Quiz Section 6: Midterm Review

Task 1 – Midterm Review: Translation

Let your domain of discourse be all coffee drinks. You should use the following predicates:

- $\texttt{soy}(x)$ is true iff x contains soy milk.
- $\texttt{whole}(x)$ is true iff x contains whole milk.
- $\texttt{sugar}(x)$ is true iff x contains sugar
- $\texttt{decaf}(x)$ is true iff x is not caffeinated.
- $\texttt{vegan}(x)$ is true iff x is vegan.
- $\texttt{KevinLikes}(x)$ is true iff Kevin likes the drink x.

Translate each of the following statements into predicate logic. You may use quantifiers, the predicates above, and usual math connectors like $=$ and $\neq$.

a) Coffee drinks with whole milk are not vegan.

b) Kevin only likes one coffee drink, and that drink is not vegan.

c) There is a drink that has both sugar and soy milk.

Translate the following symbolic logic statement into a (natural) English sentence. Take advantage of domain restriction.

$$\forall x([\texttt{decaf}(x) \wedge \texttt{KevinLikes}(x)] \rightarrow \texttt{sugar}(x))$$

Task 2 – Remains to be Seen

Prove the following for all integers $x, y \in \mathbb{Z}$:

$$\text{If } x \equiv_6 1 \text{ and } y \equiv_5 3 \text{ then } 5x + 3y \equiv_{15} 14.$$

a) Let your domain be integers. Write the predicate logic of this claim.

b) Write a formal proof for this claim.

Task 3 – Formal Proof

Show that for any integers x, y, z, where $x + y$ is even and $y + z$ is odd that $x - z$ is odd.

a) Let your domain be integers. Write the predicate logic of this claim. Define predicates Odd and Even!

b) Write a formal proof for this claim.

Task 4 – Bernoulli's Inequality

Prove that for $x \geqslant 0$ and even integer $n \geqslant 0$,

$$(1 + x)^n \geqslant 1 + nx.$$

Note that this is Bernoulli's inequality with slightly adjusted bounds for simplicity
We will use the definition of even to write $n = 2j$ and then induct on j.

Task 5 – Strong Induction

Consider the function $a(n)$ defined for $n \geqslant 1$ recursively as follows.

$$a(1) = 1$$

$$a(2) = 3$$

$$a(n) = 2a(n-1) - a(n-2) \text{ for } n \geqslant 3$$

Use strong induction to prove that $a(n) = 2n - 1$ for all $n \geqslant 1$.

Task 6 – Extra Practice: In Harmony with Ordinary Induction

Define

$$H_i = \sum_{j=1}^{i} \frac{1}{j} = 1 + \frac{1}{2} + \cdots + \frac{1}{i}$$

The numbers H_i are called the *harmonic* numbers.
Prove that $H_{2^n} \geqslant 1 + \frac{n}{2}$ for all integers $n \geqslant 0$.

Task 7 – Extra Practice: Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in year n is described by the function $f(n)$:

$$
\begin{aligned}
&f(0) = 0 \\
&f(1) = 1 \\
&f(n) = 2f(n-1) - f(n-2) \text{ for } n \geqslant 2
\end{aligned}
$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

Task 8 – Extra Practice: Donald Duck

Donald duck challenges you to a game. The rules are simple; there are a bunch of cookies on the table, you and Donald will take turns eating either 1, 2 or 3 cookies. Whoever eats the last cookie wins. You will go first. Prove that no matter how many cookies there are on the table, if the number of cookies on the table is divisible by 4 then there is a way for Donald Duck to always win.