Problem Set 2

Due: Wednesday, April 16th by 11:00pm

Instructions

Solutions submission. All parts *except* Task 3 should be submitted in Gradescope.

Task 3 should be submitted on Cozy. (See the instructions at the end of Task 3 for how to do that.) If you are unable to submit in Cozy due to technical problems or if you are unable to complete the problem, you can submit your work on Gradescope (for partial credit in the second case).

Your Gradescope submission should follow these rules:

– Each numbered task should be solved on its own page (or pages). Do not write your name on the individual pages. (Gradescope will handle that.)

– When you upload your pages, make sure each one is **properly rotated**. If not, you can use the Gradescope controls to turn them to the proper orientation.

– Follow the Gradescope prompt to **link tasks to pages**. You do not need to link tasks that you did not include, e.g., Task 7 (extra credit) or Task 3 (if you submitted on Cozy).

– You are not required to typeset your solution, but your submission must be **legible**. It is your responsibility to make sure solutions are readable — we will *not* grade unreadable write-ups.

Task 1 – With Gate Power Comes Gate Responsibility [12 pts]

For this problem, we have invented a new gate, called an "A" gate, which is defined as follows:

r	s	t	$A(r, s, t)$
1	1	1	1
1	1	0	0
1	0	1	1
1	0	0	1
0	1	1	1
0	1	0	0
0	0	1	1
0	0	0	0

Note that this gate has *three inputs*, rather than just two (like `AND` and `OR`) or one (like `NOT`). Also, note that we are using function syntax for an "A" gate. The same could be done for `AND` and `OR` gates. For example, we could write `AND`(r, s) rather than $r \wedge s$ and `OR`(r, s) rather than $r \vee s$.

Demonstrate that we can construct each of the operators below. For each one, first give a formula that is equivalent to that operator but *that uses only As and no other gates*. Second, justify your answer with a truth table containing columns for both the original operator and your formula. If you are using multiple A gates, the output of each one should have its own column in the table.

Hint: It is okay to use the same propositional variable more than once. It is also okay to hard-code inputs of an "A" gate to be a constant 0 or 1.

a) $\neg p$ using *only one* A gate

The answer should be of the form $A(r, s, t)$, where each of r, s, t is replaced by any of p, 0, 1.

b) $p \vee q$ using *only one* A gate

The answer should be of the form $A(r, s, t)$, where each of r, s, t is replaced by any of p, q, 0, 1.

c) $p \wedge q$ using *any number* of A gates

Task 2 – With a Fine-Truth Comb [15 pts]

For each of the following pairs of propositions, use truth tables to determine whether they are equivalent.

Include the full truth table and state whether they are equivalent. (In principle, only one row is needed to show non-equivalence, but please turn in the entire table so that we can give partial credit in case of errors.) Your truth table must include columns for all subformulas.

a) $(P \lor Q) \land (\neg P \lor Q)$ vs. $P \lor Q$

b) $P \oplus Q$ vs. $\neg P \oplus (P \land Q)$

c) $P \to (Q \to R)$ vs. $(P \to \neg Q) \land R$

d) $(P \to Q) \to R$ vs. $(P \lor R) \land (Q \to R)$

Task 3 – Too Cool For Rule [20 pts]

Prove the following assertions using a sequence of logical equivalences such as Absorption, Associativity, Commutativity, Contrapositive, De Morgan, Distributivity, Double Negation, Idempotency, Identity, Law of Implication, and Negation. (Our solutions, put together, use every one of those rules at least once.)

Hints: For equivalences where one side is much longer than the other, a good heuristic is to start with the longer side and try to apply the rules that will shorten it. In some cases, it may work better to work to shorten both sides to the same expression and then combine those two sequences into one.

a) $(\neg P \to P) \to \neg P \equiv \neg P$

b) $(P \to Q) \to R \equiv (\neg P \to R) \land (Q \to R)$

c) $(\neg P \to R) \land (\neg Q \to P) \equiv \neg P \to (Q \land R)$

d) $(\neg Q \to R) \land (Q \to R) \equiv R$ (proof by "simple cases")

e) $(P \lor \neg Q) \land (P \land \neg Q) \equiv P \land \neg Q$ (*Hint*: It is **not** necessary to use Distributivity here.)

> Submit and check your answers to this question here:
>
> http://cozy.cs.washington.edu
>
> You can make as many attempts as needed to find a correct answer.
>
> If technical problems prevent you from saving or if you are unable to complete the problem and wish to submit partially complete work for partial credit, you can instead submit in Gradescope. But cozy is where we'd like you to submit your answers.
>
> Documentation is available on the Cozy homepage, at the the link labelled "Docs" at the top of the page.

Task 4 – Universal Answer Machine [16 pts]

The CSE311 TA's have developed a machine that they think tells them the correct answer to any 4-option multiple choice problem! Unfortunately, they don't know how it works. In order to generate limitless profit (or find out the machine is wrong), help the TA's reconstruct the machine!
Consider the following boolean function C:

a	b	c	d	$C(a, b, c, d)$
1	1	1	1	1
1	1	1	0	1
1	1	0	1	0
1	1	0	0	0
1	0	1	1	0
1	0	1	0	0
1	0	0	1	0
1	0	0	0	0
0	1	1	1	1
0	1	1	0	1
0	1	0	1	0
0	1	0	0	0
0	0	1	1	1
0	0	1	0	1
0	0	0	1	0
0	0	0	0	0

a) Write a Boolean algebra expression for C in canonical DNF form [1].

b) Use equivalences of Propositional Logic to simplify your expression from (a) down to an expression that includes *only 3 gates* (each of which is either AND, OR, or NOT).

You should format your work like an equivalence chain with one expression per line and with the name of the identity applied to produce that line written next to it. However, since we are using Boolean algebra notation, which does not include unnecessary parentheses, you do not need to include lines that apply Associativity, Commutativity, or Identity (you may still cite them for clarity if desired).

c) Write a truth table for your simplified expression from part (b) and confirm that it matches the one used to define C originally. As always, be sure to include all subexpressions as their own columns.

d) Draw your simplified expression from part (b) as a circuit.

[1] With circuits, these are usually called sum-of-products

Task 5 – Welcome to the $\forall\text{LL}\exists n$ School! [16 pts]

Let the domain of discourse be all students and classes at UW. Define the predicates $\text{CS}(x)$ to mean that x majors in CS (presumably x is a student) and $\text{CE}(x)$ to mean that x majors in CE (presumably x is a student). Define the predicates $\text{CSE}(y)$ to mean that y is a CSE class (presumably y is a class) and $\text{MATH}(y)$ to mean that y is a MATH class (presumably y is a class). Define the predicate $\text{Wants}(x, y)$ to mean that x wants to take y (presumably x is a student and y is a class), the predicate $\text{Likes}(x, y)$ to mean that x likes y (presumably x is a student and y is a class), and the predicate $\text{HasToTake}(x, y)$ to mean that x has to take y (presumably x is a student and y is a class).

Translate each of the following logical statements into English. You should not simplify. However, you should use the techniques shown in lecture for producing more natural translations when restricting domains and for avoiding the introduction of variable names when not necessary.

a) $\neg\exists x\,(\text{CS}(x) \wedge \text{CE}(x))$

b) $\exists x\,\big(\text{CS}(x) \wedge \exists y\,(\text{CSE}(y) \wedge \neg\text{HasToTake}(x, y) \wedge \text{Likes}(x, y))\big)$

c) $\forall x\,\big(\text{CE}(x) \rightarrow \exists y\,(\text{MATH}(y) \wedge \text{HasToTake}(x, y))\big)$

d) $\exists x\,\big((\text{CS}(x) \vee \text{CE}(x)) \wedge \forall y\,(\text{CSE}(y) \rightarrow \text{Wants}(x, y))\big)$

Task 6 – The Odd One Out [12 pts]

The following parts use the predicates Odd, Prime, and IsTwo, defined as follows:

$$\begin{aligned}
\text{Odd}(x) &:= \exists k\,(x = 2k + 1)\\
\text{Prime}(x) &:= \neg(\exists j \exists k((jk = x) \wedge (j \neq 1) \wedge (j \neq x))) \wedge (x \geqslant 2)\\
\text{IsTwo}(x) &:= (x = 2)
\end{aligned}$$

Note: Prime will be true for prime numbers $\{2, 3, 5, 7, 11, \ldots\}$ and false for all other inputs (a natural is prime if it has no divisors other than 1 and itself). For this problem, you *do not* need to make your translation sound natural (literal is fine).

a) Let the domain of discourse be all non-negative integers (the naturals: $0, 1, 2, \ldots$).

i) Translate the proposition

$$\forall x\,(\mathsf{Prime}(x) \rightarrow \mathsf{Odd}(x))$$

directly into English. Then, state and justify whether the proposition is true or false (1-2 sentences).

ii) Translate the proposition

$$\forall x\,((\mathsf{Prime}(x) \wedge \neg\,\mathsf{IsTwo}(x)) \rightarrow \mathsf{Odd}(x))$$

directly into English. Then, state and justify whether the proposition is true or false (1-2 sentences).

iii) Translate the proposition

$$(\forall x\,(\mathsf{Prime}(x) \wedge \neg\,\mathsf{IsTwo}(x))) \rightarrow (\forall x\ \mathsf{Odd}(x))$$

directly into English. Then, state and justify whether the proposition is true or false (1-2 sentences).

b) Let the domain of discourse be all non-negative *even* integers (the even naturals: $0, 2, 4, 6, \ldots$).

i) Translate the proposition

$$\forall x\,(\mathsf{Prime}(x) \rightarrow \mathsf{Odd}(x))$$

directly into English. Then, state and justify whether the proposition is true or false (1-2 sentences).

ii) Translate the proposition

$$\forall x\,((\mathsf{Prime}(x) \wedge \neg\,\mathsf{IsTwo}(x)) \rightarrow \mathsf{Odd}(x))$$

directly into English. Then, state and justify whether the proposition is true or false (1-2 sentences).

iii) Translate the proposition

$$(\forall x\,(\mathsf{Prime}(x) \wedge \neg\,\mathsf{IsTwo}(x))) \rightarrow (\forall x\ \mathsf{Odd}(x))$$

directly into English. Then, state and justify whether the proposition is true or false (1-2 sentences).

c) Let P and Q be predicates, and fix a domain of discourse. Given that $\forall x(P(x) \rightarrow Q(x))$ is true, is $\forall x\,(P(x)) \rightarrow \forall x\,(Q(x))$ always true? Justify your answer with 1-2 sentences.

Task 7 – Extra Credit: Coined at the Hip

Five pirates, called Ann, Brenda, Carla, Danielle and Emily, found a treasure of 100 gold coins. On their ship, they decide to split the coins using the following scheme:

- The first pirate in alphabetical order becomes the chief pirate.
- The chief proposes how to share the coins, and all other pirates (except the chief) vote for or against it.
- If 50% or more of the pirates vote for it, then the coins will be shared that way.
- Otherwise, the chief will be thrown overboard, and the process is repeated with the pirates that remain.

Thus, in the first round Ann is the chief: if her proposal is rejected, she is thrown overboard and Brenda becomes the chief, etc; if Ann, Brenda, Carla, and Danielle are thrown overboard, then Emily becomes the chief and keeps the entire treasure.

The pirates' first priority is to stay alive: they will act in such a way as to avoid death. If they can stay alive, they want to get as many coins as possible. Finally, they are a blood-thirsty bunch, if a pirate would get the same number of coins if she voted for or against a proposal, she will vote against so that the pirate who proposed the plan will be thrown overboard.

Assuming that all 5 pirates are intelligent (and aware that all the other pirates are just as aware, intelligent, and bloodthirsty), what will happen? Your solution should indicate which pirates die, and how many coins each of the remaining pirates receives.