Section 6: More Proofs, Induction

Proof by Contrapositve

1. Even Numbers, Odd Results!

(This question was also in Section 4)
For any integer j, if $3j + 1$ is even, then j is odd

(a) Write the predicate logic of this claim

Odd(x) := x is $2k + 1$, for some integer k
Even(x) := x is $2k$, for some integer k

(b) Write the contrapositive of this claim

(c) Prove this claim using a proof by contrapositive.

2. The Trifecta

(This question was also in Section 4)
Consider the following proposition: For each integer a, if 3 divides a^2, then 3 divides a

(a) Write the contrapositive of this proposition as a sentence:

(b) Prove the proposition by proving its contrapositive.
Hint: Consider using cases based on the remainder for "division by 3." There will be two cases!

Proof by Contradiction

3. Wait, That Doesn't Add Up

(This question was also in Section 5)
Write a proof by contradiction for the following proposition: There exist no integers x and y such that $18x + 6y = 1$.
HINT: Try writing in propositional logic, then negating this statement before writing your proof.

4. Prime Checking

This question is part of a question on the Section 5 handout - take a look at the handout for the full context and question.
We will use "nontrivial divisor" to mean a factor that isn't 1 or the number itself. Formally, a positive integer k being a "nontrivial divisor" of n means that $k|n$, $k \neq 1$ and $k \neq n$.
Claim: For every positive integer n, if n has a nontrivial divisor, then it has a nontrivial divisor at most $\sqrt{n}$.

Prove the claim. Hint: we recommend a proof by contradiction.

A Set of Set Questions

5. How Many Elements?

For each of these, how many elements are in the set? If the set has infinitely many elements, say ∞.

(a) $A = \{1, 2, 3, 2\}$

(b) $B = \{\emptyset,\ \{\emptyset\},\ \{\emptyset, \emptyset\},\ \{\emptyset, \emptyset, \emptyset\},\ \ldots\}$

(c) $C = A \times (B \cup \{7\})$

(d) $D = \emptyset$

(e) $E = \{\emptyset\}$

(f) $F = \mathcal{P}(\{\emptyset\})$

6. Just The Setup

For this statement,

- Translate the sentence into predicate logic.
- Write the first few sentences and last few sentences of the English proof.

If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$ for any sets A, B, C.

7. Set Theory

Suppose that $A \subseteq B$. Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

8. Set = Set

Prove the following set identities. Write an English proof.

(a) Let the universal set be $\mathcal{U}$. Prove $A \cap \overline{B} \subseteq A \backslash B$ for any sets A, B.

(b) Prove that $(A \cap B) \times C \subseteq A \times (C \cup D)$ for any sets A, B, C, D.

9. Set Equality

Prove that $A \cap (A \cup B) = A$ for any sets A, B.

Intro to Induction

10. Induction with Equality

(a) Show using induction that $0 + 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \in \mathbb{N}$.

(b) Define the triangle numbers as $\triangle_n = 1 + 2 + \cdots + n$, where $n \in \mathbb{N}$. In part (a) we showed $\triangle_n = \frac{n(n+1)}{2}$. Prove the following equality for all $n \in \mathbb{N}$:

$$0^3 + 1^3 + \cdots + n^3 = \triangle_n^2$$

11. Induction with Divides

Prove that $9 \mid (n^3 + (n+1)^3 + (n+2)^3)$ for all $n > 1$ by induction.

12. Induction with Inequality

Prove that $6n + 6 < 2^n$ for all $n \geq 6$.