CSE 311 Section 7

Induction

Administrivia

Announcements & Reminders

- HW4
  - Grades out now
  - If you think something was graded incorrectly, submit a regrade request!

- HW5
  - due yesterday (11/05 @ 11:59 pm)

- HW6
  - due 11/12 @ 11:59 pm on Gradescope

Induction

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds for all n by induction on n.

Base Case: Show $P(b)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds for all n by the principle of induction.

(Weak) Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds **for all n** by induction on n.

Note: often you will condition n here, like "all natural numbers n" or "$n \geq 0$"

Base Case: Show $P(b)$ is true.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Show $P(k+1)$ (i.e. get $P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds **for all n** by the principle of induction.

Match the earlier condition on n in your conclusion!

Problem 1 – Induction with Equality

a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

Lets walk through part (a) together.

We can “fill in” our induction template to construct our proof by induction.

Problem 1 – Induction with Equality

Show using induction that
$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
for all $n \in \mathbb{N}$.

Let $P(n)$ be "". We show $P(n)$ holds for (some) n by induction on n.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for (some) n by the principle of induction.

Problem 1 – Induction with Equality

Show using induction that
$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
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Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k + 1)$:

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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Show using induction that
$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
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Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Inductive Step: Goal: Show $P(k+1)$:

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Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

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Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 1 – Induction with Equality

Show using induction that
$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$, i.e. $0 + 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$

Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

$$
\begin{aligned}
0 + 1 + \cdots + k + (k+1) &= \cdots \\
&\cdots \\
&= \frac{(k+1)(k+2)}{2} \qquad ?
\end{aligned}
$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 1 – Induction with Equality

Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

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Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

$$0 + 1 + \cdots + k + (k+1) = (0 + 1 + \cdots + k) + (k+1)$$

$$\cdots$$

$$= \frac{(k+1)(k+2)}{2} \qquad ?$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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Show using induction that
$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
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Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

$$
\begin{aligned}
0 + 1 + \cdots + k + (k+1) &= (0 + 1 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by I.H.} \\
&\cdots \\
&= \frac{(k+1)(k+2)}{2} && ?
\end{aligned}
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Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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Show using induction that
$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$
for all $n \in \mathbb{N}$.

Let $P(n)$ be "$0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

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\begin{aligned}
0 + 1 + \cdots + k + (k+1) &= (0 + 1 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by I.H.} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&\dots \\
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Base Case: $P(0)$: $0 + \cdots = 0 = \frac{0(0+1)}{2}$ so the base case holds.

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Inductive Step: Goal: Show $P(k+1)$: $0 + 1 + \cdots + k + (k+1) = \frac{(k+1)(k+2)}{2}$

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\begin{aligned}
0 + 1 + \cdots + k + (k+1) &= (0 + 1 + \cdots + k) + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{by I.H.} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} && ?
\end{aligned}
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Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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Show using induction that
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&= \frac{k(k+1)+2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2} && \text{factoring out (k+1)}
\end{aligned}
$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 1 – Induction with Equality

a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

Now try part (b) with people around you, and then we'll go over it together!

$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}.$

$\triangle_n = \frac{n(n+1)}{2}$. Prove for all $n \in \mathbb{N}$:

$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$

Problem 1 – Induction with Equality

Let $P(n)$ be "". We show $P(n)$ holds for (some) n by induction on n.

Base Case: $P(b)$:

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for (some) n by the principle of induction.

Problem 1 – Induction with Equality

$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}.$
$\triangle_n = \frac{n(n+1)}{2}$. Prove for all $n \in \mathbb{N}$:
$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$

Let $P(n)$ be "$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(b)$:

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Let $P(n)$ be "$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0^3 = 0 = (0)^2$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show $P(k+1)$:

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

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$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}.$

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Let $P(n)$ be "$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0^3 = 0 = (0)^2$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. i.e. $0^3 + 1^3 + \cdots + k^3 += (0 + 1 + \cdots + k)^2$

Inductive Step: Goal: Show $P(k + 1)$:

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}.$

$\triangle_n = \frac{n(n+1)}{2}$. Prove for all $n \in \mathbb{N}$:

$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$

Problem 1 – Induction with Equality

Let $P(n)$ be "$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0^3 = 0 = (0)^2$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. i.e. $0^3 + 1^3 + \cdots + k^3 += (0 + 1 + \cdots + k)^2$

Inductive Step: Goal: Show $P(k+1)$: $0^3 + 1^3 + \cdots + k^3 + (k+1)^3 = (0 + 1 + \cdots + k + (k+1))^2$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 1 – Induction with Equality

$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}.$

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Let $P(n)$ be "$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

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Inductive Step: Goal: Show $P(k+1)$: $0^3 + 1^3 + \cdots + k^3 + (k+1)^3 = (0 + 1 + \cdots + k + (k+1))^2$

$$0^3 + 1^3 + \cdots + k^3 + (k+1)^3 = \cdots$$

$$\cdots$$

$$= (0 + 1 + \cdots + k + (k+1))^2$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Problem 1 – Induction with Equality

$\triangle_n = 1 + 2 + \cdots + n,\ n \in \mathbb{N}.$

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Let $P(n)$ be "$0^3 + 1^3 + \cdots + n^3 = (0 + 1 + \cdots + n)^2$". We show $P(n)$ holds for all $n \in \mathbb{N}$ by induction on n.

Base Case: $P(0)$: $0^3 = 0 = (0)^2$ so the base case holds.

Inductive Hypothesis: Suppose $P(k)$ holds for an arbitrary $k \geq 0$. i.e. $0^3 + 1^3 + \cdots + k^3 += (0 + 1 + \cdots + k)^2$

Inductive Step: Goal: Show $P(k+1)$: $0^3 + 1^3 + \cdots + k^3 + (k+1)^3 = (0 + 1 + \cdots + k + (k+1))^2$

$$
\begin{aligned}
0^3 + 1^3 + \cdots + k^3 + (k+1)^3 &= (0 + 1 + \cdots + k)^2 + (k+1)^3 && \text{by I.H.} \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 && \text{by (a)} \\
&= (k+1)^2\left(\frac{k^2}{2^2} + (k+1)\right) && \text{factor out } (k+1)^2 \\
&= (k+1)^2\left(\frac{k^2+4k+4}{4}\right) && \\
&= (k+1)^2\left(\frac{(k+2)^2}{4}\right) && \text{factor numerator} \\
&= \left(\frac{(k+1)(k+2)}{2}\right)^2 && \\
&= (0 + 1 + \cdots + k + (k+1))^2 && \text{by (a)}
\end{aligned}
$$

Conclusion: Therefore, $P(n)$ holds for all $n \in \mathbb{N}$ by the principle of induction.

Strong Induction

Why Strong Induction?

In **weak induction**, the inductive hypothesis only assumes that $P(k)$ is true and uses that in the inductive step to prove the implication $P(k) \to P(k+1)$.

In **strong induction**, the inductive hypothesis assumes the predicate holds for every step from the base case(s) up to $P(k)$. This usually looks something like $P(b_1) \wedge P(b_2) \wedge \cdots \wedge P(k)$. Then it uses this stronger inductive hypothesis in the inductive step to prove the implication $P(b_1) \wedge \cdots \wedge P(k) \to P(k+1)$.

Strong induction is necessary when we have multiple base cases, or when we need to go back to a smaller number than k in our inductive step.

Strong Induction Template

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds for all $n \geq b_{min}$ by induction on n.

Base Case: Show $P(b_{min}), P(b_{min+1}), \ldots, P(b_{max})$ are all true.

Inductive Hypothesis: Suppose $P(b_{min}) \land \cdots \land P(k)$ hold for an arbitrary $k \geq b_{max}$.

Inductive Step: Show $P(k+1)$ (i.e. get $P(b_{min}) \land \cdots \land P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, $P(n)$ holds for all $n \geq b_{min}$ by the principle of induction.

Problem 3 – Cantelli's Rabbits

Xavier Cantelli owns some rabbits. The number of rabbits he has in any given year is described by the function f:

$$f(0) = 0$$
$$f(1) = 1$$
$$f(n) = 2f(n-1) - f(n-2) \text{ for } n \geq 2$$

Determine, with proof, the number, $f(n)$, of rabbits that Cantelli owns in year n. That is, construct a formula for $f(n)$ and prove its correctness.

First, let's construct a formula for $f(n)$. How many rabbits does he have each year? Let's do some calculations, and see if we can find a pattern. Then, we'll use induction to prove the pattern holds for all n!

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$$f(0) = 0$$
$$f(1) = 1$$
$$f(2) = 2f(2-1) - f(2-2) = 2f(1) - f(0) = 2(1) - 0 = 2 - 0 = 2$$

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$$f(3) = 2f(3-1) - f(3-2) = 2f(2) - f(1) = 2(2) - 1 = 4 - 1 = 3$$

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$$f(3) = 2f(3-1) - f(3-2) = 2f(2) - f(1) = 2(2) - 1 = 4 - 1 = 3$$
$$f(4) = 2f(4-1) - f(4-2) = 2f(3) - f(2) = 2(3) - 2 = 6 - 2 = 4$$

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$$f(0) = 0$$
$$f(1) = 1$$
$$f(2) = 2f(2-1) - f(2-2) = 2f(1) - f(0) = 2(1) - 0 = 2 - 0 = 2$$
$$f(3) = 2f(3-1) - f(3-2) = 2f(2) - f(1) = 2(2) - 1 = 4 - 1 = 3$$
$$f(4) = 2f(4-1) - f(4-2) = 2f(3) - f(2) = 2(3) - 2 = 6 - 2 = 4$$

It seems like we have a pattern here!

$f(n) = n$

But we don't want to have to check for EVERY n, so let's see if we can prove it with induction instead!

Problem 3 – Cantelli's Rabbits

What kind of induction should we use?

Problem 3 – Cantelli's Rabbits

What kind of induction should we use?

Strong induction!

Problem 3 – Cantelli's Rabbits

What kind of induction should we use?

Strong induction!

Two big clues:

- Multiple base cases in the formula: $f(0) = 0$ and $f(1) = 1$
- Recursively defined step of formula goes back further than just n:
  - $f(n)$ based on both $f(n-1)$ and $f(n-2)$
  - for $P(n)$ to be true, both $P(n-1)$ and $P(n-2)$ must be true

Problem 3 – Cantelli's Rabbits

Let $P(n)$ be "(whatever you're trying to prove)".
We show $P(n)$ holds for all $n \geq b_{min}$ by induction on n.

<u>Base Case:</u> Show $P(b_{min}), P(b_{min+1}), \ldots, P(b_{max})$ are all true.

<u>Inductive Hypothesis:</u> Suppose $P(b_{min}) \land \cdots \land P(k)$ hold for an arbitrary $k \geq b_{max}$.

<u>Inductive Step:</u> Show $P(k+1)$ (i.e. get $P(b_{min}) \land \cdots \land P(k) \rightarrow P(k+1)$)

<u>Conclusion:</u> Therefore, $P(n)$ holds for all $n \geq b_{min}$ by the principle of induction.

Fill in the strong induction template to prove the claim!

Problem 3 – Cantelli's Rabbits

Let $P(n)$ be "".
We show $P(n)$ holds …
Base Cases:
Inductive Hypothesis:

Inductive Step:

Conclusion: Therefore, $P(n)$ holds for all … by the principle of induction.

Problem 3 – Cantelli's Rabbits

Let $P(n)$ be "$f(n) = n$".

We show $P(n)$ holds for all $n \geq 0$ by induction on n.

Base Cases:

Inductive Hypothesis:

Inductive Step:

Conclusion: Therefore, $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Problem 3 – Cantelli's Rabbits

Let $P(n)$ be "$f(n) = n$".

We show $P(n)$ holds for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0, n = 1)$: $f(0) = 0$ and $f(1) = 1$ by definition of f.

Inductive Hypothesis:

Inductive Step:

Conclusion: Therefore, $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Problem 3 – Cantelli's Rabbits

Let $P(n)$ be "$f(n) = n$".

We show $P(n)$ holds for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0, n = 1)$: $f(0) = 0$ and $f(1) = 1$ by definition of f.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ hold for an arbitrary $k \geq 1$.

Inductive Step:

Conclusion: Therefore, $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Problem 3 – Cantelli's Rabbits

Let $P(n)$ be "$f(n) = n$".

We show $P(n)$ holds for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0, n = 1)$: $f(0) = 0$ and $f(1) = 1$ by definition of f.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ hold for an arbitrary $k \geq 1$.

i.e. $f(k) = k, f(k-1) = k-1, f(k-2) = k-2$, etc.

Inductive Step:

Conclusion: Therefore, $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Problem 3 – Cantelli's Rabbits

Let $P(n)$ be "$f(n) = n$".

We show $P(n)$ holds for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0, n = 1)$: $f(0) = 0$ and $f(1) = 1$ by definition of f.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ hold for an arbitrary $k \geq 1$.

i.e. $f(k) = k, f(k-1) = k-1, f(k-2) = k-2$, etc.

Inductive Step: Goal: Show $P(k+1)$: $f(k+1) = k+1$

Conclusion: Therefore, $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Problem 3 – Cantelli's Rabbits

Let $P(n)$ be "$f(n) = n$".

We show $P(n)$ holds for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0, n = 1)$: $f(0) = 0$ and $f(1) = 1$ by definition of f.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ hold for an arbitrary $k \geq 1$.

i.e. $f(k) = k, f(k-1) = k-1, f(k-2) = k-2$, etc.

Inductive Step: Goal: Show $P(k+1)$: $f(k+1) = k+1$

$$
\begin{aligned}
f(k+1) &= \ldots \\
&\ldots \\
&= k+1
\end{aligned}
$$

Conclusion: Therefore, $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Problem 3 – Cantelli's Rabbits

Let $P(n)$ be "$f(n) = n$".

We show $P(n)$ holds for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0, n = 1)$: $f(0) = 0$ and $f(1) = 1$ by definition of f.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ hold for an arbitrary $k \geq 1$.

i.e. $f(k) = k, f(k-1) = k-1, f(k-2) = k-2$, etc.

Inductive Step: Goal: Show $P(k+1)$: $f(k+1) = k+1$

$$f(k+1) = 2f(k) - f(k-1) \quad \text{definition of } f$$

$$\ldots$$

$$= k + 1$$

Conclusion: Therefore, $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Problem 3 – Cantelli's Rabbits

Let $P(n)$ be "$f(n) = n$".

We show $P(n)$ holds for all $n \geq 0$ by induction on n.

Base Cases: $(n = 0, n = 1)$: $f(0) = 0$ and $f(1) = 1$ by definition of f.

Inductive Hypothesis: Suppose $P(0) \wedge P(1) \wedge \cdots \wedge P(k)$ hold for an arbitrary $k \geq 1$.

i.e. $f(k) = k, f(k-1) = k-1, f(k-2) = k-2$, etc.

Inductive Step: Goal: Show $P(k+1)$: $f(k+1) = k+1$

$$\begin{aligned} f(k+1) &= 2f(k) - f(k-1) && \text{definition of } f \\ &= 2(k) - (k-1) && \text{by I.H.} \\ &= k+1 \end{aligned}$$

Conclusion: Therefore, $P(n)$ holds for all $n \geq 0$ by the principle of induction.

Structural Induction

Idea of Structural Induction

Every element is built up recursively…

So to show $P(s)$ for all $s \in S$…

Show $P(b)$ for all base case elements b.

Show for an arbitrary element not in the base case, if $P()$ holds for every named element in the recursive rule, then $P()$ holds for the new element (each recursive rule will be a case of this proof).

Structural Induction Template

Let $P(x)$ be “<predicate>”. We show $P(x)$ holds for all $x \in S$ by structural induction.

Base Case: Show $P(x)$
[Do that for every base cases x in S.]

Inductive Hypothesis: Suppose $P(x)$ for an arbitrary x
[Do that for every x listed as in S in the recursive rules.]

Inductive Step: Show $P()$ holds for y.
[You will need a separate case/step for every recursive rule.]

Therefore $P(x)$ holds for all $x \in S$ by the principle of induction.

Problem 5b – Structural Induction on Trees

Definition of Tree:
Basis Step: • is a Tree.
Recursive Step: If L is a Tree and R is a Tree then Tree(•, L, R) is a Tree

Definition of leaves():
leaves(•) = 1
leaves(Tree(•, L, R)) = leaves(L) + leaves(R)

Definition of size():
size(•) = 1
size(Tree(•, L, R)) =1 + size(L) + size(R)

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

Work on this problem with the people around you.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For x ∈ S, let P(x) be "".
We show P(x) holds for all x ∈ S by structural induction on x.

Base Case: Show P(x) (for all x in the basis rules)

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules), i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(x) holds for all x ∈ S by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(x) holds for all x ∈ S by structural induction on x.

Base Case: Show P(x) (for all x in the basis rules)

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules), i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: Show P(x) (for all x in the basis rules)

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules), i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(x) (for all x in the recursive rules), i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R, i.e. (IH in terms of P(x))

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show that P(?) holds. (IS goal in terms of P(?))

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

Again, as long as you can get this far, you will get the majority of points on the problem! Go for this skeleton first, and then think about what you need to do to complete the proof.

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R)) =

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

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So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R)) = leaves(L) + leaves(R)   definition of leaves

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

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So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R))	= leaves(L) + leaves(R)	definition of leaves
	≥ (size(L)/2 + 1/2) + (size(R)/2 + 1/2)	by Inductive Hypothesis

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R))	= leaves(L) + leaves(R)	definition of leaves
	≥ (size(L)/2 + 1/2) + (size(R)/2 + 1/2)	by Inductive Hypothesis
	= (1/2 + size(L)/2 + size(R)/2) + 1/2	

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

<u>Base Case:</u> P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

<u>Inductive Hypothesis:</u> Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

<u>Inductive Step:</u> Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R))	= leaves(L) + leaves(R)	definition of leaves
	≥ (size(L)/2 + 1/2) + (size(R)/2 + 1/2)	by Inductive Hypothesis
	= (1/2 + size(L)/2 + size(R)/2) + 1/2	
	= (1 + size(L) + size(R)) / 2 + 1/2	

<u>Conclusion:</u> Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be “leaves(T) ≥ size(T)/2 + 1/2”.
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R))	= leaves(L) + leaves(R)	definition of leaves
	≥ (size(L)/2 + 1/2) + (size(R)/2 + 1/2)	by Inductive Hypothesis
	= (1/2 + size(L)/2 + size(R)/2) + 1/2	
	= (1 + size(L) + size(R)) / 2 + ½	
	= size(T)/2 + 1/2	definition of size

Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Problem 5b – Structural Induction on Trees

Prove that leaves(T) ≥ size(T)/2 + 1/2 for all Trees T

For a tree T, let P(T) be "leaves(T) ≥ size(T)/2 + 1/2".
We show P(T) holds for all trees T by structural induction on T.

Base Case: P(•): By definition of leaves(•), leaves(•) = 1 and size(•) = 1.
So, leaves(•) = 1 ≥ 1/2 + 1/2 = size(•)/2 + 1/2, so P(•) holds.

Inductive Hypothesis: Suppose P(L) and P(R) hold for some arbitrary trees L and R,
i.e., leaves(L) ≥ size(L)/2 + 1/2, leaves(R) ≥ size(R)/2 + 1/2

Inductive Step: Goal: Show P(Tree(•, L, R)): leaves(Tree(•, L, R)) ≥ size(Tree(•, L, R))/2 + 1/2

leaves(Tree(•, L, R))	= leaves(L) + leaves(R)	definition of leaves
	≥ (size(L)/2 + 1/2) + (size(R)/2 + 1/2)	by Inductive Hypothesis
	= (1/2 + size(L)/2 + size(R)/2) + 1/2	
	= (1 + size(L) + size(R)) / 2 + ½	
	= size(T)/2 + 1/2	definition of size

So, P(Tree(•, L, R)) holds!
Conclusion: Therefore P(T) holds for all trees T by the principle of induction.

Weak Induction With Inequalities

Problem 1 - Induction With Inequality

Prove that $6n + 6 < 2^n$ for all integers $n \geq 6$.

Work on this problem with the people around you.

Problem 1 - Induction With Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$

Let P(n) be "(whatever you're trying to prove)".
We show P(n) holds for all $n \in S$ by induction on n.

Base Case: Show P(b) is true.

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show that P(k+1) holds. (i.e. get $P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, P(n) holds for all $n \in S$ by the principle of induction.

Problem 1 - Induction With Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for all $n \in S$ by induction on n.

Base Case: Show P(b) is true.

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show that P(k+1) holds. (i.e. get $P(k) \rightarrow P(k+1)$)

Conclusion: Therefore, P(n) holds for all $n \in S$ by the principle of induction.

Problem 1 - Induction With Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for all integers $n \geq 6$ by induction on n.

Base Case: Show P(b) is true.

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show that P(k+1) holds. (i.e. get P(k) → P(k+1))

Conclusion: Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction With Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for all integers $n \geq 6$ by induction on n.

Base Case (n = 6): $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P (6) holds.

Inductive Hypothesis: Suppose P(k) holds for an arbitrary $k \geq b$.

Inductive Step: Goal: Show that P(k+1) holds. (i.e. get P(k) → P(k+1))

Conclusion: Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction With Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for all integers $n \geq 6$ by induction on n.

Base Case (n = 6): $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P (6) holds.

Inductive Hypothesis: Suppose that $6k + 6 < 2^k$ for an arbitrary integer $k \geq 6$.

Inductive Step: Goal: Show that P(k+1) holds. (i.e. get P(k) → P(k+1))

Conclusion: Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

Problem 1 - Induction With Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$

Let P(n) be "$6n + 6 < 2^n$".
We show P(n) holds for all integers $n \geq 6$ by induction on n.

<u>Base Case (n = 6):</u> $6 \cdot 6 + 6 = 42 < 64 = 2^6$, so P (6) holds.

<u>Inductive Hypothesis:</u> Suppose that $6k + 6 < 2^k$ for an arbitrary integer $k \geq 6$.

<u>Inductive Step:</u> Goal: Show $6(k+1) + 6 < 2^{k+1}$

<u>Conclusion:</u> Therefore, P(n) holds for all integers $n \geq 6$ by the principle of induction.

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So, P(k+1) holds.

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That’s All, Folks!

Thanks for coming to section this week!
Any questions?