Homework 3: English Proofs

Version 2: Updated 7/10 10 PM. Question 7 (Divides) asks you to prove that if $6 \mid (x + 7)$, then x is odd.
Due date: Wednesday, July 16th at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on both the clarity and accuracy of your work. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we are expecting. You are allowed to have longer explanations, but explanations significantly longer than necessary may receive deductions.

Be sure to read the grading guidelines on the assignments page for more information on what we're looking for.

1. Oddly Even

In this problem we will analyze the statement:

For all integers n, if $n + 3$ is even, then n^2 must be odd.

(a) Translate the claim into predicate logic. Let your domain of discourse be integers, and use the following predicates:

- $\texttt{Even}(x) := \exists k(x = 2k)$
- $\texttt{Odd}(x) := \exists k(x = 2k + 1)$

which were first defined using English in lecture 6.

(b) Prove the claim is true by writing a proof.

2. The Oddacity

When direct proofs fail, our logical equivalences can come to the rescue. Consider the statement

For every integer k, if $k^4 + 5$ is even then k is odd.

Proving this directly is not easy (try it for yourself to see!). Instead, we will prove the contrapositive of this statement.

(a) Write the contrapositive of the given statement (in English).

(b) Write a proof by contrapositive of the given statement.

3. Something is wrong here...

3.1. Dance or Die!

You have been kidnapped by pirates and are offered a choice: either beat the captain in a dance-off or walk the plank. (Specifically, if you beat the captain in a dance-off, you will not walk the plank.)

Assume the following things to be true.

- The captain gets to choose the style of dance used for the dance-off. Both participants will perform the same style.
- You are classically trained in ballet, tap, and contemporary dance, and are confident that you can beat the captain in these dance styles regardless of her skill. However, you will certainly lose in any other dance style.

- The captain tells you that she knows ballet and tap, and she would not lie.
- The captain will only pick a style of dance that she knows.

Consider the following (incorrect) proof of the claim: ”You will not walk the plank.” .

1. The captain gets to choose the style of dance, and will only pick a style of dance that she knows.
2. The captain only knows ballet and tap, so the dance-off must be ballet or tap.
3. You can beat the captain in both ballet and tap, so you will win in either scenario.
4. We conclude that you will win the dance-off.
5. Because you win the dance-off, you will not walk the plank.

(a) Identify the most significant error in the proof and discuss why this step is incorrect. Sentences have been labeled to easily refer back to specific portions of the proof.

3.2. What’s wrong with this proof?

Consider the following statement:

For all real numbers x, y, if $x^2 = y^2$, then $x = y$.

and the following spoof (incorrect proof) of the statement:

> Let x, y be arbitrary real numbers and suppose that $x^2 = y^2$. Taking the square root, we obtain $x = y$. Thus, our claim holds.

(a) Why is the above proof incorrect?

Here, let’s try again. This must be correct this time, right?

> Let x and y be arbitrary real numbers and suppose that $x = y$. Squaring both sides of the equation, we obtain $x^2 = y^2$.

(b) Again, why is the above proof incorrect?

(c) Is the original statement true or false? If the statement is true, write a correct proof. If it is false, provide a counterexample.

4. Disprove!

In this problem you will use proof by counterexample to disprove claims in a wider set of problems than what we’ve seen in class. For each part, provide a counterexample that disproves the given claim. Remember to provide one counterexample, not a class of them.

(a) If x and y are two prime numbers greater than 19, then $|x - y| > 2$.
Note: You may state that an integer is prime without proving it.

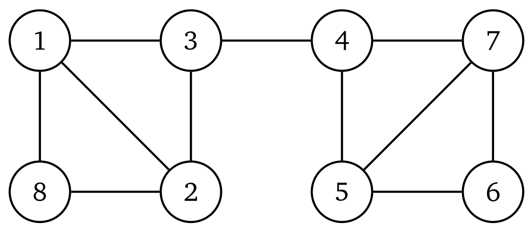
(b) **Buggy Algorithm**: Your friend claims “For every linked list whose nodes have distinct integer values, the function below will remove the node with the target value if such a node exists in the list.” Disprove this claim with a counter-example.

For your counterexample, provide a LinkedList and a target value. Briefly explain why the code fails on your example.

```
public void removeNode(ListNode head, int target) {
    ListNode curr = head;
    while(curr.next != null) {
        if (curr.next.val != target) {
            curr = curr.next;
        } else {
            curr.next = curr.next.next;
        }
    }
}
```

(c) *You may have seen graphs like the one below in an introductory programming course. In case you haven't: we call the circles in the graph "vertices" and the lines connecting two vertices "edges". No further understanding of graphs is required to complete this problem.*

Husky Edge Coloring: Your friend claims "There exists no way to color the edges of the graph below using colors purple, gold, and pink such that every vertex is connected to at most one edge of each color [for your counterexample, provide a coloring of the edges of the graph].
For this part you do not have to explain your counter-example.



5. The Campus Cat Chase

There's a mischievous campus cat that keeps sneaking into the lecture halls of your academic building. To make sure it's safe before the building closes, you're trying to find it and bring it outside. The building has four adjacent classrooms labeled C_1, C_2, C_3, C_4, laid out in a line.

Here's how the chase works:

1. At the beginning, the cat is in one of the four rooms — you don't know which.
2. On each turn, you may check one room by opening its door:
   - If the cat is in that room, you catch it.
   - Otherwise, you must close the door.
3. Immediately after you check and close a door, the cat moves into an adjacent room (either left or right, if possible).
   - The cat cannot stay in the same room.
   - For example, if the cat is in C_1, it must move to C_2; if it's in C_4, it must move to C_3.

Now that you know how the chase works, let's figure out a strategy that guarantees you'll catch the cat.

(a) Prove using cases that if you check C_2 and then C_3, then either you have caught the cat or it is now in C_1 or C_3.

(b) Prove using cases that if the cat is currently in C_1 or C_3, then if you check C_3 and then C_2, you always catch the cat.

(c) Use parts (a) and (b) to find a sequence of 4 room checks that always catches the cat. Give a 1–2 sentence explanation justifying why your sequence works.

6. Kinda Iffy

Consider the following statement:

For all integers x, x^2 is even if and only if $x^4 + 9$ is odd.

Proving an "if and only if" claim often involves proving two separate implications ($\rightarrow$).

(a) Translate the claim into predicate logic using **only** $\wedge$, $\vee$, $\neg$, and $\rightarrow$ symbols; you may **not** use the biconditional ($\leftrightarrow$) operator. Your final predicate logic statement should involve exactly two implications.

Again, you may use the predicates $\texttt{Even}(x) := \exists k(x = 2k)$ and $\texttt{Odd}(x) := \exists k(x = 2k + 1)$.

(b) Write a proof of the given statement. Your proof should involve two sub-proofs – one for each implication identified in part (a).
Hint: Try taking the contrapositive for one of your implications.

7. Divides

Write an English proof to show that if 6 divides $(x + 7)$ (i.e. $6 \mid (x + 7)$) for an integer x, then x is odd. Recall that English proofs don't have domains of discourse, so you need to state the types for your variables when you introduce them.

8. Feedback [Extra Credit]

Answer these questions on the separate gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?

9. The ONE RING [Extra Credit]

You will submit this question to the separate gradescope box for "homework 3 extra credit."

Seven Hobbits found a stash of 100 remnants of the ONE RING after it had been allegedly destroyed by Frodo. The Hobbits are coincidentally named Alice, Emma, Evan, Lisa, Parker, Rushil, and Zareef. They agreed to split the ring pieces using the following rules:

- The first Hobbit in alphabetical order becomes the leader of the Hobbits.
- The leader of Hobbits proposes how to split the remnants. For example, they might say "Alice gets all 100 pieces of the ring, all other Hobbits get none" or "Alice, Emma, Evan, Lisa, and Parker each gets 20 pieces, and Rushil and Zareef get none."
- All Hobbits (including the leader) vote for or against the proposal.

- If 2 or more Hobbits disagree to the proposal, the Ring will possess the leader for being too greedy. Once possessed, the leader no longer participates in the splitting process (they do not vote, and they cannot receive ring remnants in the split).
- Otherwise, the ring remnants will be split as proposed.

Thus, the first round Alice is the leader: if her proposal has been rejected by at least 2 Hobbits, she'd be possessed and Emma becomes the leader, etc; If Alice, Emma, Evan, Lisa, Parker, and Rushil get possessed, then Zareef will become the leader and keep all the ring remnants.

The Hobbits' first priority is not to be possessed, since being possessed means they will forever be away from power. If they don't get possessed by the ring, they will try to get as many pieces of ring as possible for themselves, since the used-to-be nice Hobbits are corrupted at the sight of the omnipotent ring. Finally, in a scary world like this, every Hobbit tries to overpower others, so if they can get the same amount of ring remnants for agreeing and disagreeing with the proposal, they will disagree with the proposal and cause the leader to be possessed.

Assuming that all 7 Hobbits are smart (and greedy and all aware of the others' intelligence and greediness), what will happen?

Your solution should indicate which Hobbits will be possessed by the ring, and how many ring remnants each of the remaining Hobbits will receive.