CSE 311 Section 5

Number Theory & Induction

Announcements & Reminders

- HW4 due yesterday @ 11:00PM on Gradescope
  - Use late days if you need to!
  - Make sure you tagged pages on gradescope correctly

- HW5
  - Releases tonight
  - Due Wednesday 5/7 @11:00 PM

Extended Euclid

Problem 2 – Extended Euclidean Algorithm

a) Find the multiplicative inverse y of $7 \bmod 33$. That is, find y such that $7y \equiv 1\ (mod\ 33)$. You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \le y < 33$.

Problem 2 – Extended Euclidean Algorithm

a) Find the multiplicative inverse y of 7 mod 33. That is, find y such that $7y \equiv 1 \ (mod\ 33)$. You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \le y < 33$.

First, we find the gcd:

```
gcd(33,7)   = gcd(7,5)      33 = 4 • 7 + 5
            = gcd(5,2)      7  = 1 • 5 + 2
            = gcd(2,1)      5  = 2 • 2 + 1
            = gcd(1,0)      2  = 2 • 1 + 0
```

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```
33 = 4 • 7 + 5
7  = 1 • 5 + 2
5  = 2 • 2 + 1
2  = 2 • 1 + 0
```

Next, we re-arrange the equations by solving for the remainder:

```
1 = 5 - 2 • 2
2 = 7 - 1 • 5
5 = 33 - 4 • 7
```

Problem 2 – Extended Euclidean Algorithm

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First, we find the gcd:

gcd(33,7)	= gcd(7,5)	$33 = 4 \cdot 7 + 5$
	= gcd(5,2)	$7 = 1 \cdot 5 + 2$
	= gcd(2,**1**)	$5 = 2 \cdot 2 + \mathbf{1}$
	= gcd(**1**,0)	$2 = 2 \cdot \mathbf{1} + 0$

Next, we re-arrange the equations by solving for the remainder:

$\mathbf{1} = 5 - 2 \cdot 2$

$2 = 7 - 1 \cdot 5$

$5 = 33 - 4 \cdot 7$

Now, we backward substitute into the boxed numbers using the equations:

$$
\begin{aligned}
\mathbf{1} &= 5 - 2 \cdot 2 \\
&= 5 - 2 \cdot (7 - 1 \cdot 5) \\
&= 3 \cdot 5 - 2 \cdot 7 \\
&= 3 \cdot (33 - 4 \cdot 7) - 2 \cdot 7 \\
&= 3 \cdot 33 + -14 \cdot 7
\end{aligned}
$$

Problem 2 – Extended Euclidean Algorithm

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$$\begin{aligned} 33 &= 4 \cdot 7 + 5 \\ 7 &= 1 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + \mathbf{1} \\ 2 &= 2 \cdot \mathbf{1} + 0 \end{aligned}$$

Next, we re-arrange the equations by solving for the remainder:

$$\begin{aligned} \mathbf{1} &= 5 - 2 \cdot 2 \\ 2 &= 7 - 1 \cdot 5 \\ 5 &= 33 - 4 \cdot 7 \end{aligned}$$

Now, we backward substitute into the boxed numbers using the equations:

$$\begin{aligned} \mathbf{1} &= 5 - 2 \cdot 2 \\ &= 5 - 2 \cdot (7 - 1 \cdot 5) \\ &= 3 \cdot 5 - 2 \cdot 7 \\ &= 3 \cdot (33 - 4 \cdot 7) - 2 \cdot 7 \\ &= 3 \cdot 33 + -14 \cdot 7 \end{aligned}$$

So, $\mathbf{1} = \mathbf{3} \cdot 33 + \mathbf{-14} \cdot 7$. Thus, $33 \mathbf{- 14} = \mathbf{19}$ is the multiplicative inverse of 7 mod 33

Problem 2 – Extended Euclidean Algorithm

a) Find the multiplicative inverse y of $7 \bmod 33$. That is, find y such that $7y \equiv 1 \ (mod\ 33)$. You should use the extended Euclidean Algorithm. Your answer should be in the range $0 \leq y < 33$.

b) Now, solve $7z \equiv 2 \pmod{33}$ for all of its integer solutions z.

Try this problem with the people around you, and then we'll go over it together!

Problem 2 – Extended Euclidean Algorithm

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If we have $7z \equiv 2 (\text{mod } 33)$, multiplying both sides by 19, we get:

$z \equiv 2 \cdot 19 (\text{mod } 33) \equiv 5 (\text{mod } 33)$.

This means that the set of solutions is $\{5 + 33k \mid k \in Z\}$

Introducing Induction (kind of)

Climb the ladder!

You are scared of heights and there is a prize at the top of a very very tall ladder.

You do not want to climb this ladder…

Climb the ladder!

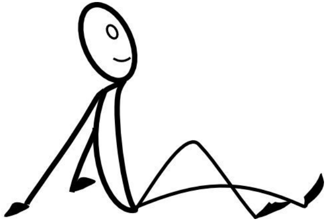
You are scared of heights and there is a prize at the top of a very very tall ladder.

You do not want to climb this ladder…

Lets convince your friend to climb it instead!!!

Climb the ladder!

You Claim: “There are n steps in the ladder. After n steps you will reach the top!” for n>= 1

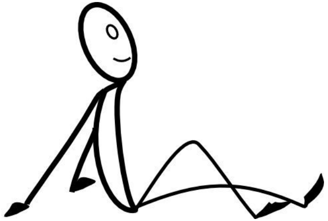


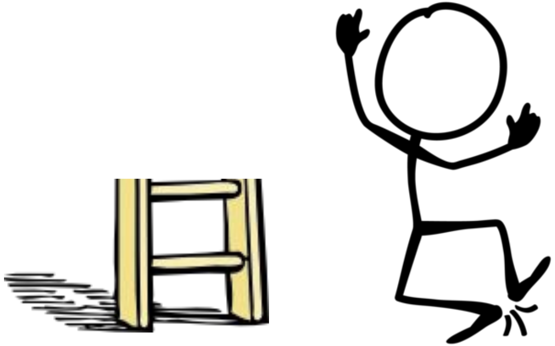
Climb the ladder!

You Claim: “There are n steps in the ladder. After n steps you will reach the top!”

“If we have a ladder with **1** step. I know you can lift your foot so after 1 step you will reach the top of a 1 step ladder!”

“So my claim holds for 1 step!”





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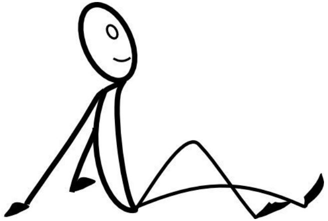

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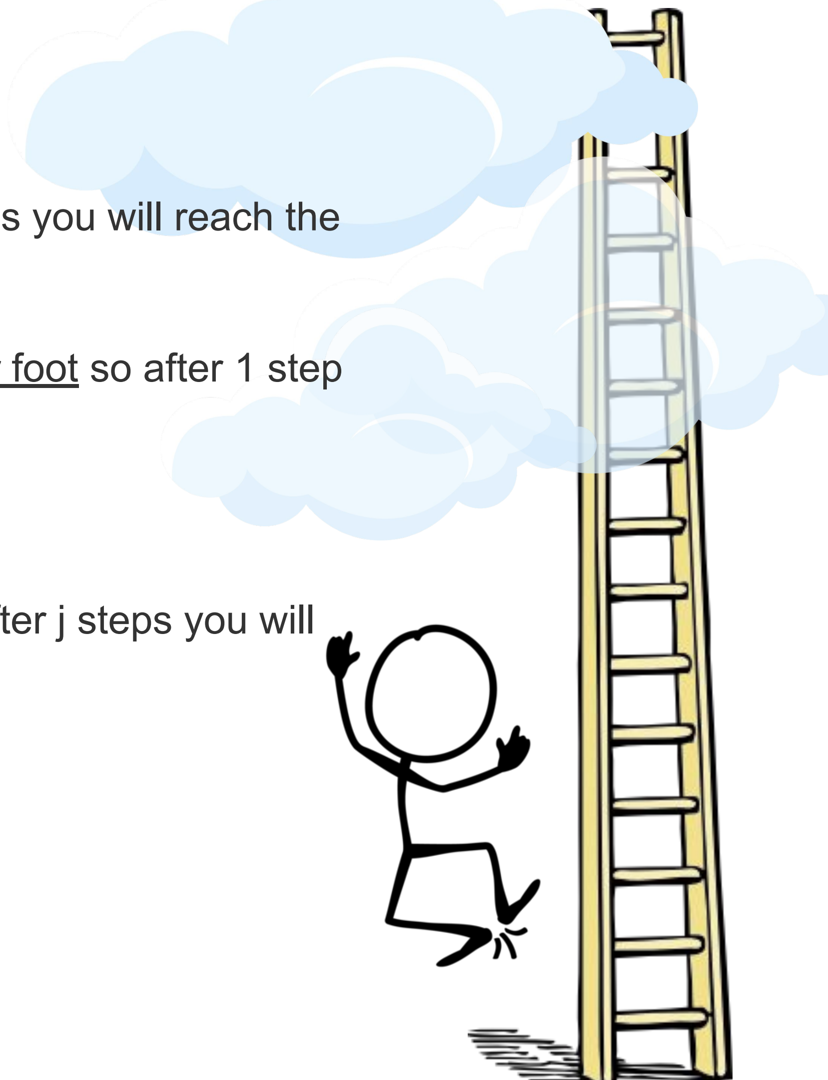
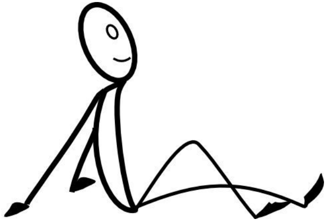




Climb the ladder!

You Claim: “There are n steps in the ladder. After n steps you will reach the top!”

For n >= 1

“If we have a ladder with **1** step. I know you can lift your foot so after 1 step you will reach the top of a 1 step ladder!”

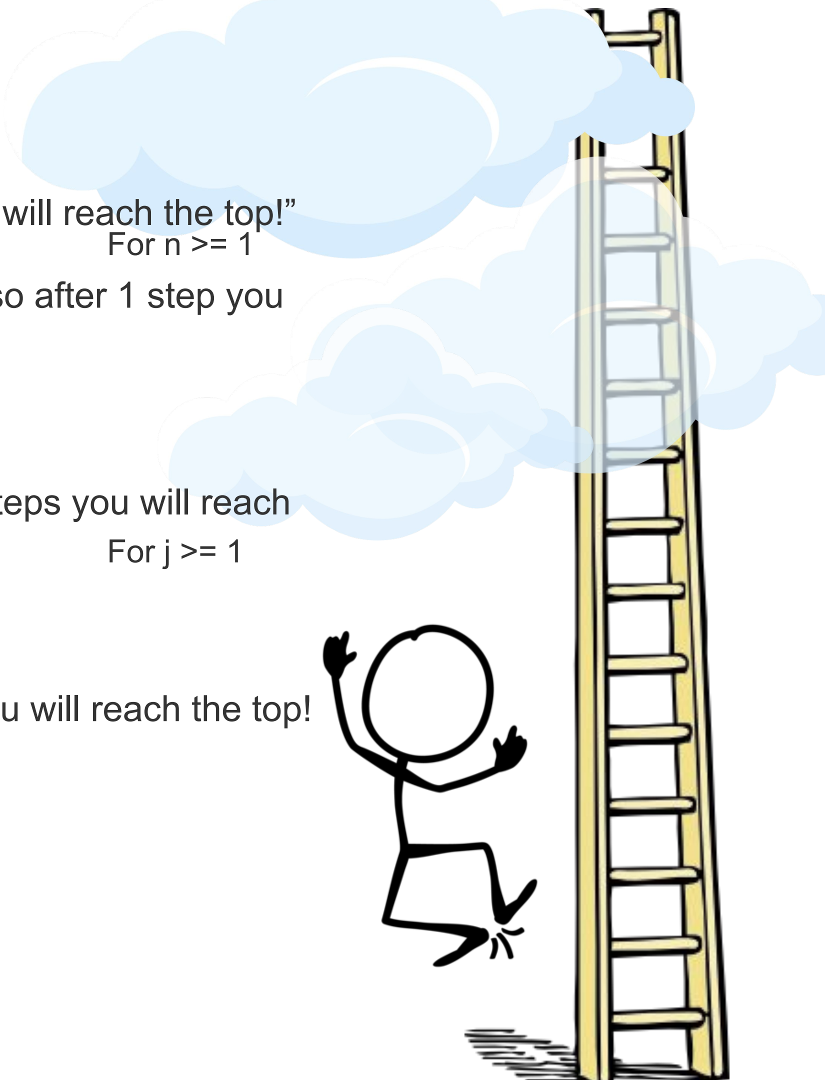
“So my claim holds for 1 step!”

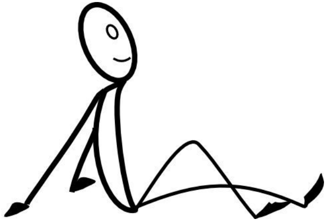
Let’s **suppose** that for an arbitrary number of steps j, after j steps you will reach the top.

For j >= 1

I can prove to you that this claim will still hold for j+1 steps!

Goal: Prove that for **j+1** steps in the ladder, after **j+1** steps you will reach the top!





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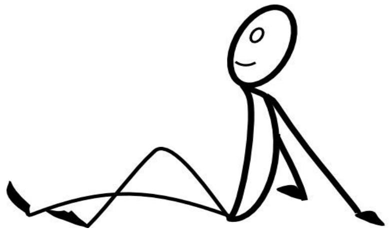

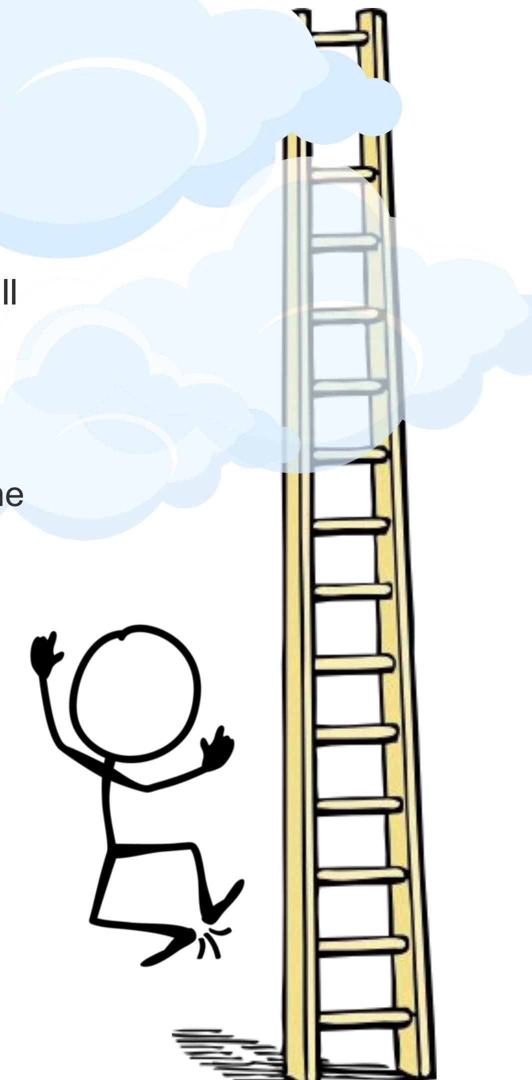

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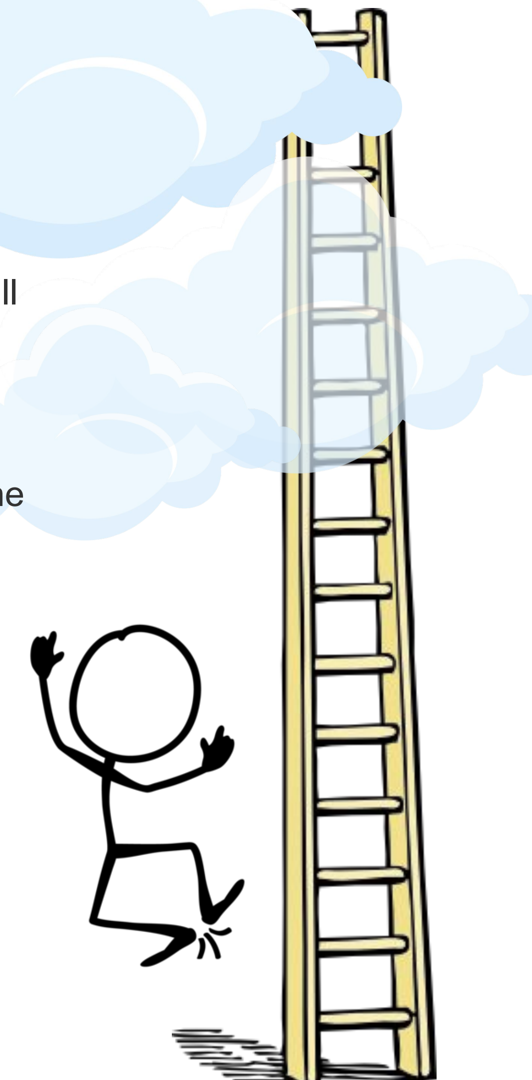

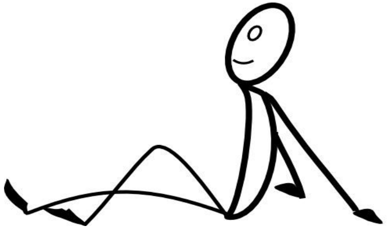

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THE CLAIM HOLDS YOUR FRIEND IS CLIMBING THE LADDER

For n >= 1

WELCOME TO PROOF BY INDUCTION

You Claim: “There are n steps in the ladder. After n steps you will reach the top!” P(n)

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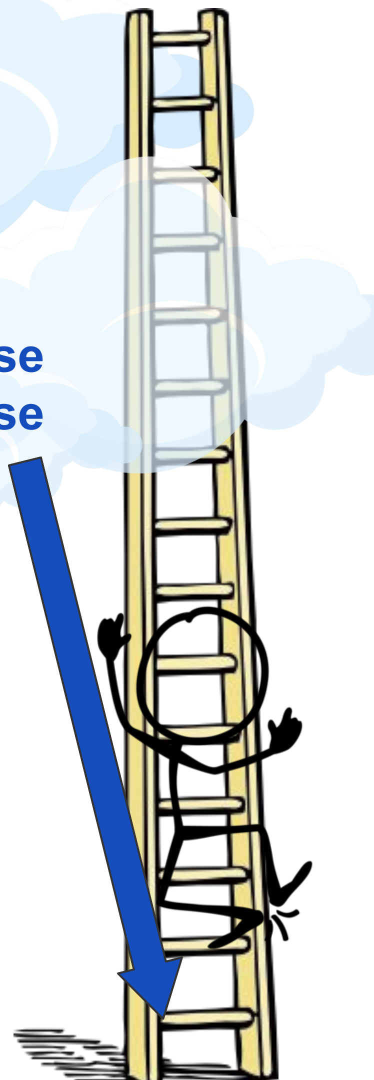

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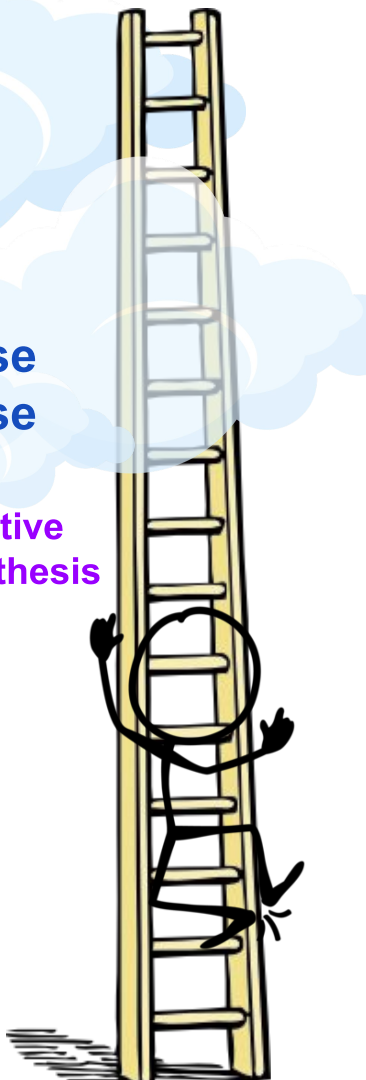

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Inductive Step P(j+1)

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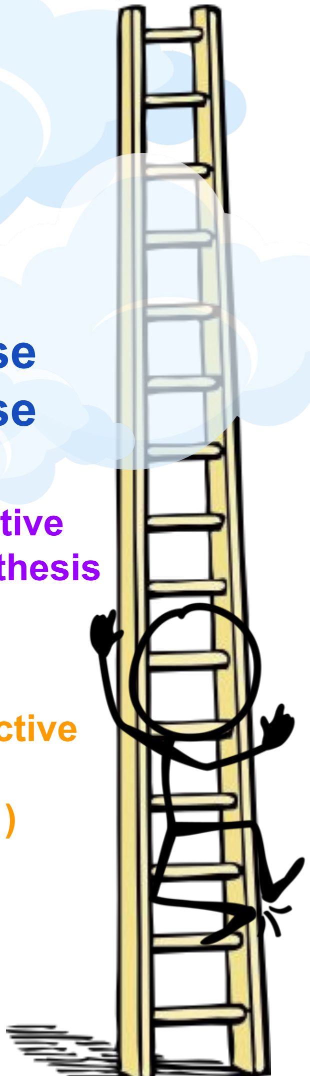

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Inductive Step

Using the IH

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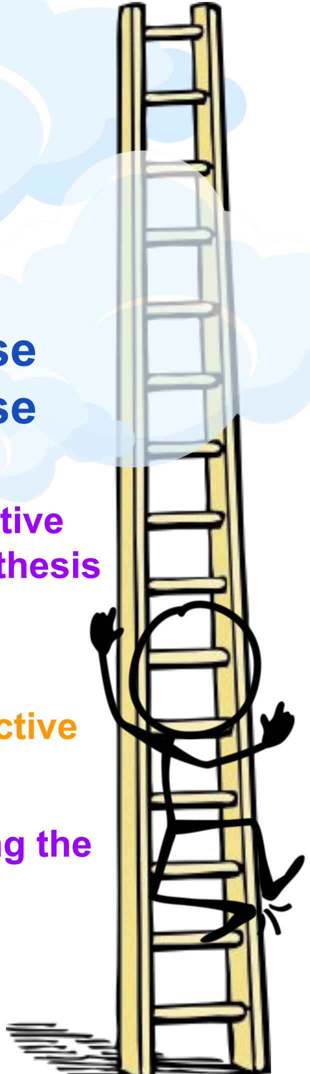

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P(n) holds!

For n >= 1

Induction: How it actually works

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n

Base Case: Show $P(b)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Show $P(k + 1)$ (i.e. get $P(k) \rightarrow P(k + 1)$)

Conclusion: Therefore, $P(n)$ holds for all n by the principle of induction.

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds **for all n** $\in$ N by induction on n

Note: often you will condition n here, like "all natural numbers n" or "$n \geq 0$"

<u>Base Case:</u> Show $P(b)$ is true.

<u>Inductive Hypothesis:</u> Suppose $P(k)$ holds for an arbitrary $k \geq b$.

<u>Inductive Step:</u> Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)

<u>Conclusion:</u> Therefore, $P(n)$ holds **for all n** by the principle of induction.

Match the earlier condition on n in your conclusion!

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P(n) IS A PREDICATE, IT HAS A BOOLEAN VALUE NOT A NUMERICAL ONE

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YOU MUST INTRODUCE AN ARBITRARY VARIABLE IN YOUR IH

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P(n) IS A PREDICATE, IT HAS A BOOLEAN VALUE NOT A NUMERICAL ONE

YOU MUST INTRODUCE AN ARBITRARY VARIABLE IN YOUR IH

START WITH LHS OF EXPRESSION AND END WITH RHS (FOR BASE CASE AND IS)

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Prove that $6n + 6 < 2^n$ for all integers $n \geqslant 6$.

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Base Case $(n = 6)$**:**

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Base Case $(n = 6)$**:**

$$6 \cdot 6 + 6 =$$

Always start with something on the left hand side like this! And go down in equivalences

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Prove that $6n + 6 < 2^n$ for all integers $n \geqslant 6$.

Let $P(n)$ be "$6n + 6 < 2^n$". We will prove $P(n)$ for all integers $n \geqslant 6$ by induction on n

Base Case $(n = 6)$**:**

$$6 \cdot 6 + 6 = 42$$

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Base Case $(n = 6)$**:**

$$\begin{aligned} 6 \cdot 6 + 6 &= 42 \\ &< 64 \end{aligned}$$

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so $P(6)$ holds.

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Base Case $(n = 6)$:

$$\begin{aligned} 6 \cdot 6 + 6 &= 42 \\ &< 64 \\ &= 2^6 \end{aligned}$$

so $P(6)$ holds.

But what if we did this?
Isn't this easier?

$6(6) + 6 < 2^6$

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so $P(6)$ holds.

NO! This is backwards reasoning (WRONG)

$6(6) + 6 < 2^6$

This uses the rule you are **proving**

rather than justifying it using algebra

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Let $P(n)$ be "$6n + 6 < 2^n$". We will prove $P(n)$ for all integers $n \geqslant 6$ by induction on n

Base Case $(n = 6)$**:**

$$\begin{aligned} 6 \cdot 6 + 6 &= 42 \\ &< 64 \\ &= 2^6 \end{aligned}$$

so $P(6)$ holds.

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It's always good to write out the goal!

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$6(k+1) + 6 =$

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$$\begin{aligned} 6(k+1) + 6 &= 6k + 6 + 6 \\ &< 2^k + 6 \qquad \text{[Inductive Hypothesis]} \end{aligned}$$

2^{k+1}

Hint: use the fact that you are proving an inequality!

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So $P(k+1)$ is true.

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Conclusion: $P(n)$ holds for all integers $n \geqslant 6$ by the principle of induction.

Weak Induction

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For all $n \in \mathbb{N}$, prove that $\sum_{i=0}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

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\sum_{i=0}^{k+1} i^2 &= \sum_{i=0}^{k} i^2 + (k+1)^2 && \text{by definition}\\
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Thus, we can conclude that $P(k+1)$ is true.

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Thus, we can conclude that $P(k+1)$ is true.

Conclusion: Therefore, $P(n)$ is true for all $n \in \mathbb{N}$ by induction.

That's All!

I hope you enjoyed it, because I know I did

Written by Aruna & Zareef