

Section 07: Solutions

1. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all integers $n \geq 6$. **Solution:**

Let $P(n)$ be “ $6n + 6 < 2^n$ ”. We will prove $P(n)$ for all integers $n \geq 6$ by induction on n

Base Case ($n = 6$): $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so $P(6)$ holds.

Inductive Hypothesis: Assume that $6k + 6 < 2^k$ for an arbitrary integer $k \geq 6$.

Inductive Step: Goal: Show $6(k + 1) + 6 < 2^{k+1}$

$$\begin{aligned} 6(k + 1) + 6 &= 6k + 6 + 6 \\ &< 2^k + 6 && \text{[Inductive Hypothesis]} \\ &< 2^k + 2^k && \text{[Since } 2^k > 6, \text{ since } k \geq 6\text{]} \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

So $P(k) \rightarrow P(k + 1)$ for an arbitrary integer $k \geq 6$.

Conclusion: $P(n)$ holds for all integers $n \geq 6$ by the principle of induction.

2. Induction with Formulas

These problems are a little more difficult and abstract. Try making sure you can do all the other problems before trying these ones.

(a) (i) Show that given two sets A and B that $\overline{A \cup B} = \overline{A} \cap \overline{B}$. (Don't use induction.)

Solution:

Let x be arbitrary. Then,

$$\begin{aligned} x \in \overline{A \cup B} &\equiv \neg(x \in A \cup B) && \text{[Definition of complement]} \\ &\equiv \neg(x \in A \vee x \in B) && \text{[Definition of union]} \\ &\equiv \neg(x \in A) \wedge \neg(x \in B) && \text{[De Morgan's Laws]} \\ &\equiv x \in \overline{A} \wedge x \in \overline{B} && \text{[Definition of complement]} \\ &\equiv x \in (\overline{A} \cap \overline{B}) && \text{[Definition of intersection]} \end{aligned}$$

Since x was arbitrary we have that $x \in \overline{A \cup B}$ if and only if $x \in \overline{A} \cap \overline{B}$ for all x . By the definition of set equality we've shown,

$$\overline{A \cup B} = \overline{A} \cap \overline{B}.$$

(ii) Show using induction that for an integer $n \geq 2$, given n sets $A_1, A_2, \dots, A_{n-1}, A_n$ that

$$\overline{A_1 \cup A_2 \cup \dots \cup A_{n-1} \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_{n-1}} \cap \overline{A_n}$$

Solution:

Let $P(n)$ be “given n sets $A_1, A_2, \dots, A_{n-1}, A_n$ it holds that $\overline{A_1 \cup A_2 \cup \dots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_{n-1}} \cap \overline{A_n}$.” We show $P(n)$ for all integers $n \geq 2$ by induction on n .

Base Case: $P(2)$ says that for two sets A_1 and A_2 that $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$, which is exactly part (a) so $P(2)$ holds.

Inductive Hypothesis: Suppose that $P(k)$ holds for some arbitrary integer $k \geq 2$.

Inductive Step: Let $A_1, A_2, \dots, A_k, A_{k+1}$ be sets. Then by part (a) we have,

$$\overline{(A_1 \cup A_2 \cup \dots \cup A_k) \cup A_{k+1}} = \overline{A_1 \cup A_2 \cup \dots \cup A_k} \cap \overline{A_{k+1}}.$$

By the inductive hypothesis we have $\overline{A_1 \cup A_2 \cup \dots \cup A_k} = \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k}$. Thus,

$$\overline{A_1 \cup A_2 \cup \dots \cup A_k} \cap \overline{A_{k+1}} = (\overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k}) \cap \overline{A_{k+1}}.$$

We’ve now shown

$$\overline{A_1 \cup A_2 \cup \dots \cup A_k \cup A_{k+1}} = \overline{A_1} \cap \overline{A_2} \cap \dots \cap \overline{A_k} \cap \overline{A_{k+1}}.$$

which is exactly $P(k+1)$.

Conclusion $P(n)$ holds for all integers $n \geq 2$ by the principle of induction.

- (b) (i) Show that given any integers a, b , and c , if $c \mid a$ and $c \mid b$, then $c \mid (a + b)$. (Don’t use induction.)

Solution:

Let a, b , and c be arbitrary integers and suppose that $c \mid a$ and $c \mid b$. Then by definition there exist integers j and k such that $a = jc$ and $b = kc$. Then $a + b = jc + kc = (j + k)c$. Since $j + k$ is an integer, by definition we have $c \mid (a + b)$.

- (ii) Show using induction that for any integer $n \geq 2$, given n numbers $a_1, a_2, \dots, a_{n-1}, a_n$, for any integer c such that $c \mid a_i$ for $i = 1, 2, \dots, n$, that

$$c \mid (a_1 + a_2 + \dots + a_{n-1} + a_n).$$

In other words, if a number divides each term in a sum then that number divides the sum.

Solution:

Let $P(n)$ be “given n numbers $a_1, a_2, \dots, a_{n-1}, a_n$, for any integer c such that $c \mid a_i$ for $i = 1, 2, \dots, n$, it holds that $c \mid (a_1 + a_2 + \dots + a_n)$.” We show $P(n)$ holds for all integer $n \geq 2$ by induction on n .

Base Case: $P(2)$ says that given two integers a_1 and a_2 , for any integer c such that $c \mid a_1$ and $c \mid a_2$ it holds that $c \mid (a_1 + a_2)$. This is exactly part (a) so $P(2)$ holds.

Inductive Hypothesis: Suppose that $P(k)$ holds for some arbitrary integer $k \geq 2$.

Inductive Step: Let $a_1, a_2, \dots, a_k, a_{k+1}$ be $k + 1$ integers. Let c be arbitrary and suppose that $c \mid a_i$ for $i = 1, 2, \dots, k + 1$. Then we can write

$$a_1 + a_2 + \dots + a_k + a_{k+1} = (a_1 + a_2 + \dots + a_k) + a_{k+1}.$$

The sum $a_1 + a_2 + \dots + a_k$ has k terms and c divides all of them, meaning we can apply the inductive hypothesis. It says that $c \mid (a_1 + a_2 + \dots + a_k)$. Since $c \mid (a_1 + a_2 + \dots + a_k)$ and $c \mid a_{k+1}$,

by part (a) we have,

$$c \mid (a_1 + a_2 + \dots + a_k + a_{k+1}).$$

This shows $P(k+1)$.

Conclusion: $P(n)$ holds for all integers $n \geq 2$ by induction the principle of induction.

3. Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f :

$$f(0) = 0$$

$$f(1) = 1$$

$$f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n . That is, construct a formula for $f(n)$ and prove its correctness.

Solution:

Let $P(n)$ be " $f(n) = n$ ". We prove that $P(n)$ is true for all $n \in \mathbb{N}$ by strong induction on n .

Base Cases ($n = 0, n = 1$): $f(0) = 0$ and $f(1) = 1$ by definition.

Inductive Hypothesis: Assume that $P(0) \wedge P(1) \wedge \dots \wedge P(k)$ hold for some arbitrary $k \geq 1$.

Inductive Step: We show $P(k+1)$:

$$\begin{aligned} f(k+1) &= 2f(k) - f(k-1) && \text{[Definition of } f] \\ &= 2(k) - (k-1) && \text{[Induction Hypothesis]} \\ &= k+1 && \text{[Algebra]} \end{aligned}$$

Conclusion: $P(n)$ is true for all $n \in \mathbb{N}$ by principle of strong induction.

4. A Horse of a Different Color

Did you know that all dogs are named Dubs? It's true. Maybe. Let's prove it by induction. The key is talking about groups of dogs, where every dog has the same name.

Let $P(i)$ mean "all groups of i dogs have the same name." We prove $\forall n P(n)$ by induction on n .

Base Case: $P(1)$ Take an arbitrary group of one dog, all dogs in that group all have the same name (there's only the one, so it has the same name as itself).

Inductive Hypothesis: Suppose $P(k)$ holds for some arbitrary k .

Inductive Step: Consider an arbitrary group of $k+1$ dogs. Arbitrarily select a dog, D , and remove it from the group. What remains is a group of k dogs. By inductive hypothesis, all k of those dogs have the same name. Add D back to the group, and remove some other dog D' . We have a (different) group of k dogs, so the inductive hypothesis applies again, and every dog in that group also shares the same name. All $k+1$ dogs appeared in at least one of the two groups, and our groups overlapped, so all of our $k+1$ dogs have the same name, as required.

Conclusion: We conclude $P(n)$ holds for all n by the principle of induction.

Recalling that Dubs is a dog, we have that every dog must have the same name as him, so every dog is named Dubs.

This proof cannot be correct (the proposed claim is false). Where is the bug?

Solution:

The bug is in the final sentence of the inductive step. We claimed that the groups overlapped, i.e. that some dog was in both of them. That's true for large k , but not when $k + 1 = 2$. When $k = 2$, D is in a group by itself, and D' was in a group by itself. The inductive hypothesis holds (D has the only name in its subgroup, and D' has the only name in its subgroup) but returning to the full group $\{D, D'\}$ we cannot conclude that they share a name.

From there everything unravels. $P(1) \not\rightarrow P(2)$, so we cannot use the principle of induction. It turns out this is the **only** bug in the proof. The argument in the inductive step is correct as long as $k + 1 > 2$. But that implication is always vacuous, since $P(2)$ is false.

5. Structural Induction

(a) Consider the following recursive definition of strings.

Basis Step: "" is a string

Recursive Step: If X is a string and c is a character then $\text{append}(c, X)$ is a string.

Recall the following recursive definition of the function len :

$$\begin{aligned} \text{len}("") &= 0 \\ \text{len}(\text{append}(c, X)) &= 1 + \text{len}(X) \end{aligned}$$

Now, consider the following recursive definition:

$$\begin{aligned} \text{double}("") &= "" \\ \text{double}(\text{append}(c, X)) &= \text{append}(c, \text{append}(c, \text{double}(X))). \end{aligned}$$

Prove that for any string X , $\text{len}(\text{double}(X)) = 2\text{len}(X)$.

Solution:

For a string X , let $P(X)$ be " $\text{len}(\text{double}(X)) = 2\text{len}(X)$ ". We prove $P(X)$ for all strings X by structural induction on X .

Base Case ($X = ""$): By definition, $\text{len}(\text{double}("")) = \text{len}("") = 0 = 2 \cdot 0 = 2\text{len}("")$, so $P("")$ holds

Inductive Hypothesis: Suppose $P(X)$ holds for some arbitrary string X .

Inductive Step: Goal: Show that $P(\text{append}(c, X))$ holds for any character c .

$$\begin{aligned} \text{len}(\text{double}(\text{append}(c, X))) &= \text{len}(\text{append}(c, \text{append}(c, \text{double}(X)))) && \text{[By Definition of double]} \\ &= 1 + \text{len}(\text{append}(c, \text{double}(X))) && \text{[By Definition of len]} \\ &= 1 + 1 + \text{len}(\text{double}(X)) && \text{[By Definition of len]} \\ &= 2 + 2\text{len}(X) && \text{[By IH]} \\ &= 2(1 + \text{len}(X)) && \text{[Algebra]} \\ &= 2(\text{len}(\text{append}(c, X))) && \text{[By Definition of len]} \end{aligned}$$

This proves $P(\text{append}(c, X))$.

Conclusion: $P(X)$ holds for all strings X by structural induction.

(b) Consider the following definition of a (binary) **Tree**:

Basis Step: \bullet is a **Tree**.

Recursive Step: If L is a **Tree** and R is a **Tree** then $\text{Tree}(\bullet, L, R)$ is a **Tree**.

The function **leaves** returns the number of leaves of a **Tree**. It is defined as follows:

$$\begin{aligned} \text{leaves}(\bullet) &= 1 \\ \text{leaves}(\text{Tree}(\bullet, L, R)) &= \text{leaves}(L) + \text{leaves}(R) \end{aligned}$$

Also, recall the definition of **size** on trees:

$$\begin{aligned} \text{size}(\bullet) &= 1 \\ \text{size}(\text{Tree}(\bullet, L, R)) &= 1 + \text{size}(L) + \text{size}(R) \end{aligned}$$

Prove that $\text{leaves}(T) \geq \text{size}(T)/2 + 1/2$ for all **Trees** T .

Solution:

For a tree T , let $P(T)$ be $\text{leaves}(T) \geq \text{size}(T)/2 + 1/2$. We prove $P(T)$ for all trees T by structural induction on T .

Base Case ($T = \bullet$): By definition of $\text{leaves}(\bullet)$, $\text{leaves}(\bullet) = 1$ and $\text{size}(\bullet) = 1$. So, $\text{leaves}(\bullet) = 1 \geq 1/2 + 1/2 = \text{size}(\bullet)/2 + 1/2$, so $P(\bullet)$ holds.

Inductive Hypothesis: Suppose $P(L)$ and $P(R)$ hold for some arbitrary trees L, R .

Inductive Step: Goal: Show that $P(\text{Tree}(\bullet, L, R))$ holds.

$$\begin{aligned} \text{leaves}(\text{Tree}(\bullet, L, R)) &= \text{leaves}(L) + \text{leaves}(R) && \text{[By Definition of leaves]} \\ &\geq (\text{size}(L)/2 + 1/2) + (\text{size}(R)/2 + 1/2) && \text{[By IH]} \\ &= (1/2 + \text{size}(L)/2 + \text{size}(R)/2) + 1/2 && \text{[By Algebra]} \\ &= \frac{1 + \text{size}(L) + \text{size}(R)}{2} + 1/2 && \text{[By Algebra]} \\ &= \text{size}(T)/2 + 1/2 && \text{[By Definition of size]} \end{aligned}$$

This proves $P(\text{Tree}(\bullet, L, R))$.

Conclusion: Thus, $P(T)$ holds for all trees T by structural induction.

(c) Prove the previous claim using strong induction. Define $P(n)$ as “all trees T of size n satisfy $\text{leaves}(T) \geq \text{size}(T)/2 + 1/2$ ”. You may use the following facts:

- For any tree T we have $\text{size}(T) \geq 1$.
- For any tree T , $\text{size}(T) = 1$ if and only if $T = \bullet$.

If we wanted to prove these claims, we could do so by structural induction.

Note, in the inductive step you should start by letting T be an arbitrary tree of size $k + 1$.

Solution:

Let $P(n)$ be “all trees T of size n satisfy $\text{leaves}(T) \geq \text{size}(T)/2 + 1/2$ ”. We show $P(n)$ for all integers $n \geq 1$ by strong induction on n .

Base Case: Let T be an arbitrary tree of size 1. The only tree with size 1 is \bullet , so $T = \bullet$. By definition,

$\text{leaves}(T) = \text{leaves}(\bullet) = 1$ and thus $\text{size}(T) = 1 = 1/2 + 1/2 = \text{size}(T)/2 + 1/2$. This shows the base case holds.

Inductive Hypothesis: Suppose that $P(j)$ holds for all integers $j = 1, 2, \dots, k$ for some arbitrary integer $k \geq 1$.

Inductive Step: Let T be an arbitrary tree of size $k + 1$. Since $k + 1 > 1$, we must have $T \neq \bullet$. It follows from the definition of a tree that $T = \text{Tree}(\bullet, L, R)$ for some trees L and R . By definition, we have $\text{size}(T) = 1 + \text{size}(L) + \text{size}(R)$. Since sizes are non-negative, this equation shows $\text{size}(T) > \text{size}(L)$ and $\text{size}(T) > \text{size}(R)$ meaning we can apply the inductive hypothesis. This says that $\text{leaves}(L) \geq \text{size}(L)/2 + 1/2$ and $\text{leaves}(R) \geq \text{size}(R)/2 + 1/2$.

We have,

$$\begin{aligned}
 \text{leaves}(T) &= \text{leaves}(\text{Tree}(\bullet, L, R)) \\
 &= \text{leaves}(L) + \text{leaves}(R) && \text{[By Definition of leaves]} \\
 &\geq (\text{size}(L)/2 + 1/2) + (\text{size}(R)/2 + 1/2) && \text{[By IH]} \\
 &= (1/2 + \text{size}(L)/2 + \text{size}(R)/2) + 1/2 && \text{[By Algebra]} \\
 &= \frac{1 + \text{size}(L) + \text{size}(R)}{2} + 1/2 && \text{[By Algebra]} \\
 &= \text{size}(T)/2 + 1/2 && \text{[By Definition of size]}
 \end{aligned}$$

This shows $P(k + 1)$.

Conclusion: $P(n)$ holds for all integers $n \geq 1$ by the principle of strong induction.

Note, this proves the claim for all trees because every tree T has some size $s \geq 1$. Then $P(s)$ says that all trees of size s satisfy the claim, including T .

6. Recursively Defined Sets of Strings

For each of the following, write a recursive definition of the sets satisfying the following properties. Briefly justify that your solution is correct.

- (a) Binary strings of even length.

Solution:

Basis: $\varepsilon \in S$.

Recursive Step: If $x \in S$, then $x00, x01, x10, x11 \in S$.

Exclusion Rule: Each element of S is obtained from the basis and a finite number of applications of the recursive step.

“Brief” Justification: We will show that $x \in S$ iff x has even length (i.e., $|x| = 2n$ for some $n \in \mathbb{N}$). (Note: “brief” is in quotes here. Try to write shorter explanations in your homework assignment when possible!)

Suppose $x \in S$. If x is the empty string, then it has length 0, which is even. Otherwise, x is built up from the empty string by repeated application of the recursive step, so it is of the form $x_1x_2\dots x_n$, where each $x_i \in \{00, 01, 10, 11\}$. In that case, we can see that $|x| = |x_1| + |x_2| + \dots + |x_n| = 2n$, which is even. Now, suppose that x has even length. If its length is zero, then it is the empty string, which is in S . Otherwise, it has length $2n$ for some $n > 0$, and we can write x in the form $x_1x_2\dots x_n$, where each $x_i \in \{00, 01, 10, 11\}$ has length 2. Hence, we can see that x can be built up from the empty string by applying the recursive step with x_1 , then x_2 , and so on up to x_n , which shows that $x \in S$.

- (b) Binary strings not containing 10.

Solution:

If the string does not contain 10, then the first 1 in the string can only be followed by more 1s. Hence, it must be of the form $0^m 1^n$ for some $m, n \in \mathbb{N}$.

Basis: $\varepsilon \in S$.

Recursive Step: If $x \in S$, then $0x \in S$ and $x1 \in S$.

Exclusion Rule: Each element of S is obtained from the basis and a finite number of applications of the recursive step.

Brief Justification: The empty string satisfies the property, and the recursive step cannot place a 0 after a 1 since it only adds 0s on the left. Hence, every string in S satisfies the property.

In the other direction, from our discussion above, any string of this form can be written as $y = 0^m 1^n$ for some $m, n \in \mathbb{N}$. We can build up the string y from the empty string by applying the rule $x \rightarrow 0x$ m times and then applying the rule $x \rightarrow x1$ n times. This shows that the string y is in S .

- (c) Binary strings not containing 10 as a substring and having at least as many 1s as 0s.

Solution:

These must be of the form $0^m 1^n$ for some $m, n \in \mathbb{N}$ with $m \leq n$. We can ensure that by pairing up the 0s with 1s as they are added:

Basis: $\varepsilon \in S$.

Recursive Step: If $x \in S$, then $0x1 \in S$ and $x1 \in S$.

Exclusion Rule: Each element of S is obtained from the basis and a finite number of applications of the recursive step.

Brief Justification: As in the previous part, we cannot add a 0 after a 1 because we only add 0s at the front. And since every 0 comes with a 1, we always have at least as many 1s as 0s.

In the other direction, from our discussion above, any string of this form can be written as xy , where $x = 0^m 1^m$ and $y = 1^{n-m}$, since $n \geq m$. We can build up the string x from the empty string by applying the rule $x \rightarrow 0x1$ m times and then produce the string xy by applying the rule $x \rightarrow x1$ $n - m$ times, which shows that the string is in S .

- (d) Binary strings containing at most two 0s and at most two 1s.

Solution:

This is the set of all binary strings of length at most 4 *except* for these:

000, 1000, 0100, 0010, 0001, 0000, 111, 0111, 1011, 1101, 1110, 1111

Since this is a **finite set**, we can define it recursively using only basis elements and no recursive step.

7. Walk the Dawgs

Suppose a dog walker takes care of $n \geq 12$ dogs. The dog walker is not a strong person, and will walk dogs in groups of 3 or 7 at a time (every dog gets walked exactly once). Prove the dog walker can always split the n dogs into groups of 3 or 7.

Solution:

Let $P(n)$ be “a group with n dogs can be split into groups of 3 or 7 dogs.” We will prove $P(n)$ for all natural numbers $n \geq 12$ by strong induction.

Base Cases $n = 12, 13, 14$, or 15 : $12 = 3 + 3 + 3 + 3$, $13 = 3 + 7 + 3$, $14 = 7 + 7$, So $P(12)$, $P(13)$, and $P(14)$ hold.

Inductive Hypothesis: Assume that $P(12), \dots, P(k)$ hold for some arbitrary $k \geq 14$.

Inductive Step: Goal: Show $k + 1$ dogs can be split into groups of size 3 or 7.

We first form one group of 3 dogs. Then we can divide the remaining $k - 2$ dogs into groups of 3 or 7 by the assumption $P(k - 2)$. (Note that $k \geq 14$ and so $k - 2 \geq 12$; thus, $P(k - 2)$ is among our assumptions $P(12), \dots, P(k)$.)

Conclusion: $P(n)$ holds for all integers $n \geq 12$ by principle of strong induction.

8. For All

For this problem, we'll see an incorrect use of induction. For this problem, we'll think of all of the following as binary trees:

- A single node.
- A root node, with a left child that is the root of a binary tree (and no right child)
- A root node, with a right child that is the root of a binary tree (and no left child)
- A root node, with both left and right children that are roots of binary trees.

Let $P(n)$ be “for all trees of height n , the tree has an odd number of nodes”

Take a moment to realize this claim is false.

Now let's see an incorrect proof:

We'll prove $P(n)$ for all $n \in \mathbb{N}$ by induction on n .

Base Case ($n = 0$): There is only one tree of height 0, a single node. It has one node, and $1 = 2 \cdot 0 + 1$, which is an odd number of nodes.

Inductive Hypothesis: Suppose $P(i)$ holds for $i = 0, \dots, k$, for some arbitrary $k \geq 0$.

Inductive Step: Let T be an arbitrary tree of height k . All trees with nodes (and since $k \geq 0$, T has at least one node) have a leaf node. Add a left child and right child to a leaf (pick arbitrarily if there's more than one), This tree now has height $k + 1$ (since T was height k and we added children below). By IH, T had an odd number of nodes, call it $2j + 1$ for some integer j . Now we have added two more, so our new tree has $2j + 1 + 2 = 2(j + 1) + 1$ nodes. Since j was an integer, so is $j + 1$, and our new tree has an odd number of nodes, as required, so $P(k + 1)$ holds.

By the principle of induction, $P(n)$ holds for all $n \in \mathbb{N}$. Since every tree has an (integer) height of 0 or more, every tree is included in some $P()$, so the claim holds for all trees.

(a) What is the bug in the proof? **Solution:**

The proof, in trying to show something about an arbitrary of height $k + 1$, builds a **particular** tree of height $k + 1$, not an arbitrary one. While the tree built is indeed of height $k + 1$ and has an odd number of nodes, it is not an *arbitrary* tree of height $k + 1$.

Why is it that in this problem we have to start with $k + 1$, but we didn't in the “Walk the Dawgs” problem above? “Walk the Dawgs” is asking you to prove an exists statement (“can split” not “for every possible split...”). When proving an exists, you just say “here's how to do it”, you don't need to introduce an arbitrary variable

(b) What should the starting point and target of the IS be (you can't write a full proof, as the claim is false).

Solution:

“Let T be an arbitrary tree of height $k + 1$ ” should be the first sentence. “ T has an odd number of nodes” is the target. Notice that the only difference