



[xkcd.com/1153](http://xkcd.com/1153)

# Equivalences and Our First Proof

CSE 311 Summer 2025  
Lecture 2

# Announcements

- HW1
  - Posted on the course website
  - Due Wednesday, 11:59 pm
  - Submit on Gradescope
- OH starts today!
  - OH times listed on the course calendar on our website
  - Zoom links are on the Pinned Thread of Important Links on Ed
- Go to section tomorrow!
- CC1 and CC2 due Friday at noon

# Homework Submissions

Make sure we can read what you submit.

We can't spend 5 minutes per submission deciding if that's a  $p$  or a  $q$ .

Typesetting guarantees we can read it.

LaTeX is the industry standard for typesetting (if you go to CS grad school, you'll use it for all your papers). Overleaf and Parchmynt are the easiest ways to get started. Look out for LaTeX HW templates on the Ed board.

Need to know the code for a symbol? [Detexify](#)! Word uses LaTeX codes...mostly...

# Announcements: Office Hours

A chance to talk to staff about the class.

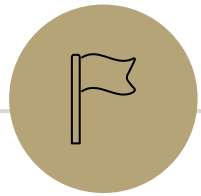
Aim early on in the “homework cycle” (i.e., long before the deadline).

We won’t answer every kind of question in office hours:

We don’t “pregrade” homework. We won’t tell you whether something is right or wrong.

We will help on homework, but not usually by “giving hints.” We’ll usually ask questions about what you’ve tried, ask questions to help you decide what to do next, or point you to good examples to look at.

You’re allowed to talk to others at office hours, as long as you’re still following the collaboration policy.



**Review**

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# Propositions

## Propositions

- **Propositions** are T/F-valued statements
- Combine using **logical connectives** ( $\neg, \vee, \wedge, \rightarrow$ , etc.)
- Can be described by a **truth table**

## Applications

- Understanding complex English sentences
- Simplifying Boolean logic in code

# Recall: Atomic Propositions

- **Atomic Propositions** are true or false statements that cannot be broken down any further
- **Compound Propositions** can be divided into simpler propositions.
- **Propositional variables:**  $p, q, r, s \dots$

# Recall: Logical Connectives

<u>Name</u>	<u>Logical Symbol</u>
Not	$\neg p$
And	$p \wedge q$
Or	$p \vee q$
XOR	$p \oplus q$
Implication	$p \rightarrow q$
Biconditional	$p \leftrightarrow q$

# Recall: Truth Tables

$p$	$\neg p$
T	F
F	T

$p$	$q$	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

$p$	$q$	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

$p$	$q$	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

# Recall: Implication

"If it's raining, then I have my umbrella"

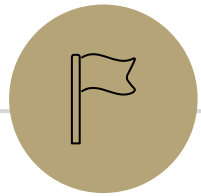
$p$ : It is raining       $q$ : I have my umbrella

$p \rightarrow q$

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Equivalently:

- Whenever it is raining, I have my umbrella.
- It is raining only if I have my umbrella.
- For it to be raining, it is necessary that I have my umbrella.



**Warm Up**

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# Warm Up

Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.

It is snowing today, and if it is raining or snowing then we won't walk to school.

# Warm Up – Solution

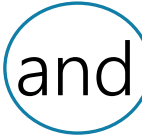
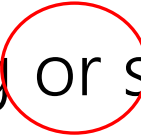
Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.

It is snowing today, and if it is raining or snowing then we won't walk to school.

Parker's process: identify connecting words, identify propositions, replace English with logic, figure out parentheses

# Warm Up – Solution

Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.

It is snowing today,   if it is raining  or snowing  we won't walk to school.

Identify connecting words: look for and, or, not, if-then, etc.

# Warm Up – Solution

Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.

$p$   
It is snowing today, and if it is raining or snowing then we won't walk to school.  
 $q$   
 $p$   
 $\neg r$

Identify propositions: What's left are propositions, look for repeats and hidden negations.

$p$ : it is snowing today.

$q$ : it is raining.

$r$ : we walk to school.

# Warm Up – Solution

Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.

$p$   
It is snowing today, and if it is raining or snowing then we won't walk to school.  
 $q$   
 $p$   
 $\neg r$

$p$ : it is snowing today.

$q$ : it is raining.

$r$ : we walk to school.

Replace English with logic

$$p \wedge q \vee p \rightarrow \neg r$$

# Warm Up – Solution

Translate this sentence into symbolic logic, and describe a weather pattern and transportation method that causes the proposition to be false.

$p$   
It is snowing today, and if it is raining or snowing then we won't walk to school.  
 $q$   
 $p$   
 $\neg r$

$p$ : it is snowing today.

$q$ : it is raining.

$r$ : we walk to school.

Figure out parentheses

$$p \wedge [(q \vee p) \rightarrow \neg r]$$

“raining or snowing” is the condition of the implication, not walking to school is the conclusion. Omitted words in other clauses hint that “It is snowing today” stands on its own.

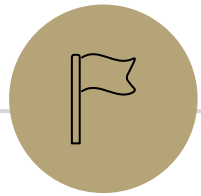
# Today

Logical Equivalences

Simplification Rules

Our first proof!

Modifying Implications



# Logical Equivalences

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# Logical Equivalence

## Definition:

Two propositions are **logically equivalent** if they have identical truth values.

The notation for A and B being logically equivalent is  $A \equiv B$ .

## Examples:

$$p \vee q \equiv q \vee p$$

$$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$$

$=$  VS.  $\equiv$  VS.  $\leftrightarrow$

$A = B$  means  $A, B$  are the exact same string.

E.g.  $p \vee q = p \vee q$ , but  $p \vee q \neq q \vee p$ . We hardly use  $=$  with propositions!

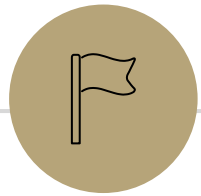
$A \equiv B$  is an assertion that  $A, B$  always have the same truth value.

E.g.  $p \vee q \equiv q \vee p$ .

$A \leftrightarrow B$  is a proposition that might be true or false.

E.g.  $p \vee q \leftrightarrow p \wedge q$  is a false proposition.

$A \equiv B$  has the same meaning as  $(A \leftrightarrow B) \equiv T$ .



# Proving Logical Equivalences

# Motivation

Given two propositions, we would like to know if they are equivalent.

E.g. one developer wrote `if ((p && p) || p) { ... }`.

Another developer wrote `if (p) { ... }`.

You want to confirm if those are the same.

Given a complicated proposition, we would like to find a simpler proposition that it's equivalent to.

E.g. you see `if ((p && p) || p){ ... }` in code.

You simplify the code to `if (p) { ... }`.

# Strategy 1: Truth Tables

Make a truth table for the two propositions and check if they are the same.

*$p$  vs.  $(p \wedge p) \vee p$*

$p$	$p \wedge p$	$(p \wedge p) \vee p$
T	T	T
F	F	F

# Strategy 1: Truth Tables

Truth tables *do* let us check if two propositions are equivalent.

Truth tables *don't* give us a good way to start from a complicated proposition, and simplify it.

# Strategy 2: Manipulating Expressions

Instead, we are going to learn logical equivalence rules to help us simplify expressions.

Similar to algebra, where we can apply rules to transform expressions into equivalent expressions:

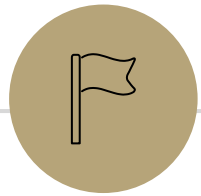
$$(x + 2)(x + 3) = x^2 + 2x + 3x + 6$$

Distributivity

$$= x^2 + 5x + 6$$

Adding like terms

By the end of the course, you'll do these "automatically" on full sentences; for now we'll practice mechanically on symbolic forms to gain *intuition* for why these rules work.



# Logical Equivalence Rules

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# De Morgan's Laws

Negate the statement

"my code compiles or there is a bug."

i.e. find a natural English sentence that says

$$\neg(p \vee q)$$

"the following is not true: my code compiles or there is a bug"

Hint: when is the original sentence false?

'or' means 'at least one is true' so to negate, we need to say 'neither is true' or equivalently 'both are false'

"my code does not compile and there is not a bug"

$$\neg p \wedge \neg q$$

# De Morgan's Laws

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

is a general rule. It's always true for any propositions  $p$  and  $q$ . This is one of De Morgan's Laws.

The other is:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

# De Morgan's Laws

Example:  $\neg(p \wedge q) \equiv \neg p \vee \neg q$

$p$	$q$	$\neg p$	$\neg q$	$\neg p \vee \neg q$	$p \wedge q$	$\neg(p \wedge q)$
T	T	F	F	F	T	F
T	F	F	T	T	F	T
F	T	T	F	T	F	T
F	F	T	T	T	F	T

Every line is the same!  
So these expressions are equivalent.

# De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

```
if (!(front != null && value > front.data))
    front = new ListNode(value, front);
else {
    ListNode current = front;
    while (current.next != null && current.next.data < value))
        current = current.next;
    current.next = new ListNode(value, current.next);
}
```

# De Morgan's Laws

$$\neg(p \wedge q) \equiv \neg p \vee \neg q$$

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

```
!(front != null && value > front.data)
```

≡

```
front == null || value <= front.data
```

**You've been using these for a while!**

# Law of Implication

Implications are hard.

AND/OR/NOT make more intuitive sense to me...

can we rewrite implications using just ANDs ORs and NOTs?

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

One approach: think “when is this implication false?” then negate it (you might want one of DeMorgan’s Laws!)

# Law of Implication

Implications are hard.

AND/OR/NOT make more intuitive sense to me...

can we rewrite implications using just ANDs ORs and NOTs?

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Seems like we might want  $\neg(p \wedge \neg q)$

$$\neg p \vee q$$

Seems like a reasonable guess.

So is it true? Is  $\neg p \vee q \equiv p \rightarrow q$ ?

# Law of Implication

$$\neg p \vee q \equiv p \rightarrow q$$

$p$	$q$	$p \rightarrow q$	$\neg p$	$\neg p \vee q$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Every line is the same!  
So these expressions are equivalent.

# Properties of Logical Connectives

We've derived two facts about logical connectives.

There's a lot more. A LOT more.

The next slide is a list of a bunch of them...

Most of these are much less complicated than the last two, so we won't go through them in detail.

DO NOT freak out about how many there are. We will always provide you the list on the next slide (no need to memorize).

# Properties of Logical Connectives

These identities hold for all propositions  $p, q, r$

- Identity

- $p \wedge T \equiv p$
- $p \vee F \equiv p$

- Domination

- $p \vee T \equiv T$
- $p \wedge F \equiv F$

- Idempotent

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

- Commutative

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

- Associative

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- Distributive

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- Absorption

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

- Negation

- $p \vee \neg p \equiv T$
- $p \wedge \neg p \equiv F$

- DeMorgan's Laws

- $\neg(p \vee q) \equiv \neg p \wedge \neg q$
- $\neg(p \wedge q) \equiv \neg p \vee \neg q$

- Double Negation

- $\neg\neg p \equiv p$

- Law of Implication

- $p \rightarrow q \equiv \neg p \vee q$

- Contrapositive

- $p \rightarrow q \equiv \neg q \rightarrow \neg p$

# Using Our Rules

WOW that was a lot of rules.

Why do we need them? Simplification!

Let's go back to the "law of implication" example.

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

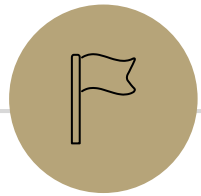
When is the implication true? Just "or" each of the three "true" lines!

$$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q)$$

Also seems pretty reasonable

So is  $(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv (\neg p \vee q)$

i.e. are these both alternative representations of  $p \rightarrow q$ ?



# Our First Proof



# Our First Proof

We could make another truth table (you should! It's a good exercise)

But we have another technique that is nicer.

Let's try that one

Then talk about why it's another good option.

We're going to give an iron-clad guarantee that:

$$(p \wedge q) \vee (\neg p \wedge q) \vee (\neg p \wedge \neg q) \equiv \neg p \vee q$$

i.e. that this is another valid "law of implication"

# Our First Proof

This will be a long proof! Longer than most of the ones on homeworks. I'm starting with a hard one so you see all the tricks.

This process will be easier if we change variables, we're going to show

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv \neg a \vee b$$

How do we write a proof?

It's not always plug-and-chug...we'll be highlighting strategies throughout the quarter.

To start with:

Make sure we know what we want to show...

# Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv$$

None of the rules look like this

Practice of Proof-Writing:

**Big Picture**...WHY do we think this might be true?

The last two "pieces" came from the  $\equiv (\neg a \vee b)$  vacuous proof lines...maybe the " $\neg a$ " came from there? Maybe that **simplifies** down to  $\neg a$

# Let's apply a rule

$$(\neg a \wedge b) \vee (\neg a \wedge \neg b)$$

The law says:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

$$(\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv \neg a \wedge (b \vee \neg b)$$

# Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv$$

None of the rules look like this

Practice of Proof-Writing:

**Big Picture**...WHY do we think this might be true?

The last two "pieces" came from the vacuous proof lines...maybe the " $\neg a$ "  $\equiv (\neg a \vee b)$  came from there? Maybe that **simplifies** down to  $\neg a$

# Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Set ourselves an intermediate goal.  
Let's try to simplify those last two  
pieces

**Associative law**

Connect up the things we're working on.

$$\equiv (\neg a \vee b)$$

# Our First Proof

$$\begin{aligned}(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) &\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)] \\ &\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]\end{aligned}$$

Set ourselves an intermediate goal.  
Let's try to simplify those last two  
pieces

## Distributive law

We think  $\neg a$  is important, let's isolate it.

$$\equiv (\neg a \vee b)$$

# Our First Proof

$$\begin{aligned}(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) &\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)] \\ &\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)] \\ &\equiv (a \wedge b) \vee [\neg a \wedge \mathbf{T}]\end{aligned}$$

Set ourselves an intermediate goal.  
Let's try to simplify those last two  
pieces

**Negation**

Should make things simpler.

$$\equiv (\neg a \vee b)$$

# Our First Proof

$$\begin{aligned}(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) &\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)] \\ &\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)] \\ &\equiv (a \wedge b) \vee [\neg a \wedge \mathbf{T}] \\ &\equiv (a \wedge b) \vee [\neg a]\end{aligned}$$

Set ourselves an intermediate goal.  
Let's try to simplify those last two  
pieces

**Identity**

Should make things simpler.

$$\equiv (\neg a \vee b)$$

# Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Stay on target:

$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$

We met our intermediate goal.

$$\equiv (a \wedge b) \vee [\neg a \wedge \mathbf{T}]$$

Don't forget the final goal!

$$\equiv (a \wedge b) \vee [\neg a]$$

We want to end up at  $(\neg a \vee b)$

If we apply the distribution rule,

We'd get a  $(\neg a \vee b)$

$$\equiv (\neg a \vee b)$$

# Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Stay on target:

$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$

We met our intermediate goal.

$$\equiv (a \wedge b) \vee [\neg a \wedge \mathbf{T}]$$

Don't forget the final goal!

$$\equiv (a \wedge b) \vee [\neg a]$$

We want to end up at  $(\neg a \vee b)$

$$\equiv [\neg a] \vee (a \wedge b)$$

If we apply the distribution rule,

We'd get a  $(\neg a \vee b)$

**Commutative**

Make the expression look exactly like the law (more on this later)

$$\equiv (\neg a \vee b)$$

# Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Stay on target:

$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$

We met our intermediate goal.

$$\equiv (a \wedge b) \vee [\neg a \wedge \mathbf{T}]$$

Don't forget the final goal!

$$\equiv (a \wedge b) \vee [\neg a]$$

We want to end up at  $(\neg a \vee b)$

$$\equiv [\neg a] \vee (a \wedge b)$$

If we apply the distribution rule,

$$\equiv (\neg a \vee a) \wedge (\neg a \vee b)$$

We'd get a  $(\neg a \vee b)$

**Distributive**

Creates the  $(\neg a \vee b)$  we were hoping for.

$$\equiv (\neg a \vee b)$$

# Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Stay on target:

We met our intermediate goal.

Don't forget the final goal!

We want to end up at  $(\neg a \vee b)$

If we apply the distribution rule,

We'd get a  $(\neg a \vee b)$

$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$

$$\equiv (a \wedge b) \vee [\neg a \wedge \mathbf{T}]$$

$$\equiv (a \wedge b) \vee [\neg a]$$

$$\equiv [\neg a] \vee (a \wedge b)$$

$$\equiv (\neg a \vee a) \wedge (\neg a \vee b)$$

$$\equiv (a \vee \neg a) \wedge (\neg a \vee b)$$

$$\equiv \mathbf{T} \wedge (\neg a \vee b)$$

**Commutative**

$$\equiv (\neg a \vee b)$$

Make the expression look exactly like the law (more on this later)

**Negation**

Simplifies the part we want to disappear.

# Simplify $\top \wedge (\neg a \vee b)$ to $(\neg a \vee b)$

These identities hold for all propositions  $p, q, r$

- Identity

- $p \wedge \top \equiv p$
- $p \vee \text{F} \equiv p$

- Domination

- $p \vee \top \equiv \top$
- $p \wedge \text{F} \equiv \text{F}$

- Idempotent

- $p \vee p \equiv p$
- $p \wedge p \equiv p$

- Commutative

- $p \vee q \equiv q \vee p$
- $p \wedge q \equiv q \wedge p$

- Associative

- $(p \vee q) \vee r \equiv p \vee (q \vee r)$
- $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$

- Distributive

- $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
- $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

- Absorption

- $p \vee (p \wedge q) \equiv p$
- $p \wedge (p \vee q) \equiv p$

- Negation

- $p \vee \neg p \equiv \top$
- $p \wedge \neg p \equiv \text{F}$

# Our First Proof

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)]$$

Stay on target:

We met our intermediate goal.

Don't forget the final goal!

We want to end up at  $(\neg a \vee b)$

If we apply the distribution rule,

We'd get a  $(\neg a \vee b)$

$$\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)]$$

$$\equiv (a \wedge b) \vee [\neg a \wedge \mathbf{T}]$$

$$\equiv (a \wedge b) \vee [\neg a]$$

$$\equiv [\neg a] \vee (a \wedge b)$$

$$\equiv (\neg a \vee a) \wedge (\neg a \vee b)$$

$$\equiv (a \vee \neg a) \wedge (\neg a \vee b)$$

$$\equiv \mathbf{T} \wedge (\neg a \vee b)$$

$$\equiv (\neg a \vee b) \wedge \mathbf{T}$$

$$\equiv (\neg a \vee b)$$

**Commutative** followed by **Identity**

Look exactly like the law, then apply it.

We're done!!!

# Commutativity

We had the expression  $(a \wedge b) \vee [\neg a]$

But before we applied the distributive law, we switched the order...why?

The law says  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$

not  $(q \wedge r) \vee p \equiv (q \vee p) \wedge (r \vee p)$

So **technically** we needed to commute first.

By week 3, we'll skip this step. For now, we're doing two separate steps. Remember this is the "training wheel" stage. The point is to be careful.

# More on Our First Proof

We now have an ironclad guarantee that

$$(a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) \equiv (\neg a \vee b)$$

Hooray! But we could have just made a truth-table. Why a proof?

Here's one reason.

Proofs don't *just* give us an ironclad guarantee. They're also an explanation of *why* the claim is true.

The key insight to our simplification was “the last two pieces were the vacuous truth parts – the parts where *p* was false”

That's in there, *in the proof*.

# Our First Proof

$$\begin{aligned} (a \wedge b) \vee (\neg a \wedge b) \vee (\neg a \wedge \neg b) &\equiv (a \wedge b) \vee [(\neg a \wedge b) \vee (\neg a \wedge \neg b)] && \text{Associative} \\ &\equiv (a \wedge b) \vee [\neg a \wedge (b \vee \neg b)] && \text{Distributive} \\ &\equiv (a \wedge b) \vee [\neg a \wedge \text{T}] && \text{Negation} \\ &\equiv (a \wedge b) \vee [\neg a] && \text{Identity} \\ &\equiv [\neg a] \vee (a \wedge b) && \text{Commutative} \\ &\equiv (\neg a \vee a) \wedge (\neg a \vee b) && \text{Distributive} \\ &\equiv (a \vee \neg a) \wedge (\neg a \vee b) && \text{Commutative} \\ &\equiv \text{T} \wedge (\neg a \vee b) && \text{Negation} \\ &\equiv (\neg a \vee b) \wedge \text{T} && \text{Commutative} \\ &\equiv (\neg a \vee b) && \text{Identity} \end{aligned}$$

The last two terms are "vacuous truth" – they simplify to  $\neg a$

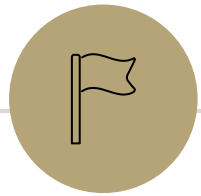
$a$  no longer matters in  $a \wedge b$  if  $\neg a$  automatically makes the expression true.

# More on Our First Proof

With practice (and quite a bit of squinting) you can see not just the ironclad guarantee, but also the reason why something is true.

That's not easy with a truth table.

Proofs can also communicate intuition about why a statement is true. We'll practice extracting intuition from proofs more this quarter.



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## Modifying Implications

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# Converse, Contrapositive

**Implication:**

$$p \rightarrow q$$

**Contrapositive:**

$$\neg q \rightarrow \neg p$$

If I don't have my umbrella, then it is not raining.

**Converse:**

$$q \rightarrow p$$

**Inverse:**

$$\neg p \rightarrow \neg q$$

If it is not raining, then I don't have my umbrella.

If it's raining, then I have my umbrella.

If I have my umbrella, then it is raining.

**How do these relate to each other?**

$p$	$q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T						
T	F						
F	T						
F	F						

# Converse, Contrapositive

**Implication:**

$$p \rightarrow q$$

**Converse:**

$$q \rightarrow p$$

**Contrapositive:**

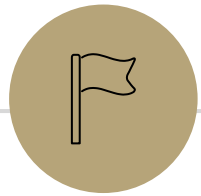
$$\neg q \rightarrow \neg p$$

**Inverse:**

$$\neg p \rightarrow \neg q$$

An **implication** and its **contrapositive**  
have the same truth value!

$p$	$q$	$p \rightarrow q$	$q \rightarrow p$	$\neg p$	$\neg q$	$\neg p \rightarrow \neg q$	$\neg q \rightarrow \neg p$
T	T	T	T	F	F	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T



**More Practice**

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# Contrapositive

We showed  $p \rightarrow q \equiv \neg q \rightarrow \neg p$  with a truth table. Let's do a proof.

Try this one on your own. Remember

1. Know what you're trying to show.
2. Stay on target – take steps to get closer to your goal.

Hint: think about your tools.

There are lots of rules with AND/OR/NOT,  
but very few with implications...

# Contrapositive

$$p \rightarrow q \equiv \neg p \vee q$$

$$\equiv q \vee \neg p$$

$$\equiv \neg\neg q \vee \neg p$$

$$\equiv \neg q \rightarrow \neg p$$

Law of Implication

Commutativity

Double Negation

Law of Implication

All of our rules deal with ORs and ANDs, let's switch the implication to just use AND/NOT/OR.

And do the same with our target

It's ok to work from both ends. In fact it's a very common strategy!

Now how do we get the top to look like the bottom?

Just a few more rules and we're done!

# Work from both ends, but...

...make sure at the end, if you read from top-to-bottom, every step makes sense.

When proving an equivalence you must:

1. Start with the left side (or right side)
2. Modify what you had in the last step (using an equivalence)
3. Derive the right side (or left side if you started with the right)

You may **not** start with the equivalence you're trying to show, and simplify to something "obviously true."

More on why later in the quarter, but tl;dr for now is you can't use your goal as a starting assumption (it's what you're trying to show! If you knew it, no need to write a proof).

# Todo

## Tonight:

Take a look at HW1 that's posted on the course website

## Thursday:

Go to section (in-person) to practice with the material we learned in lecture

## Soon:

Form a study group! Threads to organize on the Ed board.

Complete CC1 and CC2 (due Friday)