Section MR: Midterm 2 Review

1. Midterm Review: English Proof (Even Steven)

Prove that for all integers k, k(k +3) is even.
Recall that $\texttt{Even}(x) := \exists k(x = 2k)$ and $\texttt{Odd}(x) := \exists k(x = 2k + 1)$

(a) Let your domain be integers. Write this claim in predicate logic.

(b) Write an English proof for this claim.

2. Midterm Review: Proof by Contradiction

(This question was also in Section 5)
Write a proof by contradiction for the following proposition: There exist no integers x and y such that $18x + 6y = 1$.
HINT: Try writing in propositional logic, then negating this statement before writing your proof.

3. Midterm Review: Number Theory

Let p be a prime number at least 3, and let x be an integer such that $x^2\%p = 1$.

(a) Show that if an integer y satisfies $y \equiv 1 \pmod{p}$, then $y^2 \equiv 1 \pmod{p}$. (this proof will be short!)
(Try to do this without using the theorem "Raising Congruences To A Power")

(b) Repeat part (a), but don't use any theorems from the Number Theory Reference Sheet. That is, show the claim directly from the definitions.

(c) From part (a), we can see that $x\%p$ can equal 1. Show that for any integer x, if $x^2 \equiv 1 \pmod{p}$, then $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$. That is, show that the only value $x\%p$ can take other than 1 is $p - 1$.
Hint: Suppose you have an x such that $x^2 \equiv 1 \pmod{p}$ and use the fact that $x^2 - 1 = (x - 1)(x + 1)$
Hint: You may the following theorem without proof: if p is prime and $p \mid (ab)$ then $p \mid a$ or $p \mid b$.

Induction

4. Midterm Review: Induction

For any $n \in \mathbb{N}$, define S_n to be the sum of the squares of the first n positive integers, or

$$S_n = 1^2 + 2^2 + \cdots + n^2.$$

Prove that for all $n \in \mathbb{N}$, $S_n = \frac{1}{6}n(n + 1)(2n + 1)$.

5. Midterm Review: Strong Induction

Robbie is planning to buy snacks for the members of his competitive roller-skating troupe. However, his local grocery store sells snacks in packs of 5 and packs of 7.

Prove that Robbie can buy exactly n snacks for all integers $n \geq 24$

6. Midterm Review: Reversing a Binary Tree (Structural Induction)

Consider the following definition of a (binary) **Tree**.

Basis Step `Nil` is a **Tree**.

Recursive Step If L is a **Tree**, R is a **Tree**, and x is an integer, then $\texttt{Tree}(x, L, R)$ is a **Tree**.

The sum function returns the sum of all elements in a **Tree**.

$$\begin{aligned} \mathsf{sum}(\texttt{Nil}) &= 0 \\ \mathsf{sum}(\texttt{Tree}(x, L, R)) &= x + \mathsf{sum}(L) + \mathsf{sum}(R) \end{aligned}$$

The following recursively defined function produces the mirror image of a **Tree**.

$$\begin{aligned} \mathsf{reverse}(\texttt{Nil}) &= \texttt{Nil} \\ \mathsf{reverse}(\texttt{Tree}(x, L, R)) &= \texttt{Tree}(x, \mathsf{reverse}(R), \mathsf{reverse}(L)) \end{aligned}$$

Show that, for all **Tree**s T that

$$\mathsf{sum}(T) = \mathsf{sum}(\mathsf{reverse}(T))$$

Set Theory

7. Midterm Review: Unioned Intersections (Proof by [Counter]example)

Show that there exists sets A, B, C, D, E, F such that $(A \cap B) \cup (C \cap D) \cup (E \cap F) \not\subseteq (A \cup B) \cap (C \cup D) \cap (E \cup F)$. *(In other words,* ***disprove*** *the claim, "for all sets,* $(A \cap B) \cup (C \cap D) \cup (E \cap F) \subseteq (A \cup B) \cap (C \cup D) \cap (E \cup F)$*).*

8. Midterm Review: Complementary Sets (Proof by Contrapositive)

We want to write a proof for $\overline{A \cup B} \subseteq \overline{A \cap B}$.

(a) Translate $\overline{A \cup B} \subseteq \overline{A \cap B}$ to predicate logic (it should contain an implication!).

(b) Take the contrapositive of the statement in (a).

(c) Write the expression from (b) in set notation.

(d) Write an English proof for the statement in part (c).
(Note: you are effectively doing a proof by contrapositive!)

9. Midterm Review: Power Crossing (Power Sets and Cartesian Products)

Show that $\mathcal{P}(A) \times \mathcal{P}(B) \subseteq \mathcal{P}(A \cup C) \times \mathcal{P}(B \cup C)$.