Quiz Section 5: Number Theory and Induction

Review

5 Steps to an Induction Proof: To prove $\forall n \in \mathbb{N}\ P(n)$ (or equivalently $\forall n \geqslant 0\ P(n)$ for $n \in \mathbb{Z}$).

1. "Let $P(n)$ be ⟨fill in⟩. We will show that $P(n)$ is true for every $n \in \mathbb{N}$ (or equivalently integer $n \geqslant 0$) by induction."
2. "Base Case:" Prove $P(0)$
3. "Inductive Hypothesis: Suppose $P(k)$ is true for some arbitrary integer $k \geqslant 0$"
4. "Inductive Step:" Prove that $P(k+1)$ is true.

   Use the goal to figure out what you need.
   Make sure you are using I.H. and point out where you are using it.
   (Don't assume $P(k+1)$!)
5. "Conclusion: The claim follows by induction"

Task 1 – This is Really Mod

Let n and m be positive integers satisfying $n \mid m$.

Write an **English proof** of the following claim: for any integers a and b, if $a \equiv_m b$, then $a \equiv_n b$.

Task 2 – Extended Euclidean Algorithm Practice

a) Find the multiplicative inverse of y of 7 mod 33. That is, find y such that $7y \equiv 1(\text{mod } 33)$, You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \leqslant y < 33$.

b) Now solve $7z \equiv 2 \pmod{33}$ for all of its integers solutions z.

Task 3 – Formal Induction

We can use mathematical induction to prove that $P(n)$ holds for integers $n \geqslant b$ via the following rule:

Induction

$$\frac{P(b) \quad \forall n\,(P(n) \to P(n+1))}{\therefore \quad \forall n\,((n \geqslant b) \to P(n))}$$

In other words, if we know that $P(b)$ holds and we know that, whenever $P(n)$ holds, so does $P(n+1)$, then it must be the case that $P(n)$ is true for all integers $n \geqslant b$.

To gain some familiarity with this rule (called "`induction`" in Cozy), let's do a proof...

Prove, by induction, that $n^2 + n$ is even, or in other words, that $2 \mid n^2 + n$ holds for all integers $n \geqslant 0$.

Write a **formal** proof that the claim holds in cozy. https://tinyurl.com/cozysec5task3

Task 4 – Induction with Equality

a) For all $n \in \mathbb{N}$, prove that $\sum_{i=0}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

Task 5 – Induction with Inequality

Prove that $6n + 6 < 2^n$ for all integers $n \geqslant 6$.

Task 6 – Strong Induction

Consider the function $a(n)$ defined for $n \geqslant 1$ recursively as follows.

$$a(1) = 1$$

$$a(2) = 3$$

$$a(n) = 2a(n-1) - a(n-2) \text{ for } n \geqslant 3$$

Use strong induction to prove that $a(n) = 2n - 1$ for all integers $n \geqslant 1$.