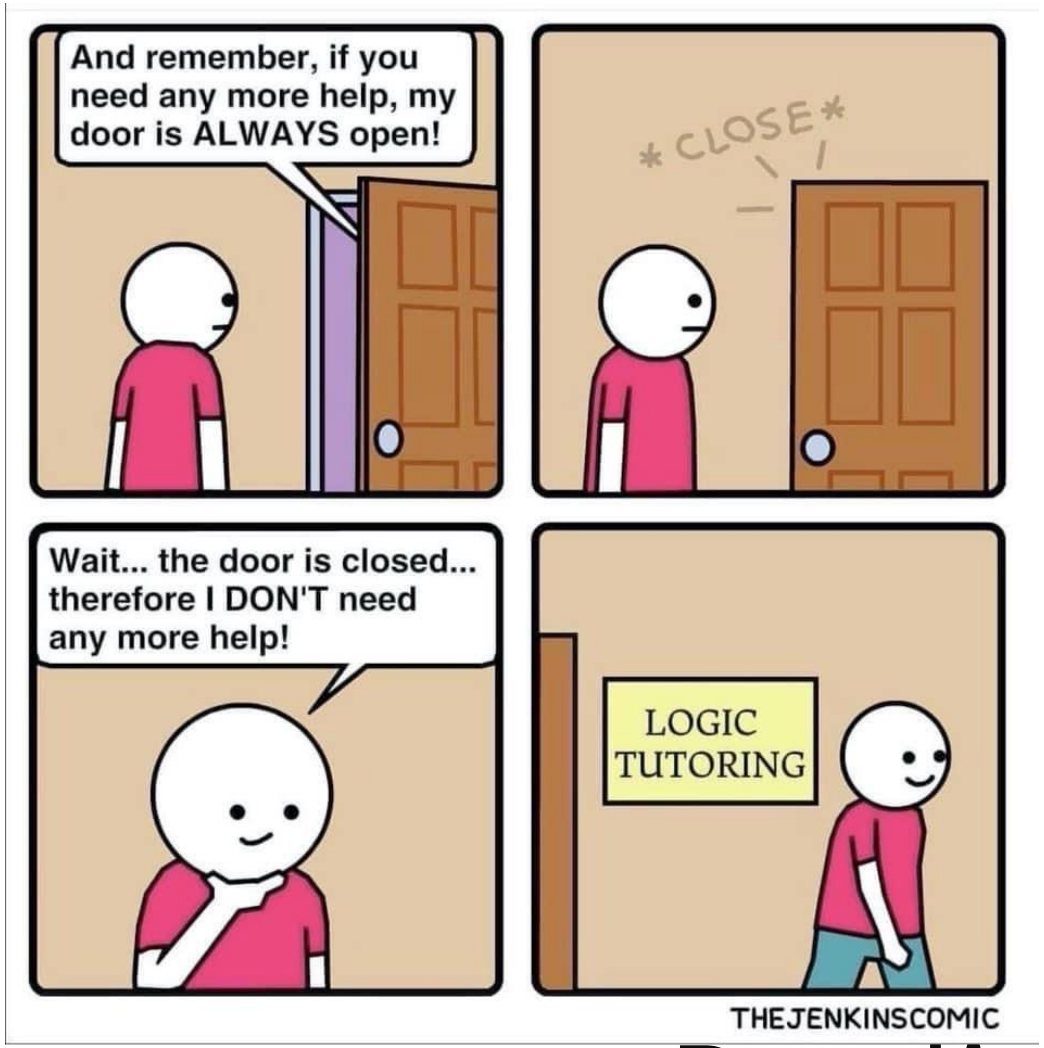


Predicate Logic Continued

CSE 311 Summer 25
Lecture 5

Announcements

- HW1
  - Due Tonight @ 11:59 pm
  - You have 6 late days for the quarter, and can use up to 3 for one assignment

- HW2 will be released later today

Review

Where were we?

A predicate is a function that outputs a Boolean

`Prime(x)` := “`x` is prime”

`LessThan(x,y)`:= “`x<y`”

The “domain of discourse” is the set of all values your variables can take.

Usually the “type” you’re allowing

Quantifiers

We have two extra symbols to indicate which way we're using the variable.

1. The statement is true for every x, we just want to put a name on it.

$\forall x\ (\mathrm{p}(\mathrm{x}) \wedge q(x))$ means "for every x in our domain, $p(x)$ and $q(x)$ both evaluate to true."

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, such that $p(x)$ and $q(x)$ are both true.

Quantifiers

We have two extra symbols to indicate which way we're using the variable.

1. The statement is true for every x, we just want to put a name on it.

$\forall x\ (\mathrm{p}(\mathrm{x}) \wedge q(x))$ means "for every x in our domain, $p(x)$ and $q(x)$ both evaluate to true."

Universal Quantifier

"$\forall x$"

"for each x", "for every x", "for all x" are common translations

Remember: upside-down-A for All.

Quantifiers

Existential Quantifier

"$\exists x$"

"there is an x", "there exists an x", "for some x" are common translations

Remember: backwards-E for Exists.

2. There's some x out there that works, (but I might not know which it is, so I'm using a variable).

$\exists x(p(x) \wedge q(x))$ means "there is an x in our domain, for which $p(x)$ and $q(x)$ are both true.

Domain Restriction

Definition:

Domain restriction is the technique of limiting our domain of discourse to a smaller set of objects.

Quantifiers

Which of these translates "For every cat: if a cat is fat then it is happy."
when our domain of discourse is "mammals"?

$\forall x[(\texttt{Cat}(x) \wedge \texttt{Fat}(x)) \rightarrow \texttt{Happy}(x)]$

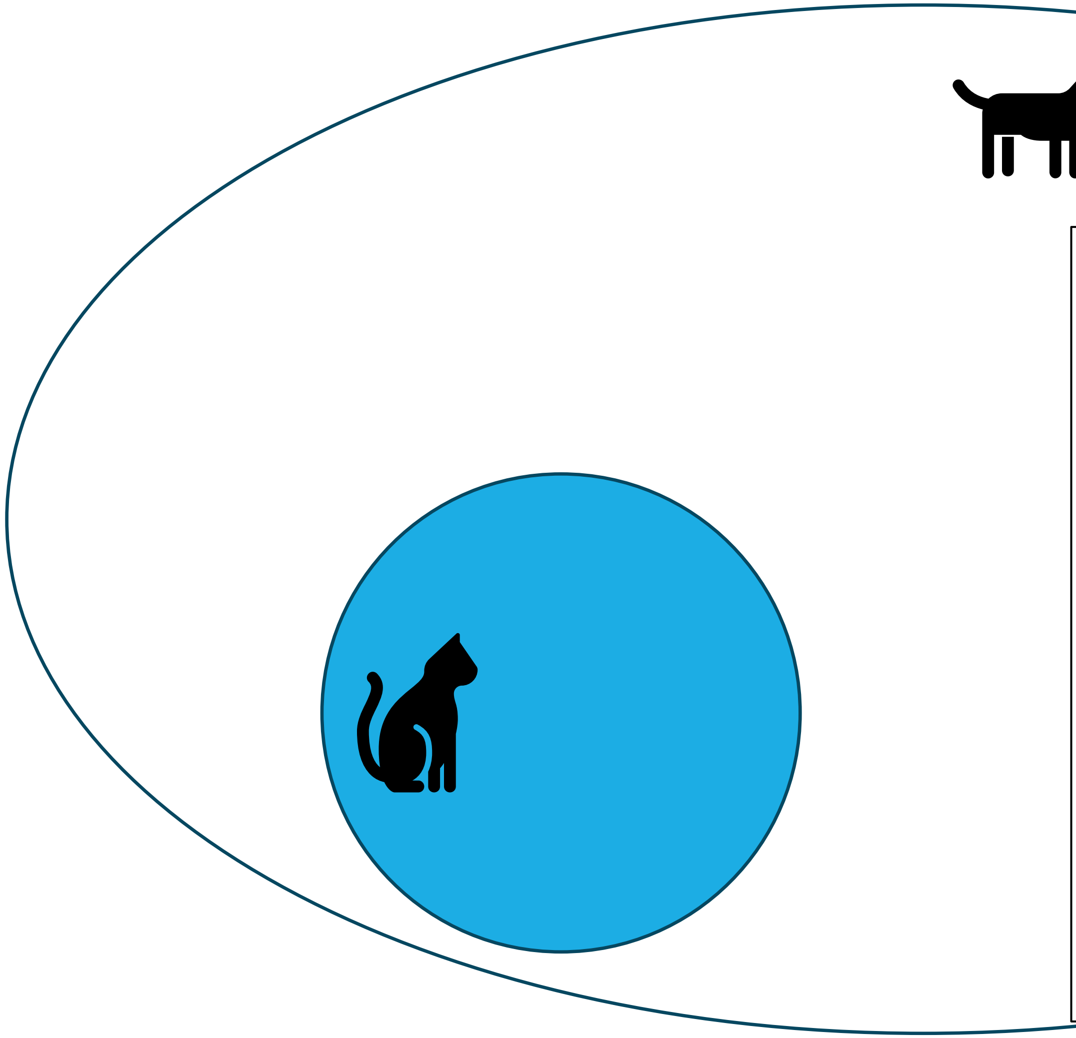
For all mammals, if x is a cat and fat then it is happy
[if x is not a cat, the claim is vacuously true, you can't use the promise for anything]

$\forall x[\texttt{Cat}(x) \wedge (\texttt{Fat}(x) \rightarrow \texttt{Happy}(x))]$

For all mammals, that mammal is a cat and if it is fat then it is happy.
[what if x is a dog? Dogs are in the domain, but…uh-oh. This isn't what we meant.]

To "limit" variables to a portion of your domain of discourse under a universal quantifier add a hypothesis to an implication.

$\forall x[(\mathtt{Cat}(x) \land \mathtt{Fat}(x)) \rightarrow \mathtt{Happy}(x)]$



x outside the subdomain

$(\mathtt{Cat}(x) \land \mathtt{Fat}(x)) \rightarrow \mathtt{Happy}(x)$

$(\mathtt{F} \land \mathtt{Fat}(x)) \rightarrow \mathtt{Happy}(x)$

$(\mathtt{F}) \rightarrow \mathtt{Happy}(x)$

$\mathtt{T}$

For x that aren't cats, the implication is vacuously true.

These won't affect whether $\forall x[(\mathtt{Cat}(x) \land \mathtt{Fat}(x)) \rightarrow \mathtt{Happy}(x)]$ is true.

$$\forall x[(\mathtt{Cat}(x) \land \ \mathtt{Fat}(x)) \to \mathtt{Happy}(x)]$$

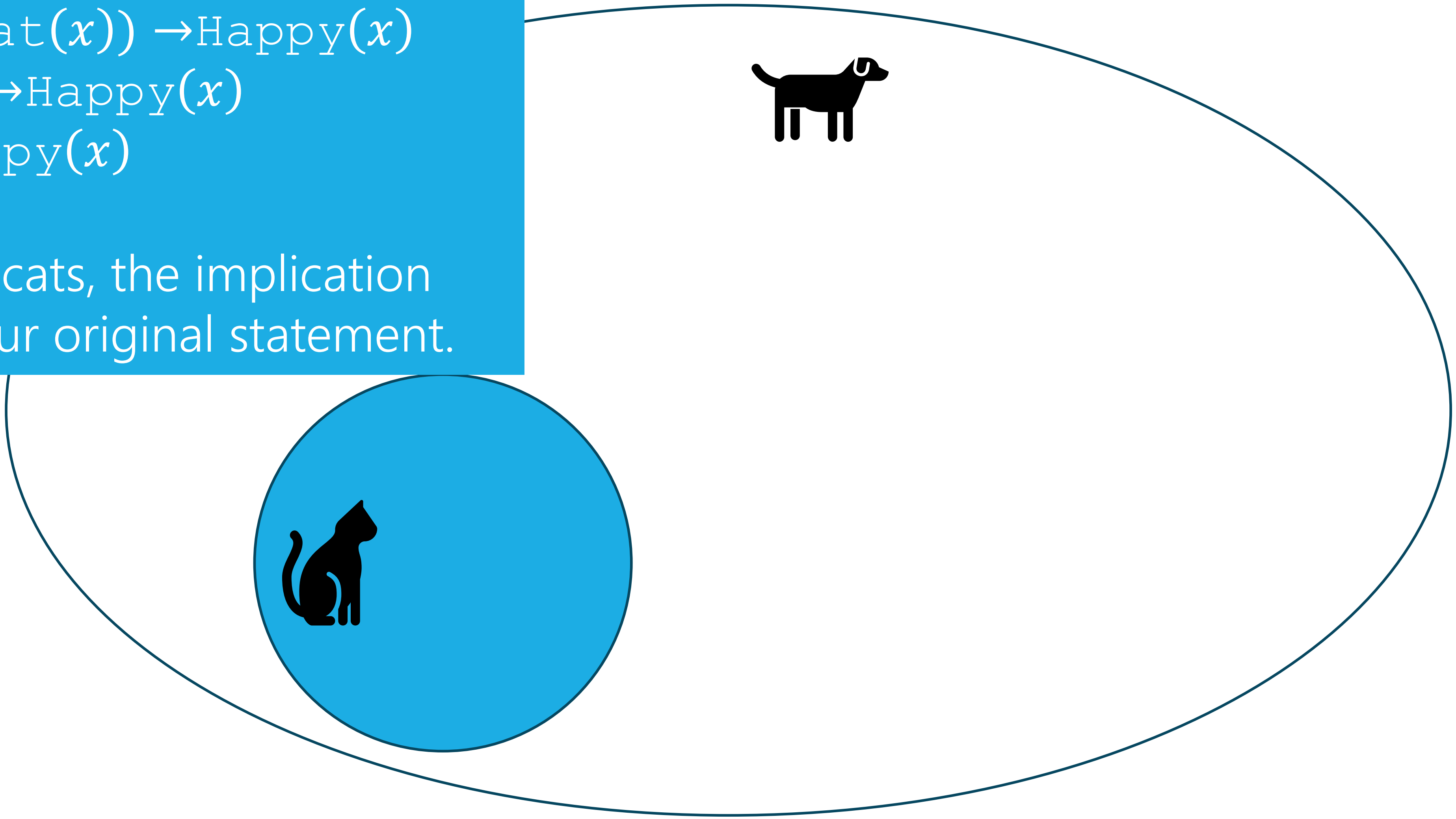
x inside the subdomain

$(\mathtt{Cat}(x) \land \ \mathtt{Fat}(x)) \to \mathtt{Happy}(x)$

$(\mathrm{T} \land \mathtt{Fat}(x)) \to \mathtt{Happy}(x)$

$\mathtt{Fat}(x) \to \mathtt{Happy}(x)$

For x that are cats, the implication simplifies to our original statement.



Quantifiers

Which of these translates "There is a dog who is not happy."
when our domain of discourse is "mammals"?

$\exists x[\mathtt{Dog}(x) \to \neg\mathtt{Happy}(x)]$

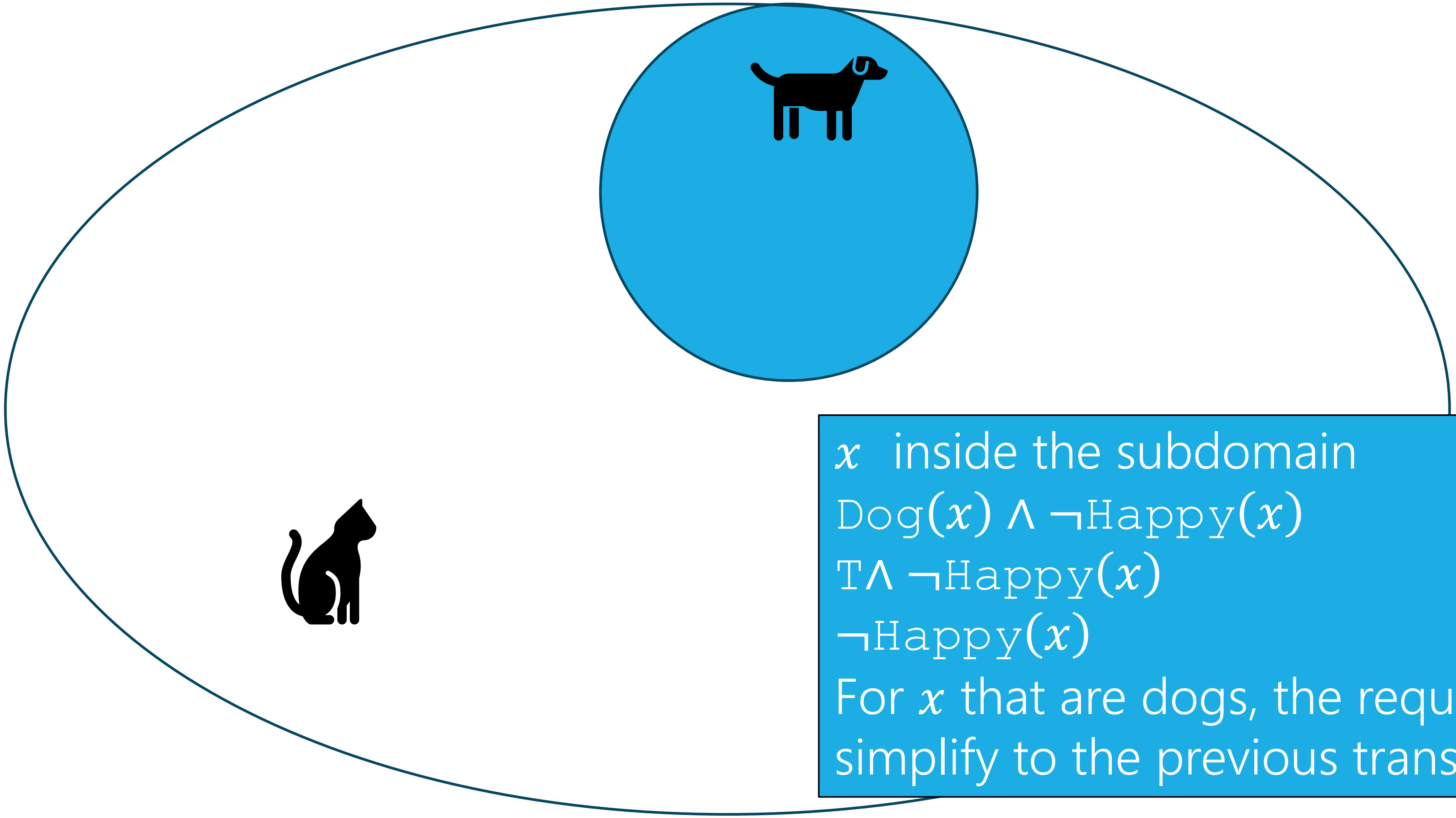
$\exists x[(\mathtt{Dog}(x) \land \neg\mathtt{Happy}(x))]$

There is a mammal, such that if x is a dog then it is not happy.
[this can't be right – plug in a cat for x and the implication is true]

There is a mammal that is both a dog and not happy.
[this one is correct!]

To "limit" variables to a portion of your domain of discourse under an existential quantifier AND the limitation together with the rest of the statement.

$\exists x[(\mathtt{Dog}(x) \land \neg \mathtt{Happy}(x))]$



x inside the subdomain

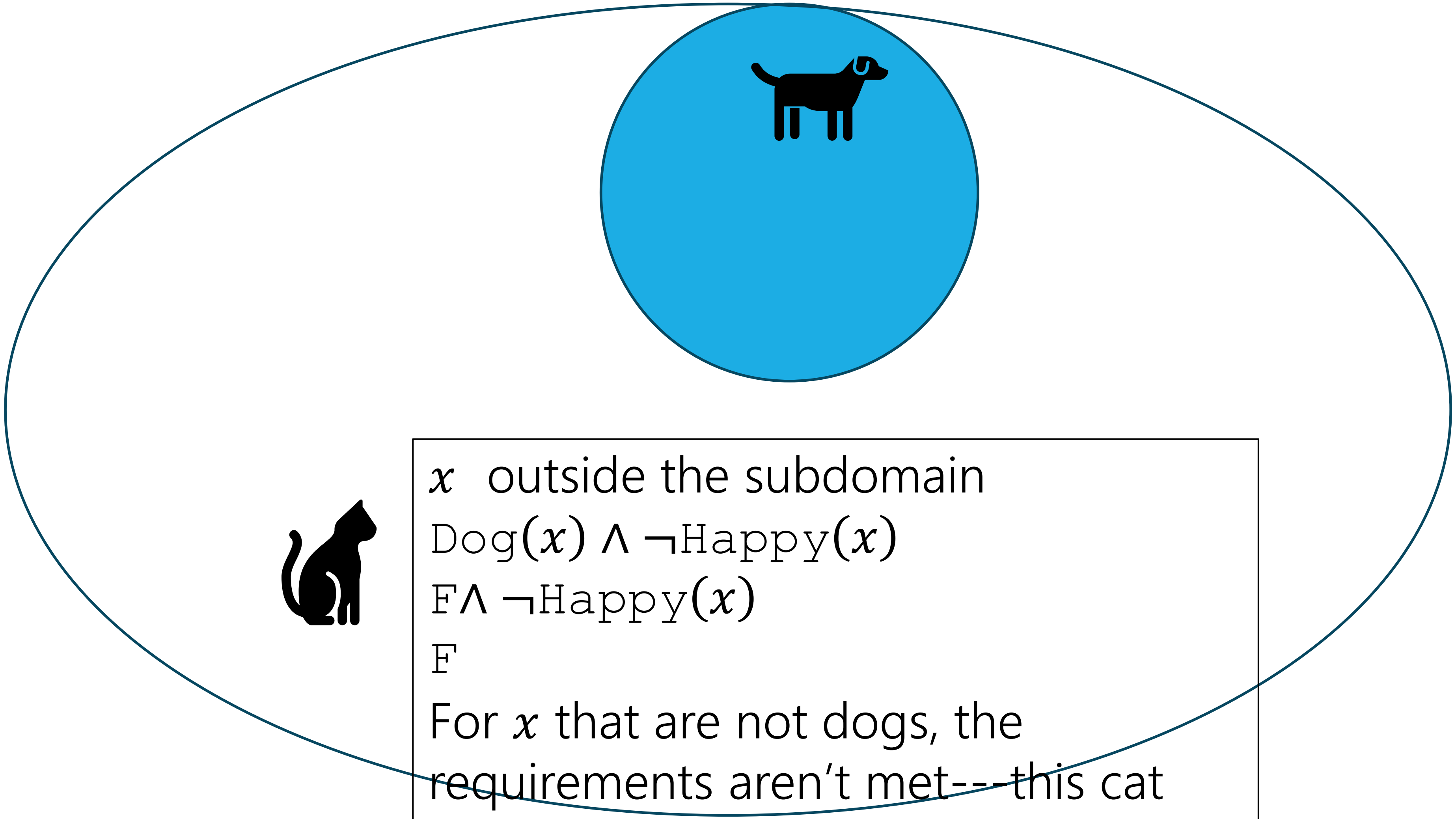
$\mathtt{Dog}(x) \land \neg \mathtt{Happy}(x)$

$\mathtt{T} \land \neg \mathtt{Happy}(x)$

$\neg \mathtt{Happy}(x)$

For x that are dogs, the requirements simplify to the previous translation

$\exists x[(\mathtt{Dog}(x) \land \neg\mathtt{Happy}(x))]$



x outside the subdomain

$\mathtt{Dog}(x) \land \neg\mathtt{Happy}(x)$

$\mathtt{F} \land \neg\mathtt{Happy}(x)$

$\mathtt{F}$

For x that are not dogs, the requirements aren't met---this cat isn't what we're looking for.

Warm Up

More Practice

Let your domain of discourse be fruits. Translate these

There is a fruit that is tasty and ripe.

For every fruit, if it is not ripe then it is not tasty.

There is a fruit that is sliced and diced.

More Practice

Let your domain of discourse be fruits. Translate these

There is a fruit that is tasty and ripe.

$$\exists x(\mathtt{Tasty}(x) \land \mathtt{Ripe}(x))$$

For every fruit, if it is not ripe then it is not tasty.

$$\forall x(\neg\mathtt{Ripe}(x) \rightarrow \neg\mathtt{Tasty}(x))$$

There is a fruit that is sliced and diced.

$$\exists x(\mathtt{Sliced}(x) \land \mathtt{Diced}(x))$$

Domain Restriction

Domain Restriction Translations

Translations often sound more natural if we:

1. Notice domain restriction patterns.
2. Avoid using variables when we can.
3. Drop the “for all” or “there exists” when we can.

For example:

$$\forall x\big(\mathrm{Cat}(x) \rightarrow \mathrm{Blue}(x)\big)$$

✗ For all animals x, if x is a cat then x is blue.

✓ All cats are blue.

Translation Practice

Domain of Discourse
Food

Predicate Definitions
$\text{Fruit}(x) \coloneqq x$ is a fruit
$\text{Tasty}(x) \coloneqq x$ is tasty
$\text{Ripe}(x) \coloneqq x$ is ripe

Translate these sentences using a natural-sounding translation.

$\exists x(\text{Fruit}(x) \land \text{Tasty}(x))$

There is a tasty fruit. OR Some fruits are tasty.

$\forall x((\text{Fruit}(x) \land \neg\text{Ripe}(x)) \rightarrow \neg\text{Tasty}(x))$

All fruits that aren't ripe aren't tasty.

Negation

DeMorgan’s Laws for Quantifiers

Consider the following sentences:

- There does not exist a green penguin.
- Every penguin is a color other than green.

Are they logically equivalent?

DeMorgan’s Laws for Quantifiers

Consider the following sentences:

- Not every person can dance.
- There is a person that cannot dance.

Are they logically equivalent?

DeMorgan's Laws for Quantifiers

$$\boxed{\begin{aligned} \neg\forall x\, \mathrm{P}(x) &\equiv \exists x\, \neg\mathrm{P}(x) \\ \neg\exists x\, \mathrm{P}(x) &\equiv \forall x\, \neg\mathrm{P}(x) \end{aligned}}$$

I.e. to negate an expression with a quantifier:

1. Switch the quantifier (∀ becomes ∃, and vice versa).
2. Negate the expression inside.

Example

Translate to predicate logic & rewrite using DeMorgan's Laws.

There is no integer which is prime and even.

$$\neg\exists x(\mathrm{Prime}(x) \wedge \mathrm{Even}(x))$$

$$\equiv \forall x \neg(\mathrm{Prime}(x) \wedge \mathrm{Even}(x))$$

$$\equiv \forall x\ (\neg\mathrm{Prime}(x) \vee \neg\mathrm{Even}(x))$$

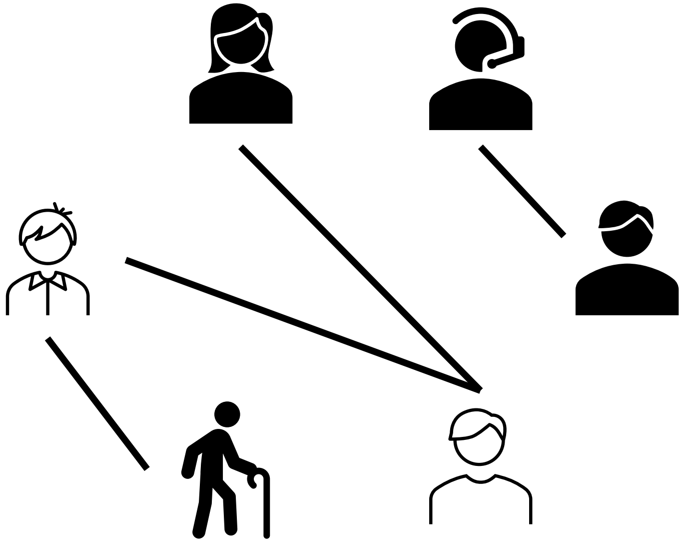
All integers are not prime or not even.

Nested Quantifiers

Nested Quantifiers

Translate these sentences using only quantifiers and the predicate `AreFriends`(x, y)

Everyone is friends with someone.



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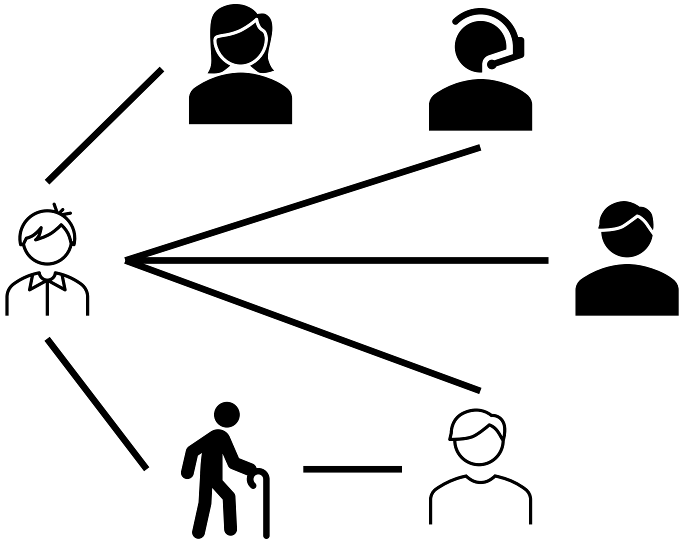

Nested Quantifiers

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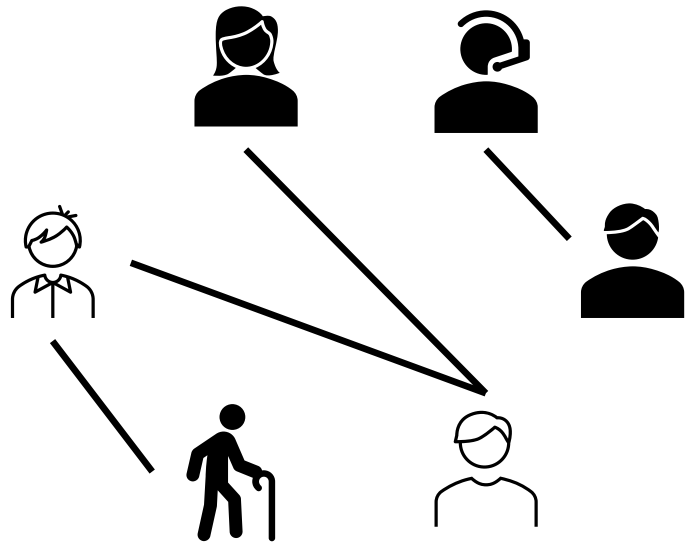
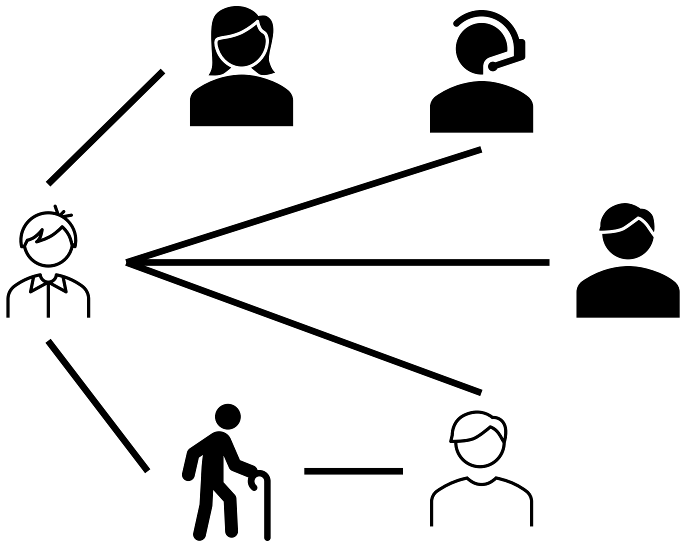


$\forall x(\exists y \texttt{ AreFriends}(x, y))$

$\forall x \exists y \texttt{ AreFriends}(x, y)$

Someone is friends with everyone.



$\exists x(\forall y \texttt{ AreFriends}(x, y))$

$\exists x \forall y \texttt{ AreFriends}(x, y)$

Nested Quantifiers

$\forall x \exists y\, P(x, y)$

"For every x there exists a y such that $P(x, y)$ is true."

y might change depending on the x (people have different friends!).

$\exists x \forall y\, P(x, y)$

"There is an x such that for all $y, P(x, y)$ is true."

There's a special, magical x value so that $P(x, y)$ is true regardless of y.

Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

$\forall x \exists y P(x, y)$

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Nested Quantifiers

Let our domain of discourse be $\{A, B, C, D, E\}$

And our proposition $P(x, y)$ be given by the table.

What should we look for in the table?

$\exists x \forall y P(x, y)$

A row, where every entry is $\mathtt{T}$

$\forall x \exists y P(x, y)$

In every row there must be a $\mathtt{T}$

$P(x,y)$	A	B	C	D	E
A	T	T	T	T	T
B	T	F	F	T	F
C	F	T	F	F	F
D	F	F	F	F	T
E	F	F	F	T	F

Keep everything in order

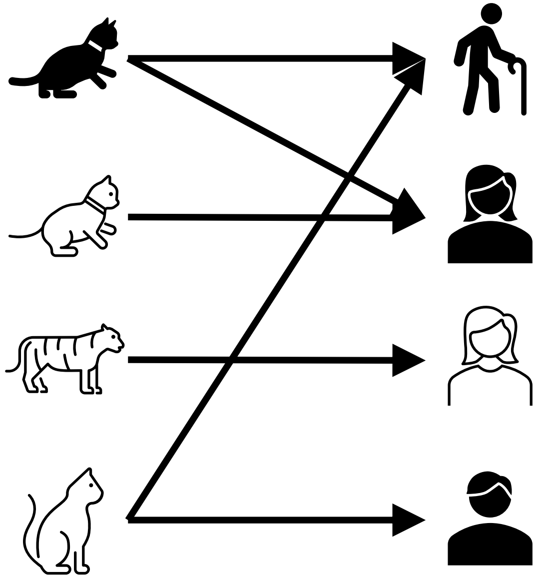
Keep the quantifiers in the same order in English as they are in the logical notation.

“There is someone out there for everyone” is a $\forall x \exists y$ statement in “everyday” English.

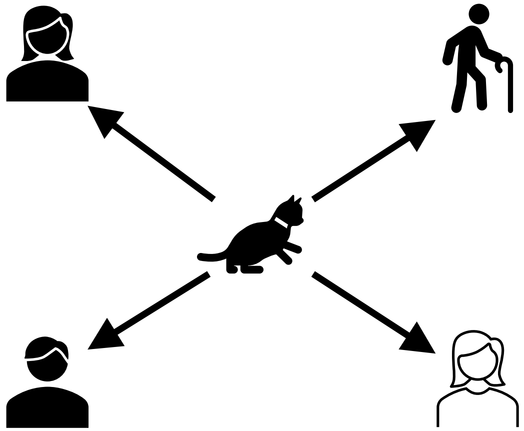
It would **never** be phrased that way in “mathematical English” We’ll only every write “for every person, there is someone out there for them.”

Try it yourselves

Every cat loves some human.



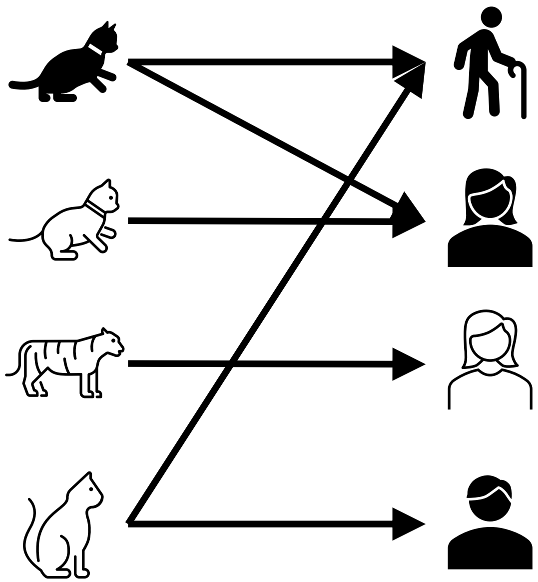
There is a cat that loves every human.



Let your domain of discourse be mammals.
Use the predicates $\texttt{Cat}(x)$, $\texttt{Dog}(x)$, and $\texttt{Loves}(x, y)$ to mean x loves y.

Try it yourselves

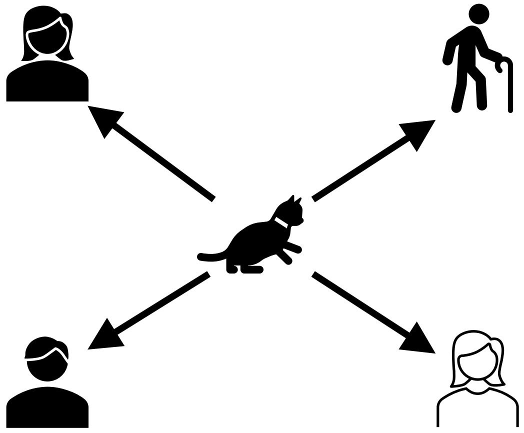
Every cat loves some human.



$\forall x\ (\mathtt{Cat}(x) \rightarrow \exists y[\mathtt{Human}(y)\ \wedge \mathtt{Loves}(x,y)])$

$\forall x \exists y(\mathtt{Cat}(x) \rightarrow [\mathtt{Human}(y) \wedge \mathtt{Loves}(x,y)])$

There is a cat that loves every human.



$\exists x\ (\mathtt{Cat}(x) \wedge \forall y[\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x,y)])$

$\exists x \forall y(\mathtt{Cat}(x) \wedge [\mathtt{Human}(y) \rightarrow \mathtt{Loves}(x,y)])$

More Translation

For each of the following, translate it, then say whether the statement is true. Let your domain of discourse be integers.

For every integer, there is a greater integer.

$\forall x \exists y(\texttt{Greater}(y, x))$ (This statement is true: y can be $x + 1$ [y depends on x])

There is an integer x, such that for all integers y, xy is equal to 1.

$\exists x \forall y(\texttt{Equal}(xy, 1))$ (This statement is false: no single value of x can play that role for every y.)

$\forall y \exists x(\texttt{Equal}(x + y, 1))$

For every integer, y, there is an integer x such that $x + y = 1$
(This statement is true, y can depend on x)

Negation

How do we negate nested quantifiers?

The old rule still applies.

To negate an expression with a quantifier, apply DeMorgan's Law:
1. Switch the quantifier (∀ becomes ∃, ∃ becomes ∀)
2. Negate the expression inside

$\neg(\forall x \exists y \forall z\ [P(x,y) \wedge Q(y,z)])$

$\exists x(\neg(\exists y \forall z\ [P(x,y) \wedge Q(y,z)]))$

$\exists x \forall y(\neg(\forall z[P(x,y) \wedge Q(y,z)]))$

$\exists x \forall y \exists z(\neg[P(x,y) \wedge Q(y,z)])$

$\exists x \forall y \exists z[\neg P(x,y) \vee \neg Q(y,z)]$

Practice

Translate to predicate logic & rewrite using DeMorgan's Law.

There is no integer greater than or equal to every other integer.

$$\neg\exists x\ \forall y\ (x \geq y)$$

$$\equiv \forall x \neg \forall y\ (x \geq y)$$

$$\equiv \forall x\ \exists y\ \neg(x \geq y)$$

$$\equiv \forall x\ \exists y\ (x < y)$$

For every integer, there is an integer greater than it.

Translating "Exactly one"

<u>Domain of Discourse</u>
Mammals

<u>Predicate Definitions</u>
$\text{Walks}(x, y) \coloneqq x$ walks y
$\text{Friends}(x, y) \coloneqq x$ and y are friends
$\text{Human}(x) \coloneqq x$ is a human
$\text{Dog}(x) \coloneqq x$ is a dog

Every human walks exactly one dog.

$$\underbrace{\forall x[\text{Human}(x) \rightarrow \exists y \Big(\text{Dog}(y) \land \text{Walks}(x, y)}_{\text{All humans walk a dog } y} \land \underbrace{\forall z\big((\text{Dog}(z) \land (z \neq y)) \rightarrow \neg\text{Walks}(x, z)\big)\Big)]}_{\substack{\text{For all other dogs that aren't a human's dog } y, \\ \text{the human does not walk it}}}$$

Predicate Logic Equivalence

Motivation

- We saw in our last practice example that there may be different predicate logic expressions that have the same meaning
- We can prove logical equivalence of Predicate Logic statements like we did for Propositional Logic
- Same equivalence rules still apply, in addition to DeMorgan's Law for Quantifiers

Proving Predicate Logic Equivalence

"No odd integer is equal to an even integer."

Alice translated this as: $\neg\exists x\exists y\ (\text{Odd}(x) \wedge \text{Even}(y) \wedge (x = y))$

Bob translated this as: $\forall x\forall y\,(\text{Odd}(x) \wedge \text{Even}(y) \rightarrow (x \neq y))$

Prove that these translations are logically equivalent.

Proving Predicate Logic Equivalence

$\neg\exists x\ \exists y\ (\mathrm{Odd}(x) \wedge \mathrm{Even}(y) \wedge (x = y))$

$\equiv \forall x\ \neg\exists y\ (\mathrm{Odd}(x) \wedge \mathrm{Even}(y) \wedge (x = y))$ DeMorgan's Law for Quantifiers

$\equiv \forall x\ \forall y\ \neg(\mathrm{Odd}(x) \wedge \mathrm{Even}(y) \wedge (x = y))$ DeMorgan's Law for Quantifiers

$\equiv \forall x\ \forall y\ (\neg\big(\mathrm{Odd}(x) \wedge \mathrm{Even}(y)\big) \vee \neg(x = y))$ DeMorgan's Law

$\equiv \forall x\ \forall y\ (\neg\big(\mathrm{Odd}(x) \wedge \mathrm{Even}(y)\big) \vee (x \neq y))$ Definition of $\neq$

$\equiv \forall x\ \forall y\ (\big(\mathrm{Odd}(x) \wedge \mathrm{Even}(y)\big) \rightarrow (x \neq y))$ Law of Implication

More Practice

Proving Predicate Logic Equivalence Practice

Prove $\neg\forall x\big(\mathrm{P}(x) \to \exists y\ \mathrm{Q}(x,y)\big)$ is equivalent to $\exists x\ \forall y(\mathrm{P}(x) \land \neg\mathrm{Q}(x,y))$.

Proving Predicate Logic Equivalence

$\neg\forall x\,\big(P(x) \rightarrow \exists y\, Q(x,y)\big)$	
$\equiv \neg\forall x\,\big(\neg P(x) \vee \exists y\, Q(x,y)\big)$	Law of Implication
$\equiv \exists x\,\neg\big(\neg P(x) \vee \exists y\, Q(x,y)\big)$	DeMorgan's Law for Quantifiers
$\equiv \exists x\,(\neg\neg P(x) \wedge \neg\exists y\, Q(x,y))$	DeMorgan's Law
$\equiv \exists x\,\big(P(x) \wedge \neg\exists y\, Q(x,y)\big)$	Double Negation
$\equiv \exists x\,\big(P(x) \wedge \forall y\, \neg Q(x,y)\big)$	DeMorgan's Law for Quantifiers
$\equiv \exists x\forall y\big(P(x) \wedge \neg Q(x,y)\big)$	

Todo

Tonight:

Come to OH today if you get stuck while working on HW1

CC5 due Monday at noon