Direct Proofs and Proof by Contrapositive

CSE 311: Summer 25
Lecture 6

Announcements

- HW2 is out!
  - Due Wednesday @ 11:59 pm
  - Come by OH for extra support!

- HW1 feedback will be released shortly
  - Please read *all* feedback that's left on your assignment
  - Regrade requests will be open for one week
  - More info on how to resubmit your work on Wednesday

Review

Where were we?

A predicate is a function that outputs a Boolean

`Prime(x)` := “`x` is prime”

`LessThan(x,y)`:= “`x<y`”

The “domain of discourse” is the set of all values your variables can take.

Usually the “type” you’re allowing

Quantifiers

We have two extra symbols to indicate which way we're using the variable.

1. The statement is true for every x, we just want to put a name on it.

$\forall x\ (\mathrm{p}(\mathrm{x}) \wedge q(x))$ means "for every x in our domain, $p(x)$ and $q(x)$ both evaluate to true."

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, such that $p(x)$ and $q(x)$ are both true.

Quantifiers

We have two extra symbols to indicate which way we're using the variable.

1. The statement is true for every x, we just want to put a name on it.

$\forall x\ (\mathrm{p}(\mathrm{x}) \wedge q(x))$ means "for every x in our domain, $p(x)$ and $q(x)$ both evaluate to true."

Universal Quantifier

"$\forall x$"

"for each x", "for every x", "for all x" are common translations

Remember: upside-down-A for All.

Quantifiers

Existential Quantifier

"$\exists x$"

"there is an x", "there exists an x", "for some x" are common translations

Remember: backwards-E for Exists.

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, for which $p(x)$ and $q(x)$ are both true.

Quantifiers

Which of these translates "For every cat: if a cat is fat then it is happy."
when our domain of discourse is "mammals"?

$\forall x[(\texttt{Cat}(x) \wedge \texttt{Fat}(x)) \rightarrow \texttt{Happy}(x)]$

For all mammals, if x is a cat and fat then it is happy
[if x is not a cat, the claim is vacuously true, you can't use the promise for anything]

$\forall x[\texttt{Cat}(x) \wedge (\texttt{Fat}(x) \rightarrow \texttt{Happy}(x))]$

For all mammals, that mammal is a cat and if it is fat then it is happy.
[what if x is a dog? Dogs are in the domain, but…uh-oh. This isn't what we meant.]

To "limit" variables to a portion of your domain of discourse under a universal quantifier add a hypothesis to an implication.

Quantifiers

Which of these translates "There is a dog who is not happy."
when our domain of discourse is "mammals"?

$$\exists x[\texttt{Dog}(x) \rightarrow \neg\texttt{Happy}(x)]$$

There is a mammal, such that if x is a dog then it is not happy.
[this can't be right – plug in a cat for x and the implication is true]

$$\exists x[(\texttt{Dog}(x) \land \neg\texttt{Happy}(x))]$$

There is a mammal that is both a dog and not happy.
[this one is correct!]

To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

Domain Restriction Translations

Translations often sound more natural if we:

1. Notice domain restriction patterns.
2. Avoid using variables when we can.
3. Drop the "for all" or "there exists" when we can.

For example:

$$\forall x(\text{Cat}(x) \rightarrow \text{Blue}(x))$$

✗ For all animals x, if x is a cat then x is blue.

✓ All cats are blue.

DeMorgan’s Laws for Quantifiers

$$\neg \forall x\ \mathrm{P}(x) \equiv \exists x\ \neg \mathrm{P}(x)$$

$$\neg \exists x\ \mathrm{P}(x) \equiv \forall x\ \neg \mathrm{P}(x)$$

I.e. to negate an expression with a quantifier:

1. Switch the quantifier (∀ becomes ∃, and vice versa).
2. Negate the expression inside.

Proving Predicate Logic Equivalence

$\neg\exists x\ \exists y\ (\mathrm{Odd}(x) \wedge \mathrm{Even}(y) \wedge (x = y))$

$\equiv \forall x\ \neg\exists y\ (\mathrm{Odd}(x) \wedge \mathrm{Even}(y) \wedge (x = y))$ — DeMorgan's Law for Quantifiers

$\equiv \forall x\ \forall y\ \neg(\mathrm{Odd}(x) \wedge \mathrm{Even}(y) \wedge (x = y))$ — DeMorgan's Law for Quantifiers

$\equiv \forall x\ \forall y\ (\neg\big(\mathrm{Odd}(x) \wedge \mathrm{Even}(y)\big) \vee \neg(x = y))$ — DeMorgan's Law

$\equiv \forall x\ \forall y\ (\neg\big(\mathrm{Odd}(x) \wedge \mathrm{Even}(y)\big) \vee (x \neq y))$ — Definition of $\neq$

$\equiv \forall x\ \forall y\ (\big(\mathrm{Odd}(x) \wedge \mathrm{Even}(y)\big) \rightarrow (x \neq y))$ — Law of Implication

Warm Up

Proving Predicate Logic Equivalence Practice

Prove $\neg\forall x\big(\mathrm{P}(x) \to \exists y\ \mathrm{Q}(x,y)\big)$ is equivalent to $\exists x\ \forall y(\mathrm{P}(x) \land \neg\mathrm{Q}(x,y))$.

Proving Predicate Logic Equivalence

$\neg\forall x(\mathrm{P}(x) \rightarrow \exists y\, \mathrm{Q}(x,y))$	
$\equiv \neg\forall x(\neg\mathrm{P}(x) \vee \exists y\, \mathrm{Q}(x,y))$	Law of Implication
$\equiv \exists x\, \neg(\neg\mathrm{P}(x) \vee \exists y\, \mathrm{Q}(x,y))$	DeMorgan's Law for Quantifiers
$\equiv \exists x\, (\neg\neg\mathrm{P}(x) \wedge \neg\exists y\, \mathrm{Q}(x,y))$	DeMorgan's Law
$\equiv \exists x\, (\mathrm{P}(x) \wedge \neg\exists y\, \mathrm{Q}(x,y))$	Double Negation
$\equiv \exists x\, (\mathrm{P}(x) \wedge \forall y \neg\mathrm{Q}(x,y))$	DeMorgan's Law for Quantifiers
$\equiv \exists x\, \forall y(\mathrm{P}(x) \wedge \neg\mathrm{Q}(x,y))$	

Theorems and Proofs

Theorems and Proofs

Theorem: A statement that has been proven to be true.

Proof: A valid argument that establishes a statement to be true.

You’ll also see
“claim” (the statement we’re about to prove)
“lemma” (small theorem, used to prove a bigger theorem)
“corollary” (small theorem, proven using a bigger theorem)

Theorems and Proofs

Examples of theorems include…

- Given a right triangle with side lengths a, b and hypotenuse c, $a^2 + b^2 = c^2$
- There are infinitely many prime numbers.
- There exists a problem that cannot be solved by a program.

Integer

We need a basic starting point to be able to prove things.

Objects to work with.

> An integer: is any real number with no fractional part.

Some **definitions** to analyze

Even

`Even(x)` := An integer, x, is even if and only if there is an integer k such that $x = 2k$.

Odd

`Odd(x)` := An integer, x, is odd if and only if there is an integer k such that $x = 2k + 1$.

A word on definitions

Definitions are fundamental. Our goal is to communicate precisely.

When you come across an edge case, a definition is the way to solve it.

Is -4 even? Well $\exists k(-4 = 2k)$ (take $k = -2$), so yes it is!

We go to the definition. Not your gut feeling about what feels right.

How do we know something is true? Usually we verify the definition!

A word on definitions

How do we know something is true? Usually we verify the definition!

In other resources (textbooks, Wikipedia, etc.)

You will see things that look like this:

Definition: An integer, x, is even if $\exists k(x = 2k)$.

Notice it says "if" not "if and only if."

A definition is **always** an if and only if. The word "definition" has the "only if" direction in it.

I really wish people didn't do this. I wish they explicitly said "if and only if" but some people insist that "definition" implies the "only if" direction. Otherwise it's a "sufficient condition" not a "definition"

Proof Strategy: Direct Proof

Direct Proof

Direct proof is one strategy for proving statements of the form

$$\forall x[P(x) \rightarrow Q(x)]$$

Our First Direct Proof

Prove: "For all integers x, if x is even, then x^2 is even."

What's the claim in logic? $\forall x\ (\text{Even(x)} \rightarrow \text{Even}(x^2))$

How would we prove this claim?

We'll see how to prove it formally in a minute; for now, just try to convince each other this statement is true.

Our First Direct Proof

Definitions

$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\big(\text{Even}(x) \to \text{Even}(x^2)\big)$

Arbitrary

An "arbitrary" variable is one that

- is part of the domain of discourse (or some sub-domain you pick).
- You know **nothing** else about.

EVERY element of the domain could be plugged into that arbitrary variable. And everything else you say in the proof will follow.

An arbitrary variable is exactly what you need to convince us of a $\forall$.

If you want to prove a for-all you must explicitly tell us the variable is arbitrary when it is introduced.

- Your reader doesn't know what you're doing otherwise.

Our First Direct Proof

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\big(\text{Even}(x) \rightarrow \text{Even}(x^2)\big)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

Now What?

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Well....what does it mean to be even?

$x = 2k$ for some integer k.

Where do we need to end up?

$\texttt{Even}(x^2)$

Our First Direct Proof

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\big(\text{Even}(x) \to \text{Even}(x^2)\big)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

So x^2 is even.

Our First Direct Proof

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Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\big(\text{Even}(x) \rightarrow \text{Even}(x^2)\big)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

By definition of even, $x = 2k$ for some integer k.

So x^2 is even.

Our First Direct Proof

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove: "For all integers x, if x is even, then x^2 is even." $\forall x\big(\text{Even}(x) \rightarrow \text{Even}(x^2)\big)$

Proof: Let x be an arbitrary integer. Suppose that x is even.

By definition of even, $x = 2k$ for some integer k.

Squaring both sides, we see that:

$x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$

Because k is an integer, $2k^2$ is also an integer.

So x^2 is two times an integer.

Which is exactly the definition of even, so x^2 is even.

Since x was an arbitrary integer, we conclude that for all integers x, if x is even then x^2 is also even.

Direct Proof Template

Prove: $\forall x(\text{Even}(x) \rightarrow \text{Even}(x^2))$

Declare an arbitrary variable for each ∀.	Let x be an arbitrary integer.
Assume the left side of the implication.	Suppose that x is even.
Unroll the predicate definitions.	Then by definition of even, there exists some integer k such that $x = 2k$.
Manipulate towards the goal.	Squaring both sides, we see that: $x^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2$
Reroll definitions into the right side of the implication.	Because k is an integer, then $2k^2$ is also an integer. So x^2 is two times an integer. So by definition of even, x^2 is even.
Conclude that you have proved the claim.	Since x was an arbitrary integer, we can conclude that for all integers x, if x is even then x^2 is even.

Direct Proof Steps

These are the usual steps. We'll see different outlines in the future!!

- Introduction
  - Declare an arbitrary variable for each $\forall$ quantifier
  - Assume the left side of the implication
- Core of the proof
  - Unroll the predicate definitions
  - Manipulate towards the goal (using creativity, algebra, etc.)
  - Reroll definitions into the right side of the implication
- Conclude that you have proved the claim

Another Direct Proof

Prove: "The product of two odd integers is odd."

What's the claim in logic? $\forall x\, \forall y\, ((\mathrm{Odd}(x) \wedge \mathrm{Odd}(y)) \rightarrow \mathrm{Odd}(xy))$

How would we prove this claim?

Direct Proof. In particular, we'll let x, y be arbitrary integers.
We'll suppose x, y are odd. We'll show that $x \cdot y$ is odd.

Another Direct Proof

Definitions

$\text{Odd}(x) := \exists k(x = 2k + 1)$

Prove: "The product of two odd integers is odd."

$$\forall x \forall y \left(\left(\text{Odd}(x) \land \text{Odd}(y) \right) \rightarrow \text{Odd}(xy) \right)$$

Another Direct Proof

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: "The product of two odd integers is odd."

$$\forall x \forall y \left(\left(\mathrm{Odd}(x) \wedge \mathrm{Odd}(y)\right) \rightarrow \mathrm{Odd}(xy)\right)$$

Let x and y be arbitrary integers. Suppose that x and y are odd.

[Unroll predicate definitions]

[Manipulate towards goal]

So by definition of odd, xy is odd. Since x, y were arbitrary, we have shown that the product of two odd integers is odd.

Another Direct Proof

Definitions
$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: "The product of two odd integers is odd."

$$\forall x \forall y \left(\left(\text{Odd}(x) \land \text{Odd}(y) \right) \rightarrow \text{Odd}(xy) \right)$$

Let x and y be arbitrary integers. Suppose that x and y are odd.

Then by definition of odd, there exists some integer k such that $x = 2k + 1$, and some integer j such that $y = 2j + 1$.

[Manipulate towards goal]

So by definition of odd, xy is odd. Since x, y were arbitrary, we have shown that the product of two odd integers is odd.

Another Direct Proof

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: "The product of two odd integers is odd."

$$\forall x \forall y \left(\left(\mathrm{Odd}(x) \wedge \mathrm{Odd}(y) \right) \rightarrow \mathrm{Odd}(xy) \right)$$

Let x and y be arbitrary integers. Suppose that x and y are odd.

Then by definition of odd, there exists some integer k such that $x = 2k + 1$, and some integer j such that $y = 2j + 1$.

Then multiplying x and y, we can see that:

$xy = (2k + 1) \cdot (2j + 1) = 4kj + 2j + 2k + 1 = 2(2kj + j + k) + 1$

Since k, j are integers, $2kj + j + k$ is an integer.

So by definition of odd, xy is odd. Since x, y were arbitrary, we have shown that the product of two odd integers is odd.

A note on Domain of Discourse

"The product of two odd integers is odd."

Domain: Integers

Translation:

$\forall x \, \forall y \, ((\text{Odd}(x) \wedge \text{Odd}(y)) \rightarrow \text{Odd}(xy))$

Proof Outline:

Let x and y be arbitrary integers.

Suppose x and y are odd.

Show xy is odd.

Domain: Odd Integers

Translation:

$\forall x \, \forall y \, (\text{Odd}(xy))$

Proof Outline:

Let x and y be arbitrary odd integers.

Show xy is odd.

A note on Translation to Logic

- We first translate the claim to predicate logic because:
  - The translation makes it precise what we are proving
  - The translation hints at the structure of the proof

    e.g. for each ∀, introduce an arbitrary variable
- In practice, computer scientists identify the proof claim and structure without predicate logic translation
- Eventually we'll stop asking you to translate to logic first

Square

Definition:

An integer x is square iff there exists an integer k such that $x = k^2$.

$$\text{Square}(x) \coloneqq \exists k(x = k^2)$$

Yet Another Direct Proof

Definitions

$\text{Square}(x) \coloneqq \exists k \left(x = k^2\right)$

Prove: The product of two square integers is square.

What's the claim in logic?

Prove this claim.

Yet Another Direct Proof

Definitions
$\text{Square}(x) \coloneqq \exists k\ (x = k^2)$

Prove: The product of two square integers is square.

What's the claim in logic?

$$\forall n \forall m\ [(\text{Square}(n) \wedge \text{Square}(m)) \rightarrow \text{Square}(nm)]$$

Prove this claim.

Yet Another Direct Proof

Definitions

$\text{Square}(x) \coloneqq \exists k\ (x = k^2)$

Prove: "The product of two square integers is square."

$$\forall n \forall m \left(\left(\text{Square}(n) \wedge \text{Square}(m) \right) \rightarrow \text{Square}(nm) \right)$$

Yet Another Direct Proof

Definitions

$\text{Square}(x) := \exists k \left(x = k^2\right)$

Prove: "The product of two square integers is square."

$$\forall n \forall m \left(\left(\text{Square}(n) \wedge \text{Square}(m)\right) \rightarrow \text{Square}(nm)\right)$$

Let n and m be arbitrary integers.

Suppose that n and m are square.

So by definition of square, nm is square.

Since n and m were arbitrary, we have shown that the product of two square integers is square.

Yet Another Direct Proof

Definitions
$\text{Square}(x) \coloneqq \exists k\, (x = k^2)$

Prove: "The product of two square integers is square."

$$\forall n \forall m \left(\left(\text{Square}(n) \wedge \text{Square}(m)\right) \rightarrow \text{Square}(nm) \right)$$

Let n and m be arbitrary integers.

Suppose that n and m are square.

[Unroll predicate definitions]

[Manipulate towards goal]

So by definition of square, nm is square.
Since n and m were arbitrary, we have shown that the product of two square integers is square.

Yet Another Direct Proof

Definitions

$\text{Square}(x) \coloneqq \exists k\ (x = k^2)$

Prove: "The product of two square integers is square."

$$\forall n \forall m \left(\left(\text{Square}(n) \land \text{Square}(m) \right) \rightarrow \text{Square}(nm) \right)$$

Let n and m be arbitrary integers.

Suppose that n and m are square.

Then by definition of square, $n = k^2$ for some integer k, and $m = j^2$ for some integer j.

[Manipulate towards goal]

So by definition of square, nm is square.
Since n and m were arbitrary, we have shown that the product of two square integers is square.

Yet Another Direct Proof

Definitions
$\text{Square}(x) := \exists k \left(x = k^2\right)$

Prove: "The product of two square integers is square."

$$\forall n \forall m \left(\left(\text{Square}(n) \land \text{Square}(m)\right) \to \text{Square}(nm)\right)$$

Let n and m be arbitrary integers.

Suppose that n and m are square.

Then by definition of square, $n = k^2$ for some integer k, and $m = j^2$ for some integer j.

Then multiplying n and m, we can see: $nm = k^2 \cdot j^2 = (kj)^2$

Since k and j are integers, kj is an integer.

So by definition of square, nm is square.
Since n and m were arbitrary, we have shown that the product of two square integers is square.

Proof Strategy: Contrapositive

A Direct Proof?

<u>Definitions</u>
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

What's the claim in logic? $\forall x(\mathrm{Odd}(3x + 2) \rightarrow \mathrm{Odd}(x))$

Prove this claim.

Let x be an arbitrary integer. Suppose that $3x + 2$ is odd. Then $3x + 2 = 2k + 1$ for some integer k. Subtracting both sides by 2, we have $3x = 2k - 1$. Then $x = \frac{2k}{3}$...?

Proof by Contrapositive

Proof by contrapositive is another strategy for proving statements of the form $\forall x(P(x) \rightarrow Q(x))$.

The strategy is to prove the contrapositive, i.e. prove $\forall x \left(\neg \mathrm{Q}(x) \rightarrow \neg \mathrm{P}(x)\right)$

Remember, an implication is equivalent to its contrapositive!

Proof by Contrapositive

Definitions

$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

Proof by Contrapositive

Definitions
$\mathrm{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

$$\forall x\,\big(\mathrm{Odd}(3x+2) \to \mathrm{Odd}(x)\big) \equiv \forall x\,(\mathrm{Even}(x) \to \mathrm{Even}(3x+2))$$

Proof by Contrapositive

Definitions
$\mathrm{Odd}(x) := \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

$$\forall x\, \big(\mathrm{Odd}(3x + 2) \rightarrow \mathrm{Odd}(x)\big) \equiv \forall x\, (\mathrm{Even}(x) \rightarrow \mathrm{Even}(3x + 2))$$

We prove by contrapositive.
Let x be an arbitrary integer. Suppose that x is even.

So by definition of even, $3x + 2$ is even.
Since x was arbitrary, we have shown that for all integers x, if x is even then $3x + 2$ is even. Thus the contrapositive also holds: for all integers x, if $3x + 2$ is odd, then x is odd.

Proof by Contrapositive

Definitions
$\text{Odd}(x) \coloneqq \exists k(x = 2k + 1)$

Prove: For an integer x, if $3x + 2$ is odd, then x is odd.

$$\forall x\ \big(\text{Odd}(3x+2) \rightarrow \text{Odd}(x)\big) \equiv \forall x\ (\text{Even}(x) \rightarrow \text{Even}(3x+2))$$

We prove by contrapositive.
Let x be an arbitrary integer. Suppose that x is even.

[Unroll predicate definitions]

[Manipulate towards goal]

So by definition of even, $3x + 2$ is even.
Since x was arbitrary, we have shown that for all integers x, if x is even then $3x + 2$ is even. Thus the contrapositive also holds: for all integers x, if $3x + 2$ is odd, then x is odd.

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Let x be an arbitrary integer. Suppose that x is even.

Then by definition of even, $x = 2k$ for some integer k.

[Manipulate towards goal]

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We prove by contrapositive.
Let x be an arbitrary integer. Suppose that x is even.

Then by definition of even, $x = 2k$ for some integer k.

Consider $3x + 2$:

$$3x + 2 = 3(\,2k) + 2 = 6k + 2 = 2(3k + 1)$$

Since k is an integer, $3k + 1$ is an integer. So by definition of even, $3x + 2$ is even.
Since x was arbitrary, we have shown that for all integers x, if x is even then $3x + 2$ is even. Thus the contrapositive also holds: for all integers x, if $3x + 2$ is odd, then x is odd.

More Practice

Another Proof by Contrapositive

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove by Contrapositive: For an integer n, if n^3 is even, then n is even.

Another Proof by Contrapositive

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$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove by Contrapositive: For an integer n, if n^3 is even, then n is even.

$$\forall n \left(\text{Even}(n^3) \rightarrow \text{Even}(n)\right) \equiv \forall n(\text{Odd}(n) \rightarrow \text{Odd}(n^3))$$

Another Proof by Contrapositive

Definitions
$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove by Contrapositive: For an integer n, if n^3 is even, then n is even.

$$\forall n\left(\text{Even}(n^3) \to \text{Even}(n)\right) \equiv \forall n(\text{Odd}(n) \to \text{Odd}(n^3))$$

We prove by contrapositive.
Let n be an arbitrary integer. Suppose that n is odd.

Thus by definition of odd, n^3 is odd. Since n was arbitrary, we have shown that for all integers n, if n is odd then n^3 is odd. Thus the contrapositive also holds: for all integers n, if n^3 is even, then n is even.

Another Proof by Contrapositive

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We prove by contrapositive.
Let n be an arbitrary integer. Suppose that n is odd.

[Unroll predicate definitions]

[Manipulate towards goal]

Thus by definition of odd, n^3 is odd. Since n was arbitrary, we have shown that for all integers n, if n is odd then n^3 is odd. Thus the contrapositive also holds: for all integers n, if n^3 is even, then n is even.

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We prove by contrapositive.
Let n be an arbitrary integer. Suppose that n is odd.

Then by definition of odd, $n = 2k + 1$ for some integer k.

[Manipulate towards goal]

Thus by definition of odd, n^3 is odd. Since n was arbitrary, we have shown that for all integers n, if n is odd then n^3 is odd. Thus the contrapositive also holds: for all integers n, if n^3 is even, then n is even.

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$\text{Even}(x) \coloneqq \exists k(x = 2k)$

Prove by Contrapositive: For an integer n, if n^3 is even, then n is even.

$$\forall n\left(\text{Even}(n^3) \rightarrow \text{Even}(n)\right) \equiv \forall n(\text{Odd}(n) \rightarrow \text{Odd}(n^3))$$

We prove by contrapositive.
Let n be an arbitrary integer. Suppose that n is odd.

Then by definition of odd, $n = 2k + 1$ for some integer k.

Consider n^3:

$$n^3 = (2k+1)^3 = 8k^3 + 8k^2 + 4k + 1 = 2\left(4k^3 + 4k^2 + 2k\right) + 1$$

Since k is an integer, $4k^3 + 4k^2 + 2k$ is an integer. Thus by definition of odd, n^3 is odd. Since n was arbitrary, we have shown that for all integers n, if n is odd then n^3 is odd. Thus the contrapositive also holds: for all integers n, if n^3 is even, then n is even.

Todo

Tonight:

Come to OH today if need extra support while working on HW2

CC6 due Wednesday at noon