Proof of a Biconditional

Recall that biconditionals are statements of the form:
$\forall x(\mathrm{P}(x) \leftrightarrow \mathrm{Q}(x))$

The strategy is to prove such statements is to prove an implication in both directions. I.e. prove $\forall x(\mathrm{P}(x) \rightarrow \mathrm{Q}(x)) \wedge \forall x(\mathrm{Q}(x) \rightarrow \mathrm{P}(x))$.

$$\begin{aligned}\forall x\, (P(x) \leftrightarrow Q(x)) &\equiv \forall x\, (P(x) \rightarrow Q(x) \wedge Q(x) \rightarrow P(x)) \\ &\equiv \forall x(P(x) \rightarrow Q(x)) \wedge \forall x(Q(x) \rightarrow P(x))\end{aligned}$$

Proof by Cases

Proof by cases is the strategy of:

1. Breaking your assumption(s) into smaller cases.
   Be careful to make sure that your cases are exhaustive (cover all of the possible scenarios). It's ok if they have overlap though.

2. Proving that the claim holds in all of these cases.

Formally: $(\mathrm{P} \vee \mathrm{Q}) \rightarrow \mathrm{R} \equiv (\mathrm{P} \rightarrow \mathrm{R}) \wedge (\mathrm{Q} \rightarrow \mathrm{R})$.

5 numbers: Proof by Cases

Suppose that $x_1, \dots, x_5$ are real numbers such that $x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5$ and $x_1 + x_2 + x_3 + x_4 + x_5 = 50$. Prove that $x_1 + x_2 \leq 20$.

Proof by Counterexample

For all real numbers $a, b, c,$ if $|a + c| = |b + c|$, then $|a| = |b|$.

This claim is false. Disprove!