Problem Set 5

Due: Wednesday, May 7th by 11:00pm

Instructions

Submit your solutions in Gradescope. Your Gradescope submission should follow these rules:

- Each numbered task should be solved on its own page (or pages). Do not write your name on the individual pages. (Gradescope will handle that.)
- When you upload your pages, make sure each one is **properly rotated**. If not, you can use the Gradescope controls to turn them to the proper orientation.
- Follow the Gradescope prompt to **link tasks to pages**. You do not need to link tasks that you did not include, e.g., Task 3 (if you submitted on Cozy) and Task 7 (extra credit).
- You are not required to typeset your solution, but your submission must be **legible**. It is your responsibility to make sure solutions are readable — we will *not* grade unreadable write-ups.

Task 1 – Modd and Even [12 pts]

In HW4, we have seen many formal proofs and corresponding English proofs. In this question, we will dive straight into an English proof!

Let the domain of discourse be integers. Consider the following claim:

$$\forall a\, \forall b\, ((6 \mid 3a \land 5b \equiv_{15} 10) \to (\mathsf{Even}(a + 2b) \land (3a + b) \equiv_3 2)$$

Write an **English** proof of this claim. In doing so, you are free to use the DivideEqn theorem from Homework 4.

Task 2 – Mod(ulo) Pizza [24 pts]

The CSE311 TA's are exhausted from a hard day of grading, and have decided to take a trip to their favorite restaurant, Mod(ulo) Pizza! However, the owners are rather eccentric, and the pizzas are sliced into what can only be described as an 'incredibly unconventional slice count per pizza.' The TA's need exactly enough pizzas to feed everyone, so they have set up a series of modular equations that will tell them exactly how many to order. Help the CSE311 TA's solve these to figure out exactly how many pizzas they will need to feed everyone, and in exchange, they may save a slice for you...

We say that an equation is in "**standard form**" if it looks like $Ax \equiv_n B$ for some constants A, B, and n. The first equation below is in standard form, but the latter two are *not*.

Solve each of the modular equations by following these steps, showing your work as described next.

1. If the modular equation is *not* in standard form, then **transform** it into standard form.

   Show the sequence of operations, either adding to both sides or simplifying (e.g., algebraically modifying terms on individual sides as done in lecture).

2. **Calculate** *one solution* to the modular equation in standard form using the Extended Euclidean Algorithm.

   Show your work by writing out the sequence of quotients and remainders, the resulting tableau, and the sequence of substitutions needed to calculate the relevant multiplicative inverse. Then, show how multiplying the initial equation on both sides by the multiplicative inverse gives you a solution to the equation.

3. **State** *all integer solutions* to the modular equation in standard form.

   Your answer should be of the form "$x = C + Dk$ for any integer k", where C and D are integers with $0 \leqslant C < D$.

4. If the original modular equation was *not* in standard form, then **transform** the modular equation in standard form back into the original. As done in Step 1, show the sequence of operations. [1]

a) $9x \equiv_{41} 7$

b) $54x - 6 \equiv_{42} 7 - 19x$

c) $30(3x + 2) \equiv_{11} 63 - 3x$

[1] Steps 1 and 4 combined prove that the original equation and the one in standard form have identical solutions.

Task 3 – Winnie the Two [10 pts]

We can use mathematical induction to prove that $P(n)$ holds for integers $n \geqslant b$ via the following rule:

Induction

$$\frac{P(b) \quad \forall n\,(P(n) \to P(n+1))}{\therefore \quad \forall n\,((n \geqslant b) \to P(n))}$$

In other words, if we know that $P(b)$ holds and we know that, whenever $P(n)$ holds, so does $P(n+1)$, then it must be the case that $P(n)$ is true for all integers $n \geqslant b$.

To gain some familiarity with this rule (called "`induction`" in Cozy), let's do a proof...

Prove, by induction, that $3 \mid n^3 + 2n$ holds for all integers $n \geqslant 0$.

Write a **formal** proof that the claim holds.

Submit and check your **formal proof** here:

http://cozy.cs.washington.edu

You can make as many attempts as needed to find a correct answer.

If technical problems prevent you from saving or if you are unable to complete the problem and wish to submit partially complete work for partial credit, you can instead submit in Gradescope. But `cozy` is where we'd like you to submit your answers.

Task 4 – Sum Kind of Wonderful [20 pts]

Prove, by induction, that

$$\sum_{i=0}^{n}(11(12)^i + 2) = (12)^{n+1} + 2n + 1$$

holds for all integers $n \geqslant 0$.

Write an **English** proof, following the template given in lecture.

Task 5 – to $\infty \wedge$ beyond! a 311 induction space odyssey [20 pts]

Aruna is the chief astronaut on the O.L.I.V.E. spacecraft on a crucial space voyage with her co-astronauts Danielle, Donovan, Emma, **and you, the chief theoretician**. You are peacefully flying along when you suddenly encounter Andre, a disgruntled alien flying an evil UFO. Jealous of your logic skills, at 12:04 AM, he launches a space attack on the O.L.I.V.E., releasing 2^t fighter drones every tth minute past midnight. Aruna starts up the O.L.I.V.E.'s onboard anti-drone defense system immediately; the system can eliminate $t!$ drones every tth minute past midnight. However, she is unsure if the system will be able to fend off Andre's exponentially increasing fighter drones forever. She urgently asks you to provide an induction proof that the number of drones that the system can eliminate at time t ($t!$ drones) will always be higher than the number of drones Andre releases at time t (2^t drones) starting at 12:04 AM (for $t \geqslant 4$). With a clear head and a heart full of courage, you take on the task.

> *Your mission, should you choose to accept it* — Prove, by induction, that $t! > 2^t$ holds for all integers $t \geqslant 4$. Write an English proof, following the template given in lecture in the long-ago year 2025. This proof is of crucial importance to ensuring the integrity of our spacecraft. Chief Theoretician, this is your moment to shine — now is the time to put your 311 skills into action. Does it get any more real than this, here, now? We wish you all the best with your mission. We are all rooting for you. To $\infty \wedge$ beyond!
>
> Yours, Chief Astronaut Aruna
> Your Co-Astronauts – Danielle$^{\heartsuit}$, Donovan$^{\heartsuit}$, Emma
> The O.L.I.V.E. spacecraft
> May 4th, 2311

Note that $t!$ refers to the factorial function, defined such that $0! = 1$, $1! = 1$, $2! = 2 \cdot 1 = 2$, $3! = 3 \cdot 2 \cdot 1 = 6$, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$, $t = t \cdot (t-1) \cdot \ldots \cdot 2 \cdot 1$, etc. Additionally, be sure to explain your reasoning in detail, especially in the inductive step, justifying why each reasoning step is valid.

Task 6 – One Small Step for Recursion, One Strong Leap for Induction [20 pts]

When you first learned **recursion** in CSE 123, a mysterious person gave you the following recursive Java method and claimed that it behaves like the natural-number version of Math.pow():

```
int mysteriousPow(int b, int m) {  /* Assumes: b >= 1 and m >= 0 */
    if (m == 0) {
        return 1;
    } else if (m == 1) {
        return b;
    } else {
        return (b - 1) * mysteriousPow(b, m - 1) + b * mysteriousPow(b, m - 2);
    }
}
```

You wrote some tests and realized that this method might be correct, but you didn't know how to prove it rigorously...until you are taking CSE 311 and learn strong induction! Now, let's try to prove the correctness of this method. Not sure what I mean? Let's put it another way:

Let b be a positive integer. The function $f(m)$ is defined for all integers $m \geqslant 0$ recursively as follows:

$$\begin{aligned} f(0) &= 1 \\ f(1) &= b \\ f(m) &= (b-1) \cdot f(m-1) + b \cdot f(m-2) & \text{if } m \geqslant 2 \end{aligned}$$

Use strong induction to prove that the following holds for all integers $n \geqslant 0$:

$$f(n) = b^n$$

Write an **English** proof, following the template given in lecture.

Task 7 – Extra Credit: Stone By the Company He Keeps [0 pts]

Consider an infinite sequence of positions $1, 2, 3, \ldots$ and suppose we have a stone at position 1 and another stone at position 2. In each step, we choose one of the stones and move it according to the following rule: Say we decide to move the stone at position i; if the other stone is not at any of the positions $i+1, i+2, \ldots, 2i$, then it goes to $2i$, otherwise it goes to $2i+1$.

For example, in the first step, if we move the stone at position 1, it will go to 3 and if we move the stone at position 2 it will go to 4. Note: no matter how we move the stones, they will never be at the same position.

Use induction to prove that, for any given positive integer n, it is possible to move one of the stones to position n. For example, if $n = 7$ first we move the stone at position 1 to 3. Then, we move the stone at position 2 to 5 Finally, we move the stone at position 3 to 7.