Quiz Section 10: Final Review

Task 1 – Irregularity

Show that the language $L = \{0^a 1^b : b \geqslant 2a \geqslant 0\}$.

a) Show that L is context free.

b) Show that the language L is irregular. Suppose, ________________________, that some DFA M recognizes L.

Define $S =$ _______________. Since S has infinitely many strings and M has finitely many states, there must be two different strings _______________ and _______________ in S that go to the same state p in M, where ________________________.

We go by cases, _______________.

Case 1 Suppose that _______________.

Consider appending _______________ to both _______________.

_______________ $\in L$ because:

_______________ $\notin L$ because:

Since _______________ were taken to the same state p, _______________ must be take to the same state q by M. Since _______________ $\in L$, q must be accepting. But _______________ $\notin L$, so M would incorrectly accept a string not in L, which is a contradiction!

Case 2 Suppose that _______________.

Consider appending _______________ to both _______________.

_______________ $\notin L$ because:

_______________ $\in L$ because:

Since _______________ were taken to the same state p, _______________ must be take to the same state q by M. Since _______________ $\in L$, q must be accepting. But _______________ $\notin L$, so M would incorrectly accept a string not in L, which is a contradiction!

Since we have a contradiction in all cases, we have a contradiction overall.

Therefore, there's no DFA that recognizes L, showing that L is irregular.

Task 2 – Relations

a) Let $A = \{1, 2, 3, 4\}$. Let R be the relation on A defined by $\{(x, y) : x \in A \text{ and } y \in A \text{ and } y \equiv_3 2x\} \cup \{(x, x) : gcd(x, 3) = 2\}$. Draw R as a directed graph on the vertices below.

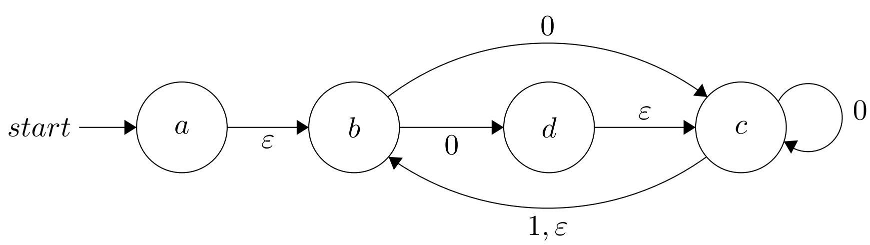
b) Draw the transitive reflexive closure of R below:

c) Compute R^2. Draw your answer as a directed graph.

d) Let $A = \mathcal{P}(3)$. Define $U \subseteq A \times A$ by $(X, Y) \in U$ iff $X \subseteq Y$. Determine if this relation has the properties of reflexivity, symmetry, antisymmetry, and transitivity.

Task 3 – NFA to DFA

Let $\Sigma = \{0, 1\}$. Use the algorithm from lecture to convert the following NFA to a DFA. Label each state of your DFA with the set of NFA states it corresponds to. Do not simplify or minimize the resulting DFA.



Task 4 – DFA/REGEXP/CFG

For the following language, construct a DFA, Regular Expression, and CFG for it. $A = \{w \in \{0,1\}^* :$ the number of 0's minus the number of 1's in w is divisible by 3$\}$.

Task 5 – Languages

a) Construct a regular expression that represents binary strings where no occurence of 11 is followed by a 0.

b) Convert the regular expression "$1(0 \cup 11)^*$" to an NFA using the algorithm from lecture. You may skip adding ε-transitions for concatenation if they are obviously unnecessary, but otherwise, you should follow the construction from lecture.

c) Construct a CFG that represents the following language: $\{1^x 2^y 3^y 4^x : x, y \geqslant 0\}$.

d) Create a DFA that recognizes all binary strings that **either** have every occurrence of a 1 immediately followed by a 0 **or** contain at least two 0s **but not both**.

Task 6 – Structural Induction

Consider the S defined recursively as follows:

Basis: $1 \in S$.
Recursive Step: If $x \in S$ and $a \in \{0, 1\}$, then $xa \in S$.

and the set T of strings that start with a 1, which is defined formally as follows:

$$T := \{x \in \{0,1\}^* : \exists y \in \{0,1\}^* \, (x = 1 \bullet y)\}$$

Use structural induction to prove that $\forall x \in S \, (x \in T)$.

Task 7 – Review: Strong Induction

Define a sequence of positive integers a_n with $n \geqslant 1$ as follows:

$$
\begin{aligned}
a_1 &= 1 \\
a_2 &= 2 \\
a_3 &= 5 \\
a_n &= 3a_{n-1} + 4a_{n-2} + a_{n-3} && \text{for } n \geqslant 4
\end{aligned}
$$

Prove that $a_n \geqslant 4^{n-2}$ for all integers $n \geqslant 1$.

Task 8 – A Set Theory Interlude

1. Prove or disprove: For all sets A, B, C if $A \cap C = B \cap C$ then $A = B$.

2. Prove or disprove: For all sets A, B, C if $A \cup C = B \cup C$ then $A = B$.

3. Prove or disprove: Let A, B, C be arbitrary sets. For all sets A, B, C if $A \cup C = B \cup C$ and $A \cap C = B \cap C$ then $A = B$.

Task 9 – Powerful Sets

Show that for any set X and any set A with $A \in \mathcal{P}(X)$, there exists a set $B \in \mathcal{P}(X)$ such that $A \cap B = \varnothing$ and $A \cup B = X$.