Section 08: Solutions

1. Regular Expressions

(a) Write a regular expression that matches base 10 numbers (e.g., there should be no leading zeroes).

Solution:

$$0 \cup ((1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)(0 \cup 1 \cup 2 \cup 3 \cup 4 \cup 5 \cup 6 \cup 7 \cup 8 \cup 9)^*)$$

(b) Write a regular expression that matches all base-3 numbers that are divisible by 3.

Solution:

$$0 \cup ((1 \cup 2)(0 \cup 1 \cup 2)^*0)$$

(c) Write a regular expression that matches all binary strings that contain the substring "111", but not the substring "000".

Solution:

$$(01 \cup 001 \cup 1^*)^*(0 \cup 00 \cup \varepsilon)111(01 \cup 001 \cup 1^*)^*(0 \cup 00 \cup \varepsilon)$$

(d) Write a regular expression that matches all binary strings that do not have any consecutive 0's or 1's.

Solution:

$$((01)^*(0 \cup \varepsilon)) \cup ((10)^*(1 \cup \varepsilon))$$

(e) Write a regular expression that matches all binary strings of the form $1^k y$, where $k \geq 1$ and $y \in \{0,1\}^*$ has at least k 1's.

Solution:

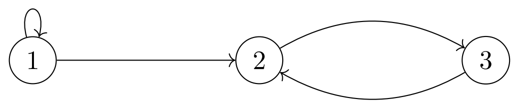
$$1(0 \cup 1)^*1(0 \cup 1)^*$$

Explanation: While it may seem like we need to keep track of how many 1's there are, it turns out that we don't. Convince yourself that strings in the language are exactly those of the form $1x$, where x is any binary string with at least one 1. Hence, x is matched by the regular expression $(0 \cup 1)^*1(0 \cup 1)^*$.

2. Graphs

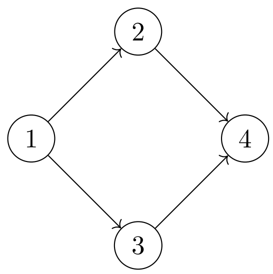
For each graph $G = (V, E)$, draw the graph. Additionally, determine if the graph has a cycle or simple cycle.

(a) $G = (\{1, 2, 3\}, \{(1, 2), (2, 3), (3, 2), (1, 1)\})$ **Solution:**



$2 \to 3 \to 2$ is both a cycle and a simple cycle.

(b) $G = (\{1, 2, 3, 4\}, \{(1, 2), (1, 3), (2, 4), (3, 4)\})$ **Solution:**



There are no cycles in this graph. Remember that we cannot traverse against the direction of an edge.

3. Classifying Functions

For each of these functions, state whether it is one-to-one, onto, both, or neither.

(a) $f : \mathbb{N} \to \mathbb{N}$, $f(x) = x^2$ **Solution:**

For this domain and co-domain, the function is one-to-one, but not onto.
It is one-to-one: For a natural number output (i.e., x^2), the possible inputs are $x, -x$, but only one of those can be a natural number (since natural numbers are all positive).
It is not onto: for example, $5 \in \mathbb{N}$ (i.e., the codomain) but no natural number can be put into the function to produce 5.

(b) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ **Solution:**

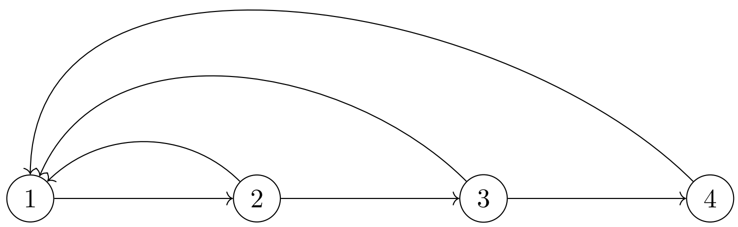
For this domain and co-domain, the function is neither one-to-one nor onto.
It is not one-to-one: 16 can be produced by both $4, -4$ as inputs.
It is not onto, -5 (for example) cannot be produced as output, since real-numbers when squared are always non-negative.

(c) $f : \mathbb{R}^+ \to \mathbb{R}^+$, $f(x) = x^2$, where $\mathbb{R}^+ = \{x : x \in \mathbb{R} \wedge x \geq 0\}$, i.e., the set of non-negative real numbers.
Solution:

For this domain and co-domain, the function is both one-to-one and onto.
It is one-to-one, if $x^2 = y^2$, then $\pm x = \pm y$, but since all inputs to f are non-negative, we must have that $x = y$.
It is onto, for an arbitrary output z, by def of f, $z = x^2$, that x is a real number, choosing x to be positive will give us a valid element of the domain.

4. More Graphs

The following questions in this section are about the following graph:



(a) What are V and E in the graph above? **Solution:**

$V = \{1, 2, 3, 4\}, E = \{(1, 2), (2, 3), (3, 4), (2, 1), (3, 1), (4, 1)\}$

(b) Find a path in the graph that is not simple. **Solution:**

Many correct answers but one example is $3 \to 1 \to 2 \to 1$.

(c) Find a simple path in the graph. **Solution:**

Many correct answers but one example is $1 \to 2 \to 3$.

(d) Find a cycle in the graph that is not simple. **Solution:**

Many correct answers but one example is $3 \to 1 \to 2 \to 1 \to 2 \to 3$.

(e) Find a simple cycle in the graph. **Solution:**

Many correct answers but one example is $1 \to 2 \to 1$.

5. Onto and One-to-One

(a) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is onto but not one-to-one. Be specific. **Solution:**

Let $f(n) = \lfloor \frac{n}{2} \rfloor$. Then f is onto. But f isn't one-to-one because (for example) both 0 and 1 are mapped onto 0.

(b) Give an example of a function $f : \mathbb{N} \to \mathbb{N}$ which is one-to-one but not onto. Be specific.
Solution:

Let $f(n) = 3n$. Observe that $f(n)$ is injective because, if $f(n_1) = f(n_2)$ for arbitrary $f(n_1), f(n_2)$, $3n_1 = 3n_2$ implies $n_1 = n_2$ by algebra. However, $f(n)$ is not onto since $2 \in \mathbb{N}$ but there is no $n \in \mathbb{N}$ such that $3n = 2$.

6. A Bijection Proof

Let A be the set of negative integers, i.e., $A = \{-1, -2, -3, ...\}$; let B be the set of integers at least 10, i.e., $B = \{10, 11, 12, 13, ...\}$. Show that $f : A \to B$ defined by $f(x) = |x| + 9$ is a bijection.

You may use these facts:

- for negative numbers x, y: $|x| = |y| \to x = y$
- for negative numbers $|x| = -x$

Solution:

One-to-One: Let x, y be arbitrary elements of A such that $f(x) = f(y)$. By definitions of f, $|x| + 9 = |y| + 9$. Cancelling the 9's, we have $|x| = |y|$. By the fact $x = y$. Since x, y were arbitrary, we have met the definitions of one-to-one.

Onto: Let y be an arbitrary element in B. Note that y is a positive integer and $y \geq 10$. Consider the value $x = -y + 9$. Observe that x is negative (since y is positive but greater than 9, negating it will give a negative greater in absolute value than 9, leaving a negative result). Since x is negative, when we calculate $|x| + 9$ we get $|x| + 9 = |-y + 9| + 9 = -(-y + 9) + 9 = y$, as required. Further note that the x we chose is in the set A: we already showed it was negative, and since integers are closed under multiplication and addition, x is an integer. Thus x is a value which gives $f(x) = y$; since y was arbitrary, f is onto.

Since f is both onto and one-to-one it is a bijection.