cases
- Proof of Existence
- Proof by Counterexample



# Proof by Cases

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# Shaking Hands

Suppose there are six people in a room. Some of them shake hands. Consider the claim:

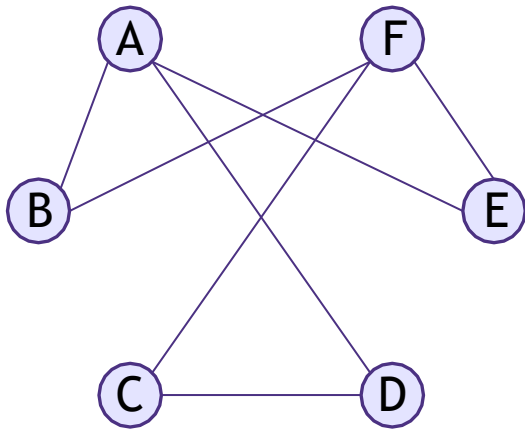
There are at least three people who all shook each other's hands, or three people such that no pair of them shook hands.

Is it true? Can you prove or disprove it?

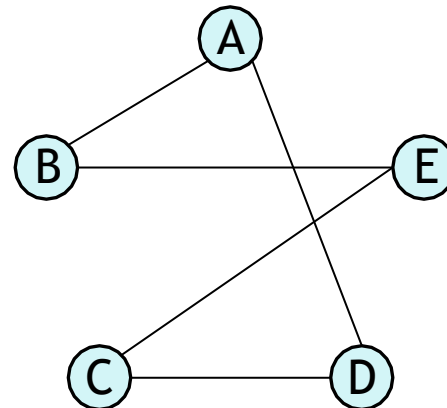
# Shaking Hands

Suppose there are six people in a room. Some of them shake hands. Consider the claim:

There are at least three people who all shook each other's hands, or three people such that no pair of them shook hands.



Not obvious! Doesn't work with 5 people.



There are six people in a room. Prove that there are at least three people who all shook each other's hands, or three people such that no pair of them shook hands.

There are six people in a room. Prove that there are at least three people who all shook each other's hands, or three people such that no pair of them shook hands.

Idea behind proof: We will choose one person, call them  $A$ . They must have either shook hands with 3 or more other people in the room or shook hands with 2 or fewer people. In both cases, we will show our claim holds.

There are six people in a room. Prove that there are at least three people who all shook each other's hands, or three people such that no pair of them shook hands.

Choose one person, call them  $A$ . Note that  $A$  has 5 people around them in the room.

Case 1:  $A$  shook 3 or more of the others' hands.

[Proof that there are at least three people who shook each other's hands, or three people such that no pair of them shook hands]

Case 2:  $A$  shook 2 or fewer of the others' hands.

[Proof that there are at least three people who shook each other's hands, or three people such that no pair of them shook hands]

Because the claim holds in both cases and our cases are exhaustive, we've proven our claim.

There are six people in a room. Prove that there are at least three people who all shook each other's hands, or three people such that no pair of them shook hands.

Choose one person, call them  $A$ . Note that  $A$  has 5 people around them in the room.

Case 1:  $A$  shook 3 or more of the others' hands.

Pick three of them, call them  $B, C, D$ .

[There are two cases for  $B, C, D$ : either two of  $B, C, D$  shook hands with each other (proving our claim) or none of them shook hands with each other (also proving our claim)]

Case 2:  $A$  shook 2 or fewer of the others' hands.

[Proof that there are at least three people who shook each other's hands, or three people such that no pair of them shook hands]

Because the claim holds in both cases and our cases are exhaustive, we've proven our claim.

There are six people in a room. Prove that there are at least three people who all shook each other's hands, or three people such that no pair of them shook hands.

Choose one person, call them  $A$ . Note that  $A$  has 5 people around them in the room.

Case 1:  $A$  shook 3 or more of the others' hands.

Pick three of them, call them  $B, C, D$ .

Then if any of  $B, C$  or  $D$  shook hands with each other, we have 3 people who have all shaken hands. If none of  $B, C$ , or  $D$  shook hands with each other, then we have 3 people who have not shaken any hands.

Case 2:  $A$  shook 2 or fewer of the others' hands.

[Proof that there are at least three people who shook each other's hands, or three people such that no pair of them shook hands]

Because the claim holds in both cases and our cases are exhaustive, we've proven our claim.

There are six people in a room. Prove that there are at least three people who all shook each other's hands, or three people such that no pair of them shook hands.

Choose one person, call them  $A$ . Note that  $A$  has 5 people around them in the room.

Case 1:  $A$  shook 3 or more of the others' hands.

Pick three of them, call them  $B, C, D$ .

Then if any of  $B, C$  or  $D$  shook hands with each other, we have 3 people who have all shaken hands. If none of  $B, C$ , or  $D$  shook hands with each other, then we have 3 people who have not shaken any hands.

Case 2:  $A$  shook 2 or fewer of the others' hands.

Pick three of the people  $A$  did not shake hands with, and call them  $X, Y, Z$ .

[There are two cases for  $X, Y, Z$ : either they all shook hands (proving our claim) or at least two of them did not shake hands (also proving our claim)]

Because the claim holds in both cases and our cases are exhaustive, we've proven our claim.

There are six people in a room. Prove that there are at least three people who all shook each other's hands, or three people such that no pair of them shook hands.

Choose one person, call them  $A$ . Note that  $A$  has 5 people around them in the room.

Case 1:  $A$  shook 3 or more of the others' hands.

Pick three of them, call them  $B, C, D$ .

Then if any of  $B, C$  or  $D$  shook hands with each other, we have 3 people who have all shaken hands. If none of  $B, C$ , or  $D$  shook hands with each other, then we have 3 people who have not shaken any hands.

Case 2:  $A$  shook 2 or fewer of the others' hands.

Pick three of the people  $A$  did not shake hands with, and call them  $X, Y, Z$ .

Then if any of  $X, Y, Z$  also did not shake with each other, we have 3 people who have all not shaken hands. If all of  $X, Y$ , or  $Z$  shook hands with each other, then we have 3 people who have all shaken hands.

Because the claim holds in both cases and our cases are exhaustive, we've proven our claim.

# Proof by Cases

Proof by cases is the strategy of:

1. Breaking your assumption(s) into smaller cases.

Be careful to make sure that your cases are **exhaustive** (cover all of the possible scenarios). It's ok if they have overlap though.

2. Proving that the claim holds in all of these cases.

Formally:  $(P \vee Q) \rightarrow R \equiv (P \rightarrow R) \wedge (Q \rightarrow R)$ .

# 5 numbers: Proof by Cases

Suppose that  $x_1, \dots, x_5$  are real numbers such that  $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ . Prove that  $x_1 + x_2 \leq 20$ .

# 5 numbers: Proof by Cases

Suppose that  $x_1, \dots, x_5$  are real numbers such that  $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ . Prove that  $x_1 + x_2 \leq 20$ .

Let  $x_1, x_2, x_3, x_4, x_5$  be arbitrary real numbers such that  $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ .

Since  $x_1, \dots, x_5$  were arbitrary, our cases were exhaustive, and the claim holds in each case, we've proven our claim.

# 5 numbers: Proof by Cases

Suppose that  $x_1, \dots, x_5$  are real numbers such that  $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ . Prove that  $x_1 + x_2 \leq 20$ .

Let  $x_1, x_2, x_3, x_4, x_5$  be arbitrary real numbers such that  $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ .

[We can split into two cases based on the value of  $x_2$ : when  $x_2 \leq 10$  (our claim immediately follows) and when  $x_2 > 10$ ]

Since  $x_1, \dots, x_5$  were arbitrary, our cases were exhaustive, and the claim holds in each case, we've proven our claim.

# 5 numbers: Proof by Cases

Suppose that  $x_1, \dots, x_5$  are real numbers such that  $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ . Prove that  $x_1 + x_2 \leq 20$ .

Let  $x_1, x_2, x_3, x_4, x_5$  be arbitrary real numbers such that  $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ .

Case 1:  $x_2 \leq 10$ . Then since  $x_1 \leq x_2$ ,  $x_1 \leq 10$ . So  $x_1 + x_2 \leq 20$ , as desired.

Case 2:  $x_2 > 10$ . [Proof that  $x_1 + x_2 \leq 20$ ]

Since  $x_1, \dots, x_5$  were arbitrary, our cases were exhaustive, and the claim holds in each case, we've proven our claim.

# 5 numbers: Proof by Cases

Suppose that  $x_1, \dots, x_5$  are real numbers such that  $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ . Prove that  $x_1 + x_2 \leq 20$ .

Let  $x_1, x_2, x_3, x_4, x_5$  be arbitrary real numbers such that  $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$  and  $x_1 + x_2 + x_3 + x_4 + x_5 = 50$ .

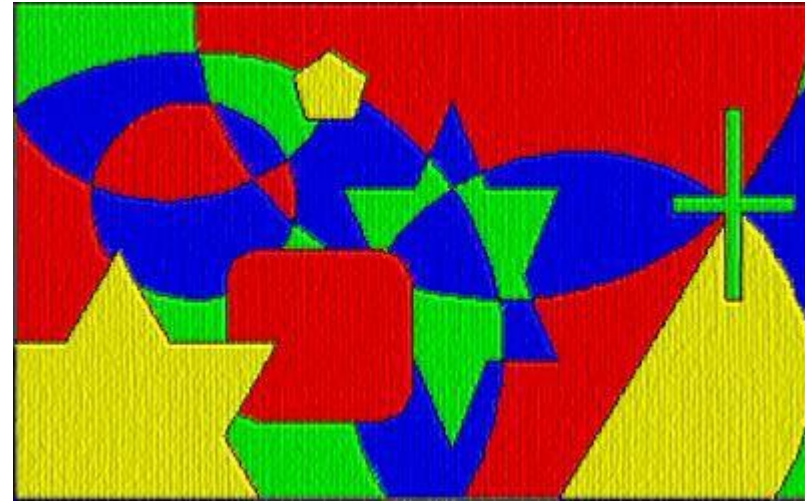
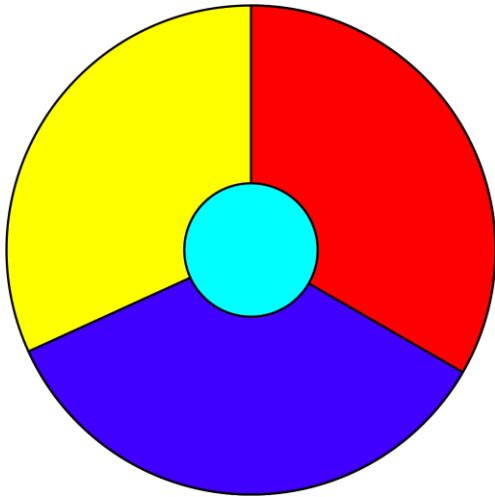
Case 1:  $x_2 \leq 10$ . Then since  $x_1 \leq x_2$ ,  $x_1 \leq 10$ . So  $x_1 + x_2 \leq 20$ , as desired.

Case 2:  $x_2 > 10$ . Then since  $x_3, x_4, x_5 \geq x_2$ , we have that  $x_3 > 10$ ,  $x_4 > 10$ ,  $x_5 > 10$ . So  $x_3 + x_4 + x_5 > 30$ . Thus  $x_1 + x_2 < 20$ , as desired.

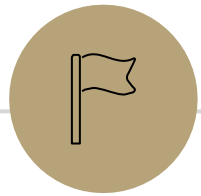
Since  $x_1, \dots, x_5$  were arbitrary, our cases were exhaustive, and the claim holds in each case, we've proven our claim.

# Four Color Theorem: Proof by Cases

Theorem (Four Color): Any plane surface with regions in it can be colored in four colors or less. Two regions that have a common border must not get the same color.



The first proof had 1,936 cases. The shortest known proof today has over 600 cases.



# Existence Proof



# Existence Proof

To prove a statement of the form  $\exists x P(x)$ , we give one example of  $x$  in the domain that makes  $P(x)$  true.

# Existence Proof

Prove: There is some prime number  $p$  such that  $p + 6$  and  $p + 8$  are also prime.

What's the claim in logic?

$$\exists p(\text{Prime}(p) \wedge \text{Prime}(p + 6) \wedge \text{Prime}(p + 8))$$

How would we prove this claim?

Provide such a prime number.

# Existence Proof

Prove: There is some prime number  $p$  such that  $p + 6$  and  $p + 8$  are also prime.

Consider  $p = 5$ . Then  $p + 6 = 11$  is also prime, as is  $p + 8 = 13$ .

# When are Existence Proofs often helpful?

To **disprove** a claim, we prove the negation of the claim.

Existence proofs are often helpful to disprove “for all” claims.

Another term for this is proof by counterexample.

# Proof by Counterexample

A single example can't *prove* a  $\forall$  statement.

A single counterexample can *disprove* a  $\forall$  statement.

For example, to disprove "all professors like pizza", you must find a professor who does not like pizza.

In formal logic:

$$\begin{aligned}\neg\forall x (P(x) \rightarrow Q(x)) &\equiv \exists x \neg(P(x) \rightarrow Q(x)) && \text{DeMorgan's Law for Quantifiers} \\ &\equiv \exists x \neg(\neg P(x) \vee Q(x)) && \text{Law of Implication} \\ &\equiv \exists x (\neg\neg P(x) \wedge \neg Q(x)) && \text{DeMorgan's Law} \\ &\equiv \exists x (P(x) \wedge \neg Q(x)) && \text{Double Negation}\end{aligned}$$

# Proof by Counterexample

For all real numbers  $a, b, c$ , if  $|a + c| = |b + c|$ , then  $|a| = |b|$ .

This claim is false. Disprove!

# Proof by Counterexample

For all real numbers  $a, b, c$ , if  $|a + c| = |b + c|$ , then  $|a| = |b|$ .

This claim is false. Disprove!

Consider  $a = -6, b = 4, c = 1$ . Certainly  $|a| \neq |b|$ . Observe that:

$$|a + c| = |-6 + 1| = |-5| = 5$$

$$|b + c| = |4 + 1| = |5| = 5$$

So this is a counterexample to the claim.

# Practice: Proof by Counterexample

You are given 1¢, 5¢, 10¢, 12¢ and 25¢ coins.

Your boss says the following process will always make change with the least amount of coins: first use as many 25¢ coins that will fit, then 12¢ coins, then 10¢, then 5¢, then 1¢ cent.

Disprove this with a counterexample.

# Practice: Proof by Counterexample

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Your boss says the following process will always make change with the least amount of coins: first use as many 25¢ coins that will fit, then 12¢ coins, then 10¢, then 5¢, then 1¢ cent.

Disprove this with a counterexample.

Consider making 21¢ of change. Your boss's strategy would involve using 12¢, 5¢, 1¢, 1¢, 1¢, 1¢ coins, i.e. 6 coins. However you can make this much change using only 3 coins: 10¢, 10¢, 1¢.

# Proof Strategies So Far

- Direct Proof
- Proof by Contrapositive
- Proof of Biconditional
- Proof by Cases
- Proof of Existence
- Proof by Counterexample

# Todo

## Tonight:

Come to OH today if need extra support while finishing HW2

Read your HW1 feedback and revise your work with it in mind if you're resubmitting

CC7 due Friday at noon