Homework 2: Predicate Logic and Alternate Notation

Due date: Wednesday July 9 at 11:59 PM
If you work with others (and you should!), remember to follow the collaboration policy outlined in the syllabus.
In general, you are graded on your work's clarity and accuracy. Your solution should be clear enough that someone in the class who had not seen the problem before would understand it.
We sometimes describe approximately how long our explanations are. These are intended to help you understand approximately how much detail we expect. You can have longer explanations, but explanations significantly longer than necessary may receive deductions.

Be sure to read the grading guidelines on the assignments page.

1. Circuit du Soleil

In this problem, we'll construct two propositions in terms of the variables p, q, r and then use these propositions to build a circuit that computes a binary function $M(p, q, r)$.

(a) Give a propositional logic formula containing only the variables q and r which evaluates to true when q is true and evaluates to r when q is false.

(b) Give a propositional logic formula containing only the variables p and q which evaluates to $\neg p$ when q is true and evaluates to false when q is false.

(c) Now consider the binary function $M(p, q, r)$ which is defined as:

$$M(p, 0, r) := \neg r$$

$$M(p, 1, r) := \neg p$$

Draw a circuit that takes p, q, r as input, uses only AND, OR, and NOT gates, and outputs $M(p, q, r)$. Your gates should not take more than two inputs.
Your answer for this part **must** combine your answers from (a) and (b)!

2. Red Carpet Reasoning

(a) Timothée Chalamet attends the Met Gala whenever Kylie Jenner attends and the Knicks are not playing.

(i) Convert this sentence to propositional logic (as on homework 1, ensure you're assigning variables to **atomic propositions**, not compound ones).

(ii) Take the contrapositive symbolically, and simplify so that any $\neg$ signs are next to atomic propositions (i.e. only single variables). You are not required to show work for this part.

(iii) Translate the contrapositive back to English.

(iv) Compare your English sentence from (iii) to the original implication. Do they mean the same thing? (Just say "yes" or "no" here)

(b) Gracie Abrams plays the piano at night only if the windows are open.

Repeat steps (i)-(iv) from (a) for this sentence.

3. Two of a Kind

(a) Translate the Boolean Algebra expression $(X \cdot Y)' + (X' + Y')' \cdot [(X \cdot Y) + (X \cdot Y') + X']$ to Propositional Logic. Use the variables x and y to represent the propositions $X = 1$ and $Y = 1$, respectively.

(b) Prove that your solution to (a) is a tautology using a chain of equivalences.

4. A Tale of Two $\forall$

Consider the following two expressions:

$$\forall x(\texttt{P}(x) \lor \texttt{Q}(x)) \qquad \forall x(\texttt{P}(x)) \lor \forall x(\texttt{Q}(x))$$

(a) Give one domain of discourse and definitions of `P` and `Q` such that these expressions **are not** equivalent. Explain why your examples work (1-2 sentences).

(b) Give one domain of discourse and definitions of `P` and `Q` such that these expressions **are** equivalent. Explain why your examples work (1-2 sentences).

5. Professors and Protein Bars

We've written a few sentences about protein bars and professors.

Let the domain of discourse be protein bars and professors. Use the following definitions of these predicates for this problem:

- `ProteinBar(x)`: x is a protein bar
- `Professor(x)`: x is a professor
- `Tasty(x)`: x is tasty
- `PlusTwenty(x)`: x has more than 20 grams of protein
- `ContainsDriedFruit(x)`: x contains dried fruit
- `ContainsNuts(x)`: x contains nuts
- `Likes(x, y)`: x likes y

5.1. Round One

Translate the following observations into English. Your translations should take advantage of "restricting the domain" to make more natural translations when possible, but you should not otherwise simplify the expression before translating. Specifically, we have these requirements for translations in this problem:

- You must not use variable names in your English translation (e.g., don't say "for every x...")
- For every quantified variable where one or more of the predicates can be interpreted as a domain restriction, you must use at least one of them to make your translation more natural. So with a domain of discourse of all integers, $\forall x([\texttt{Even}(x) \land \texttt{Prime}(x)] \to \texttt{IsEqual}(x, 2))$ could be translated as "For every even integer, if it is prime it is equal to 2" or "Every prime and even integer is equal to 2" but could not be translated as "For every integer, if the integer is prime and even then it is equal to 2."

(a) $\forall x \forall y(\texttt{Professor(x)} \land \texttt{ProteinBar(y)} \land \texttt{Tasty(y)} \to \texttt{Likes(x, y)})$

(b) $\neg \exists x(\texttt{ProteinBar(x)} \land \texttt{PlusTwenty(x)} \land \neg \texttt{ContainsNuts(x)}) \land \forall x(\texttt{ProteinBar(x)} \to \texttt{ContainsDriedFruit(x)})$

5.2. Round Two

You realize that the first sentence (i.e., part a) is false. State the negation of (a) in English. You should simplify the negation so that the English sentence is natural. ”Simplifying” here mainly means that negations should be applied only to individual predicates. For example, you must say “x is not prime and it is not even” rather than “it is not the case that x is prime or x is even.”

5.3. Round Three

Translate the following statement into predicate logic, specifying and defining any new predicates you use (which you will need to do!). Then provide a domain of discourse where the statement is true and another domain of discourse where the statement is false.

Also include 1-2 sentences for each domain for why the statement has the truth value it does.

If you wish to make extra assumptions about the world you may do so as long as you state them.

(a) There is a professor that will buy any protein bar that is on sale.

6. Nested Quantifiers

Fix your domain of discourse to be all real numbers. Define the following predicates:

- Positive(x) : x is positive
- Integer(x) : x is an integer
- Natural(x) : x is a natural number
- Even(x) : x is even
- Odd(x) : x is odd
- LessThan(x, y) : $x < y$ (note the order)

In this problem, an example of something you might give for a “scenario” might be “There is at least one number in the domain of discourse that is not prime”. You can assume that the given statements are true facts about numbers.

(a) Your friend tried to translate “Every integer is even or odd” and got

$$\forall x(\texttt{Integer}(x) \wedge [\texttt{Even}(x) \vee \texttt{Odd}(x)])$$

The translation is incorrect. Give a correct translation, and describe a scenario in which your translation and their translation evaluate to different truth values.

(b) Your friend tried to translate “There is a natural number smaller than all positive numbers” and got

$$\exists x \forall y([\texttt{Natural}(x) \wedge \texttt{Positive}(y)] \rightarrow \texttt{LessThan}(x, y))$$

The translation is incorrect. Give a correct translation, and describe a scenario in which your translation and their translation evaluate to different truth values.

(c) Translate the sentence “For every number x, there is a number y such that for every number z: x is less than y or z is less than y” into predicate logic.

7. Nope

For this question, our domain of discourse is "People", and you may use the following predicates:

- $\texttt{Student}(x)$: x is a student
- $\texttt{Taking311}(x)$: x is taking CSE 311
- $\texttt{FinishHW}(x)$: x will submit their homework on time
- $\texttt{StudyingLeetcode}(x)$: x is studying Leetcode

(a) Translate the following sentence to predicate logic:

"All students taking CSE 311 will finish their homework on time unless they are studying Leetcode."

(b) Negate the predicate logic sentence that you translated above in part (a); give your answer in predicate logic (i.e., symbols not in English). In your final answer, make sure that each $\neg$ symbol is applied only to individual predicates. For example instead of $\neg(\texttt{P}(x) \wedge \texttt{Q}(x))$, you should simplify to $(\neg\,\texttt{P}(x) \vee \neg\,\texttt{Q}(x))$.
Note: For this part, a completely correct final answer will receive full credit, but we encourage you to show work so we can give partial credit.

8. The Truth Will Set You Free

Below is the truth table for the propositional expression $\neg p \wedge (q \vee r)$.

p	q	r	$\neg p \wedge (q \vee r)$
T	T	T	F
T	T	F	F
T	F	T	F
T	F	F	F
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	F

The DNF of it is $(\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$. Use propositional equivalences to show that the DNF is equivalent to the original expression.
That is, show $(\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) \equiv \neg p \wedge (q \vee r)$.

We've done the first few steps of the proof for you.

$$\begin{aligned}
(\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r) &\equiv (\neg p \wedge [q \wedge r]) \vee (\neg p \wedge [q \wedge \neg r]) \vee (\neg p \wedge [\neg q \wedge r]) && \text{Associative law 3x} \\
&\equiv \neg p \wedge ((q \wedge r) \vee (q \wedge \neg r)) \vee (\neg p \wedge [\neg q \wedge r]) && \text{Distributivity} \\
&\equiv \neg p \wedge ((q \wedge r) \vee (q \wedge \neg r) \vee (\neg q \wedge r)) && \text{Distributivity} \\
&\equiv \ldots
\end{aligned}$$

9. Feedback [Extra Credit]

Answer these questions on the separate gradescope box for this question.

Please keep track of how much time you spend on this homework and answer the following questions. This can help us calibrate future assignments and future iterations of the course, and can help you identify which areas are most challenging for you.

- How many hours did you spend working on this assignment (excluding any extra credit questions, if applicable)? Report your estimate to the nearest hour.
- Which problem did you spend the most time on?
- Any other feedback for us?