Homework 6: Structural Induction

Due date: Wednesday August 6th at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus. In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.

We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. You are allowed to have longer explanations, but explanations significantly longer than necessary may receive deductions.

To help with formatting of English proofs, we've published a style guide on the website containing some tips. **Unless otherwise noted in a problem, all proofs must be English proofs.** You should not have numbered steps (e.g., you should not be doing an inference proof.)

Finally, be sure to read the grading guidelines for more information on what we're looking for.

You must use structural induction for problems 2 and 3.

1. Stringing Things Together

For each of the following, write a recursive definition of the set of strings satisfying the given properties. Briefly justify (2-4 sentences per part) that your solution is correct. You do not need to mention the exclusion rule.
All problems have relatively simple solutions (e.g., at most 4 basis steps and recursive steps). We may deduct for solutions which are not simple (but you do **not** need the simplest solution).

(a) Binary strings with an odd number of 0s.

(b) Binary strings where every 0 is followed by an even number of 1s.

(c) Binary strings in the form $x0y$ where x, y are binary strings and y is the reversal of x.

2. Walk the walk, talk the talk

Let S be a subset of $\mathbb{Z} \times \mathbb{Z}$ defined recursively as:

Basis Step: $(1, 0) \in S$ and $(0, 1) \in S$
Recursive Step: if $(a, b) \in S$ and $(c, d) \in S$, then $(a, b) + (c, d) = (a + c, b + d) \in S$.

We claim that $S = (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}) \setminus \{(0, 0)\}$; that is, S is the set of integer coordinates in the upper right quadrant (excluding the origin). It is easy to show that any such point is in S: if (x, y) is in this quadrant, then $x, y \geq 0$, meaning that:

$$(x, y) = x(1, 0) + y(0, 1) = (1, 0) + \cdots + (1, 0) + (0, 1) + \cdots + (0, 1)$$

where we add x copies of $(1, 0)$ and y copies of $(0, 1)$. This point is in S by the additive recursive step. To show that points not in the upper right quadrant are not in the set S, we will prove the following claim by structural induction.

Prove: For every $(a, b) \in S$, we have $a \geq 0$ and $b \geq 0$, but also $a > 0$ or $b > 0$.

3. The Leaves Don't Fall Far From The... `Tree`

In this problem, we'll use a definition of trees that looks a little different from the one we saw in class.

Basis Step: $(\texttt{null}, \bullet, \texttt{null})$ is a `tree`.
Recursive Step: If L, R are `trees` then $(L, \bullet, R)$ is also a `tree`

We will also use the following recursively defined function for nodes:

$$\texttt{nodes}((\texttt{null}, \bullet, \texttt{null})) = 1$$
$$\texttt{nodes}((L, \bullet, R)) = 1 + \texttt{nodes}(L) + \texttt{nodes}(R) \qquad \text{for two arbitrary trees } L, R$$

Show that every tree has an odd number of nodes. In other words, show that for all trees t, $\texttt{nodes}(t)$ is odd.

4. Brewing Battle

Harry and Draco are locked in a magical competition in Professor Snape's dungeon. Before them lies a powerful enchanted potion cauldron, which will brew a rare Elixir of Triumph—but only for one wizard. The cauldron requires a perfectly balanced amount of two rare ingredients: mandrake roots and unicorn hairs.

After closely observing the cauldron, Harry notices that it remembers how many of each ingredient it has received since the last successful brew. Once someone brews the Elixir, the cauldron resets to its initial state: requiring r mandrake roots and g unicorn hairs. If both requirements are met *exactly*, the potion is brewed and the cauldron resets. If only one requirement is met but the other is not, the potion fizzles, the ingredients are discarded, and the cauldron resets with no Elixir produced.

The cauldron is currently in state (k, k)—that is, it needs the same number of mandrake roots and unicorn hairs. Harry and Draco take turns adding ingredients. On their turn, a player must choose *one type* of ingredient (either mandrake roots or unicorn hairs), and must add *at least one* unit of it. They may not mix ingredient types during a single turn.

Formally, from state (a, b), a player may move to any of:

- $(a - i, b)$ where $1 \leq i \leq a$, or
- $(a, b - j)$ where $1 \leq j \leq b$

The goal is to be the one who adds the final ingredients that brings the cauldron to state $(0, 0)$ and brews the Elixir.

You (Harry) are feeling generous and let Draco go first.

(a) Using induction, prove that in any brewing duel where the initial state is (a, a) for $a > 0$, **the second player** (Harry) can always force a win and brew the potion.

Be sure to explicitly and clearly define a predicate $P()$! We **strongly** recommend your predicate includes a phrase like "It is not my turn" or "the second player can". The predicate you define should only take in one input.

(b) Describe your winning strategy (i.e., describe how Harry should put in ingredients based on Draco's moves in order to win the potion, assuming Harry goes **second**). A strategy would be something like: "If Draco puts in i unicorn hairs, then I will..."

5. Bijections?

Determine if the following functions are (1) one-to-one and (2) onto. For each claim: provide a proof if true, otherwise give a counterexample (you must also explain why the counterexample works).

(a) $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ f((x, y)) = x + y$.

(b) $g : \mathbb{Z} \to \{-1, 1\} \times \mathbb{N},\ g(x) = (\text{sgn}(x), |x|)$. Here, we define the function:

$$\text{sgn}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

(c) $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$, $h(x, y) = (4y + 5, 5x + 4)$.

6. A Fun Proof [Extra Credit]

Prove that $(\frac{n}{e})^n \leq n!$ for all $n \in \mathbb{N}$.

Some hints/guidelines:

- You must use proof templates we've covered in class.
- You may use the fact that if $\log(x) \leq \log(y)$, then $x \leq y$ without proof.
- You may find it helpful to review basic calculus concepts, such as Riemann sums and taking the integral of log functions.

7. Feedback [Extra Credit]

Answer these questions on the separate gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?